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Survey of Periodic Solutions of the Nonlinear Ordinary Differential Equations and Study of Periodic Solutions of the Duffing Type Equation with the Square Wave External Force

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1 Introduction

In the cource of studying on nonlinear ordinary differential equations, the existence of periodic solutions has been forcused on as one of main subjects. The nontrivial periodic solutions have important meanings in a variety of field such as engineering, medical, economic area and so on. The question of whether a natural or social phenomenon has a certain periodicity is important and interesting for us. For example, in the climatology, the past periodicity of global climate change has been researched well but its future periodicity is a significant issure. Another example is the terrestrial magnetism which turned the other way by the time rate of 1.5 times per a million years [10],[14]. These are earth-scale examples but using a familiar one, there exists a rise or fall in the exchange rate and stock market.

Moreover some phenomena in nature have a multiple-time periodicity, which means that multiple-time oscillations, like double-time, triple-time, quadruple-time oscillations and so on, exist in a period. The electrocardiogram of human beings is a good example. The healthy heart, roughly speaking, beats triple-time oscillations in a period. The normal heart beating consists of a P wave, QRS complex and a T wave[7] in a period. However man with heart defect does not always beat triple-time oscillations.

The objective of this paper is to survey studies of periodic solutions of the nonlinear ordinary differential equations and present the explicit form for periodic solutions of a nonlinear ordinary differential equation(Eq.(2.12)) with the external force. Also we show the nontrivial periodic solutions for Eq.(2.12) create multiple-time oscillations in a period depending on the period of the external force.

The paper is organized as follows: first, we survey periodic solutions of the nonlinear ordinary differential equations in the literature. In the following section, we treat the linear case $q=0$ in Eq.(2.12). There exist such types of periodic solutions as the ω-periodic(Definition 3.1), hidden periodic(Definition 3.2) and quasi-periodic one(Remark 3.5) even in the linear case. The explicit forms of such solutions are shown as well as the periodic conditions. Then in the first half of Section 4, the solution when $e(t)=const.$ is obtained and in the latter half we construct the nontrivial periodic solution in the Farkas sense using the result of the first half. The 'Farkas sense'[5] means that it is periodic with a period ω of the external force which can be chosen appropriately. The numerical simulations are also presented at important positions.

2 Survey of periodic solutions of the nonlinear ordinary differential equations

In this section we survey studies of periodic solutions of the nonlinear ordinary differential equations in the literature [3],[5],[6],[8],[9],[13],[15],[17],[19],[20]. First we state the following proposition: Let $\dot{u}=\frac{du}{dt}, \ddot{u}=\frac{d^2u}{dt^2}$.

Proposition 2.1.

$$\ddot{u} + \varphi(u)\dot{u} + \psi(u) = 0 \tag{2.1}$$

has an 'essentially unique' periodic solution under the following four conditions[6]: (a) $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are continuous and ψ satisfies the Lipschitz condition. (b) $\varphi(u) = \varphi(-u)$. (c) $\psi(u) = -\psi(-u)$ and $\psi(u) > 0$ for $u > 0$. (d) $\int_0^u \varphi(s)ds < 0$ for $0 < u < u_0$. $\int_0^u \varphi(s)ds > 0$ for $u > u_0$. $\int_0^u \varphi(s)ds$ is a monotone increasing function and $\int_0^u \varphi(s)ds \to \infty$ as $u \to \infty$.

Remark 2.1. *The 'essentially unique' means that if $u = \xi(t)$ is a nontrivial periodic solution of Eq.(2.1), then all other nontrivial periodic solutions of Eq.(2.1) are of the form $u = \xi(t - \tau)$, where τ is a real number.*

Proposition 2.1 was firstly proven by Levinson and Smith[8]. The important fact of this proposition is that there exists only one periodic solution in Eq.(2.1). Moreover the periodic orbit created by such an 'essentially unique' periodic solution becomes a unique limit cycle of Eq.(2.1), which is globally orbital stable. The fact that there exist a lot of periodic solutions if the conditon (d) is not satisfed is known. Under weaker hypotheses, some improvements of this proposition have been accomplished[5].

We call the special case in Eq.(2.1): $\psi(u) = u$, i.e.,

$$\ddot{u} + \varphi(u)\dot{u} + u = 0 \tag{2.2}$$

the Liénard equation[9]. Moreover letting $\varphi(u) = \varepsilon(u^2 - 1)$ yields

$$\ddot{u} + \varepsilon(u^2 - 1)\dot{u} + u = 0, \tag{2.3}$$

which is the van der Pol equation[17].

Example 2.1. *The van der Pol equation (2.3) satisfies the conditions of Proposition 2.1. Therefore the van der Pol equation has an essentially unique nontrivial periodic solution.*

Also letting $\varphi(u) = -(a - b\dot{u}^2)$ leads to

$$\ddot{u} - (a - b\dot{u}^2)\dot{u} + u = 0, \tag{2.4}$$

which is the Rayleigh equation[13]. This equation also satisfies the conditions of Proposition 2.1.

Example 2.2. *The Rayleigh equation (2.4) has an essentially unique nontrivial periodic solution.*

Remark 2.2. *By differentiating Eq.(2.4) w.r.t. t and letting $\frac{du}{dt} = v$, the equation of v is identical with the van der Pol equation.*

Eq.(2.5), which is the generalized Liénard equation and is based on a more realistic modelling, has been studied so far[20]. However we can only state that Eq.(2.5) has a periodic solution which is not essentially unique.

Proposition 2.2.

$$\ddot{u} + \varphi(u, \dot{u})\dot{u} + \psi(u) = 0 \tag{2.5}$$

has at least a periodic solution under the following four conditions: (a) $\varphi : \mathbb{R}^2 \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are continuous and satisfy the Lipschitz condition. (b) $u\psi(u) > 0$ for $u \neq 0$. $\psi(u)$ is a monotone increasing function and $|\psi(u)| \to \infty$ as $|u| \to \infty$ for $|u| \geq u_0$. Moreover

$$\frac{\psi(u)}{\int_0^u \psi(s)ds} = \mathcal{O}\Big(\frac{1}{|u|}\Big). \tag{2.6}$$

(c) $\exists u_0 > 0$ and $v_0 > 0$, s.t. $\varphi(u, v) \geq M > 0$ for $|u| \geq u_0, |v| \geq v_0$ and $\varphi(u, v) \geq -m$, $(m > 0)$ for $\forall u, v$. (d) $\varphi(0, 0) < 0$.

Next we follow up periodic solutions which synchronize with a period of an external force.

Proposition 2.3.

$$\ddot{u} + \varphi(u, \dot{u})\dot{u} + \psi(u) = e(t) \tag{2.7}$$

has at least a periodic solution(period ω) under the following four conditions: (a) $\varphi : \mathbb{R}^2 \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are continuous and satisfy the Lipschitz condition. (b) $u\psi(u) > 0$ for $|u|$ is large. $|\psi(u)|$ is a monotone increasing function for $|u|$ is large and and $|\psi(u)| \to \infty$ as $|u| \to \infty$. Moreover

$$\frac{\psi(u)}{\int_0^u \psi(s)ds} = \mathcal{O}\Big(\frac{1}{|u|}\Big). \tag{2.8}$$

(c) $\exists u_0 > 0$ and $v_0 > 0$, s.t. $\varphi(u, v) \geq M > 0$ for $|u| \geq u_0, |v| \geq v_0$ and $\varphi(u, v) \geq -m$, $(m > 0)$ for $\forall u, v$. (d) $e : \mathbb{R} \to \mathbb{R}$ is continuous and $e(t) = e(t + \omega)$.

Proposition 2.3 was also proven by Levinson and Smith. The following proposition was proven by Yamaguti[19].

Proposition 2.4.

$$\ddot{u} + \varphi(u)\dot{u} + \psi(u, t) = e(t) \tag{2.9}$$

has at least a periodic solution(period ω) under the following six conditions: (a) $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$ are continuous and satisfy the Lipschitz condition w.r.t. u. (b) $\psi(x,t)$ has a partial derivative coefficient $g_t(x, t)$, which is continuous w.r.t. t. (c) $\psi(x, t) = \psi(x, t+\omega), e(t) = e(t+\omega)$ and $\int_0^\omega e(t)dt = 0$. (d) $\int_0^u \varphi(s)ds$ $\mathrm{sgn} u \to \infty$ as as $|u| \to \infty$ and $|\int_0^t e(s)ds| < E_0$. (e) $\psi(u,t)$ $\mathrm{sgn} u \geq k_0 > 0$ for $|u| > u_0$. (f) $|\int_0^u \varphi(s)ds| > \frac{1}{k_1}|\frac{\partial t \int_0^u \psi(s,t)ds}{\psi(u,t)}|$ for $|u| > u_1$, where k_0, u_0, k_1, u_1 are positive definite and $0 < k_1 < 1$.

Moreover, the perturbed Liénard equation

$$\ddot{u} + \varphi(u)\dot{u} + \psi(u) = \varepsilon f\big(\frac{t}{\omega}, u, \dot{u}\big) \tag{2.10}$$

has been studied recently and the existence of a nontrivial periodic solution of Eq.(2.10) is proven under the mild conditions[3].

Finally, in this section, we shall introduce the study of Taam[16], which stimulates the authors' motivation. The equation is based on the Duffing equation with a periodic external force.

Proposition 2.5. *Let $p, q > 0$. The equation*

$$\ddot{u} + pu + 2qu^3 = e(t), \tag{2.11}$$

where $e(t) = e(t + \omega), e(t) = -e(-t)$ and $e(t) > 0$ for $0 < t < \frac{\omega}{2}$,$e(0) = e(\omega) = 0$, has a periodic solution of period ω such as $\omega \leq \frac{2\pi}{\sqrt{2qM^2+p}}$. Here M is a constant number which is obtained by solving an algebraic equation induced by coefficients of Eq.(2.11).

Also Taam derived such a condition that Eq.(2.11) has $\frac{\omega}{2}$-out-of-phase solutions comparing with an external force.

In this paper, our objective is to present concrete solutions in order to understand the solution structure of Eq.(2.11). To do so, we let an external force a definite function. Therefore, our target equation is the following Duffing type equation[4],[11]:

$$\ddot{u} + pu + 2qu^3 = f(t), \tag{2.12}$$

where $p, q > 0$. Also we impose the external force as follows: let $\nu = 0, 1, 2, \ldots$

$$f(t) = \begin{cases} \frac{e}{2}, & \nu\omega \le t < \left(\nu + \frac{1}{2}\right)\omega \\ -\frac{e}{2}, & \left(\nu + \frac{1}{2}\right)\omega \le t < (\nu+1)\omega \end{cases} \tag{2.13}$$

where $e > 0$ and $\omega > 0$, which indicate the amplitude and period of the external force, respectively. If the external force $f = 0$, Eq.(2.12) is the standard Duffing equation[4], which is a nonlinear oscillator with a cubic stiffness term to describe the hardening spring effect observed in many mechanical problems. We have the following fact concerning the standard Duffing equation.

Fact 2.1. *[12] The standard Duffing equation*

$$\ddot{u} + pu + qu^3 = 0, \tag{2.14}$$

where $p > 0, q > 0$, has the following essential unique periodic solution for any initial condition:

$$u(t) = \sqrt{\frac{\sqrt{p^2+qE}-p}{q}}\,\mathrm{cn}\Big(\big(\sqrt{p^2+qE}\big)^{\frac{1}{4}}t, \sqrt{\frac{\sqrt{p^2+qE}-p}{2\sqrt{p^2+qE}}}\Big). \tag{2.15}$$

Here $E > 0$ is determined by the initial condition. The period τ of the solution is presented as

$$\tau = \frac{4K\Big(\sqrt{\frac{\sqrt{p^2+qE}-p}{2\sqrt{p^2+qE}}}\Big)}{\big(\sqrt{p^2+qE}\big)^{\frac{1}{4}}}. \tag{2.16}$$

Many studies concerning the Duffing equation have been carried out, in particular, the chaos related researches[18] have been used to study after the discovery of chaos phenomenon in Eq.(2.12) with a damping factor($\dot{u}$) and a sinesoidal fuction for the external force. However the problem of whether Eq.(2.12) itself has a nontrivial periodic solution or not had been almostly forgotten except such a few researches as Taam stated before.

3 The linear case($q = 0$): harmonic oscillation with the external force

First we study the linear case in Eq.(2.12). That is, in the differential equation:

$$\ddot{u} + pu = F, \quad t \ge 0, \tag{3.1}$$

where we suppose that $F = F(t) = F(t+\omega), \omega > 0, p > 0$. We categorize the relations of ω and p to clarify periodic solutions as follows:
(1) $\omega\sqrt{p} \ne 0 (\mathrm{mod}\ 2\pi)$, moreover more precicely,
 (1-1) $\frac{\omega}{2\pi/\sqrt{p}} = \frac{g}{h}$, $g, h \in \mathbb{Z}^+$, $h \ne 1$, g and h are irreducible.
 (1-2) $\frac{\omega}{2\pi/\sqrt{p}}$ =irrational number.
(2) $\omega\sqrt{p} = 0 (\mathrm{mod}\ 2\pi)$.
Some definitions are presented before describing theorems.

Definition 3.1. *Let $g : \mathbb{R}^{n+1} \to \mathbb{R}(n \geq 1)$ and $F : \mathbb{R} \to \mathbb{R}$. Also let $u = u(t)$, which is n times differentiable function defined in $t \in \mathbb{R}$, and $u^{(n)}(t)$ be the n-th order derivative of $u(t)$. In the following, differential equation:*

$$g\big(u(t), \dot{u}(t), \ddot{u}(t), \ldots, u^{(n)}(t)\big) = F(t), \tag{3.2}$$

where $F(t) = F(t + \omega)$, we call the solution $u^*(t)$ the ω-periodic solution if $u(t) = u(t + \omega)$.

Remark 3.1. *We consider g is a polynominal of $u(t), \dot{u}(t), \ldots,$ in this paper. We distinguish the ω-periodic solution from another solution by indicating $*$ like u^* if necessary.*

Remark 3.2. *If $u(t)$ has period T, then the solution has also period $2T, 3T, \ldots$ Suppose T is the smallest period, then we call this smallest T the period of $u(t)$.*

Definition 3.2. *In the differential equation (3.2), we call the solution $u^\sharp(t)$ the hidden periodic solution if $u(t) = u(t + \hat{\omega}), \hat{\omega} \neq \omega$. We call $\hat{\omega}$ the hidden period of the solution.*

Remark 3.3. *We distinguish the hidden periodic solution from another solution by indicating $\sharp$ like $u^\sharp$ if necessary.*

Theorem 3.1. *Suppose that $\dfrac{\omega}{2\pi/\sqrt{p}} = \dfrac{g}{h}$. Here $g, h \in \mathbb{Z}^+$, $h \neq 1$, and g and h are irreducible.*
(1) The differential equation (3.1) has the ω-periodic solution, that is, $u^*(t) = u^*(t + \omega)$ iff the initial condition satisfies $\big(u(0), \dot{u}(0)\big) = \big(u^*(0), \dot{u}^*(0)\big)$.
(2) The ω-periodic solution is presented by

$$u^*(t) = \frac{1}{2\sqrt{p}\tan(\sqrt{p}\omega/2)} \int_t^{t+\omega} \cos\sqrt{p}(t-s)F(s)ds - \frac{1}{2\sqrt{p}} \int_t^{t+\omega} \sin\sqrt{p}(t-s)F(s)ds. \tag{3.3}$$

(3) Suppose ω isn't the period of the solution of Eq.(3.1). The differential equation (3.1) has the hidden periodic solution, that is, $u^\sharp(t) = u^\sharp(t + \hat{\omega})$ where $\hat{\omega}(= \omega h$ or $\dfrac{2\pi g}{\sqrt{p}})$ indicates the hidden period which is the least common multiple of ω and $\dfrac{2\pi}{\sqrt{p}}$ iff

$$\int_0^{\omega h} \sin\sqrt{p}sF(s)ds = 0 \text{ and } \int_0^{\omega h} \cos\sqrt{p}sF(s)ds = 0. \tag{3.4}$$

(4) The hidden periodic solution is presented by

$$u^\sharp(t) = u(0)\cos\sqrt{p}t + \frac{\dot{u}(0)}{\sqrt{p}}\sin\sqrt{p}t + \frac{1}{\sqrt{p}}\int_0^t \sin\sqrt{p}(t-s)F(s)ds. \tag{3.5}$$

Proof. (1) Let

$$\mathbf{u}(t) = \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}, \mathbf{F}(t) = \begin{pmatrix} 0 \\ F(t) \end{pmatrix}, \tag{3.6}$$

then Eq.(3.1) can be rewritten as

$$\dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t) + \mathbf{F}(t). \tag{3.7}$$

We have the solution of Eq.(3.7) as the following form:

$$\mathbf{u}(t) = e^{t\mathbf{A}}\mathbf{u}(0) + e^{t\mathbf{A}}\int_0^t e^{-s\mathbf{A}}\mathbf{F}(s)ds. \tag{3.8}$$

If this solution is ω-periodic, then $\mathbf{u}^*(t) = \mathbf{u}^*(t+\omega)$. Using this fact, we easily obtain

$$\mathbf{u}^*(t) = e^{\omega\mathbf{A}}\big(1 - e^{\omega\mathbf{A}}\big)^{-1} e^{t\mathbf{A}} \int_t^{t+\omega} e^{-s\mathbf{A}}\mathbf{F}(s)ds. \tag{3.9}$$

Eqs.(3.8) and (3.9) leads to

$$\mathbf{u}(t) - \mathbf{u}^*(t) = e^{t\mathbf{A}}\big(\mathbf{u}(0) - \mathbf{u}^*(0)\big). \tag{3.10}$$

This relation follows the theorem.
(2) By simple calculations we have

$$e^{\omega\mathbf{A}} = \begin{pmatrix} \cos\sqrt{p}\omega & \dfrac{\sin\sqrt{p}\omega}{\sqrt{p}} \\ -\sqrt{p}\sin\sqrt{p}\omega & \cos\sqrt{p}\omega \end{pmatrix}, \big(1 - e^{\omega\mathbf{A}}\big)^{-1} = \frac{1}{2(1-\cos\sqrt{p}\omega)} \begin{pmatrix} 1-\cos\sqrt{p}\omega & \dfrac{\sin\sqrt{p}\omega}{\sqrt{p}} \\ -\sqrt{p}\sin\sqrt{p}\omega & 1-\cos\sqrt{p}\omega \end{pmatrix} \tag{3.11}$$

Using these equations, we obtain Eq.(3.3) easily.
(3) Eq.(3.8) directly yields

$$u(t) = u(0)\cos\sqrt{p}t + \frac{\dot{u}(0)}{\sqrt{p}}\sin\sqrt{p}t + \frac{1}{\sqrt{p}}\int_0^t \sin\sqrt{p}(t-s)F(s)ds. \tag{3.12}$$

So we obtain

$$u(t+\hat{\omega}) = u(0)\cos\sqrt{p}(t+\hat{\omega}) + \frac{\dot{u}(0)}{\sqrt{p}}\sin\sqrt{p}(t+\hat{\omega}) + \frac{1}{\sqrt{p}}\int_0^{t+\hat{\omega}} \sin\sqrt{p}(t+\hat{\omega}-s)F(s)ds. \tag{3.13}$$

From the assumption $\dfrac{\omega}{2\pi/\sqrt{p}} = \dfrac{g}{h}$, we have the hidden period $\hat{\omega} = \omega h = \dfrac{2\pi g}{\sqrt{p}}$. Using the relation $\hat{\omega}\sqrt{p} = 2\pi g$, then we have

$$\begin{aligned} u(t+\hat{\omega}) &= u(0)\cos\sqrt{p}t + \frac{\dot{u}(0)}{\sqrt{p}}\sin\sqrt{p}t + \frac{1}{\sqrt{p}}\int_0^{t+\hat{\omega}} \sin\sqrt{p}(t-s)F(s)ds = \\ &= u(0)\cos\sqrt{p}t + \frac{\dot{u}(0)}{\sqrt{p}}\sin\sqrt{p}t + \frac{1}{\sqrt{p}}\int_0^{t} \sin\sqrt{p}(t-s)F(s)ds + \frac{1}{\sqrt{p}}\int_t^{t+\hat{\omega}} \sin\sqrt{p}(t-s)F(s)ds. \end{aligned} \tag{3.14}$$

The most right term can be rewritten as

$$\int_t^{t+\hat{\omega}} \sin\sqrt{p}(t-s)F(s)ds = \sin\sqrt{p}t\int_t^{t+\hat{\omega}} \cos\sqrt{p}sF(s)ds - \cos\sqrt{p}t\int_t^{t+\hat{\omega}} \sin\sqrt{p}sF(s)ds. \tag{3.15}$$

Here let $t \in [(m-1)\hat{\omega}, m\hat{\omega}), m \in \mathbb{N}$ without loss of generality then we have

$$\begin{aligned} \int_t^{t+\hat{\omega}} \cos\sqrt{p}sF(s)ds &= \int_t^{m\hat{\omega}} \cos\sqrt{p}sF(s)ds + \int_{m\hat{\omega}}^{t+\hat{\omega}} \cos\sqrt{p}sF(s)ds = \\ &= \int_t^{m\hat{\omega}} \cos\sqrt{p}sF(s)ds + \int_{(m-1)\hat{\omega}}^{t} \cos\sqrt{p}sF(s)ds = \\ &= \int_{(m-1)\hat{\omega}}^{m\hat{\omega}} \cos\sqrt{p}sF(s)ds = \\ &= \int_0^{\hat{\omega}} \cos\sqrt{p}sF(s)ds = const. \end{aligned} \tag{3.16}$$

Similarly,

$$\int_t^{t+\hat{\omega}} \sin\sqrt{p}sF(s)ds = \int_0^{\hat{\omega}} \sin\sqrt{p}sF(s)ds = const. \tag{3.17}$$

Consequently, the relations Eqs.(3.14), (3.15), (3.16) and (3.17) imply that

$$u(t+\hat{\omega}) = u(t) \tag{3.18}$$

iff Eq.(3.4) holds. This equation means that $u(t)$ is the hidden periodic solution.
(4) This is obvious in the proof of (3). ■

Remark 3.4. *The fact that the hidden periodic solution depends on the initial condition is clear from the expression of $u^\sharp(t)$. The hidden periodic solution has the period $\hat{\omega}$ and isn't unique depending on the initial condition.*

Theorem 3.2. *Suppose that $\dfrac{\omega}{2\pi/\sqrt{p}}$ =irrational number. The differential equation (3.1) has the ω-periodic solution presented by Eq.(3.3) if the initial condition satisfies $\big(u(0),\dot{u}(0)\big) = \big(u^*(0),\dot{u}^*(0)\big)$.*
There doesn't exist the hidden periodic solution.

Proof. The first half of the statement can be proven by the same manner of the proof of Theorem 3.1(1). See [18](p147-149) for the proof of the latter half.

Remark 3.5. *The solution except the ω-periodic one in Theorem 3.2 is called the quasi-periodic one.*

Theorem 3.3. *Suppose that $\omega\sqrt{p} = 0 (\mathrm{mod}\ 2\pi)$. All solutions in the differential equation (3.1) are ω-periodic and don't depend on the initial condition iff*

$$\int_0^{\omega} \sin\sqrt{p}sF(s)ds = 0 \ \text{ and } \ \int_0^{\omega} \cos\sqrt{p}sF(s)ds = 0. \tag{3.19}$$

The solutions formula is presented by

$$u^*(t) = u(0)\cos\sqrt{p}t + \frac{\dot{u}(0)}{\sqrt{p}}\sin\sqrt{p}t + \frac{1}{\sqrt{p}}\int_0^t \sin\sqrt{p}(t-s)F(s)ds. \tag{3.20}$$

Proof. We can prove this simply by letting $h = 1$ in Theorem 3.1(2).

We give some examples of the theorems of this section. Let f of Eq.(2.13) be F in Eq.(3.1), that is, we consider the following linear differential equation:

$$\ddot{u} + pu = \begin{cases} \dfrac{e}{2}, & \nu\omega \le t < \left(\nu + \dfrac{1}{2}\right)\omega \\ -\dfrac{e}{2}, & \left(\nu + \dfrac{1}{2}\right)\omega \le t < (\nu+1)\omega \end{cases} \tag{3.21}$$

Example 3.1. *Let $p = 1, \omega = \pi, e = 1$ in Eq.(3.21), then this case corresponds to Theorem 3.1. We obtain the following, $\omega(=\pi)$-periodic solution by computing Eq.(3.3) concretely:*

$$u^*(t) = \begin{cases} \dfrac{1}{2}\Big(\sin\big(t-(\nu+\tfrac{1}{2})\pi\big) - \sin(t-\nu\pi) + 1\Big), & \nu\pi \le t < (\nu+\dfrac{1}{2})\pi \\ \dfrac{1}{2}\Big(\sin\big(t-(\nu+\tfrac{1}{2})\pi\big) - \sin\big(t-(\nu+1)\pi\big) - 1\Big), & (\nu+\dfrac{1}{2})\pi \le t < (\nu+1)\pi \end{cases}$$

Also the hidden periodic solution, in which period is 2π, is computed by Eq.(3.5) as follows:

$$u^{\sharp}(t) = \begin{cases} \left(u(0) - \frac{1}{2}\right)\cos t + \dot{u}(0)\sin t + \frac{1}{2}, & \nu\pi \le t < (\nu + \frac{1}{2})\pi \\ \left(u(0) - \frac{1}{2}\right)\cos t + \left(\dot{u}(0) + 1\right)\sin t - \frac{1}{2}, & (\nu + \frac{1}{2})\pi \le t < (\nu + 1)\pi \\ \left(u(0) + \frac{1}{2}\right)\cos t + \left(\dot{u}(0) + 1\right)\sin t + \frac{1}{2}, & (\nu + 1)\pi \le t < (\nu + \frac{3}{2})\pi \\ \left(u(0) + \frac{1}{2}\right)\cos t + \dot{u}(0)\sin t - \frac{1}{2}, & (\nu + \frac{3}{2})\pi \le t < (\nu + 2)\pi \end{cases}$$

Figures 3.1 and 3.2 show the numerical results directly computed from the differental equation. In Figure 3.1, the phase portrait of the $\omega(=\pi)$-periodic solution in the initial condition: $u(0) = 0, \dot{u}(0) = -\frac{1}{2}$ and the time histories up to four periods are shown. On the other hand, Figure 3.2 shows the hidden periodic one. Note that the hidden periodic solution depends on the initial condition, so the orbit of the hidden periodic one varies with the initial condition.

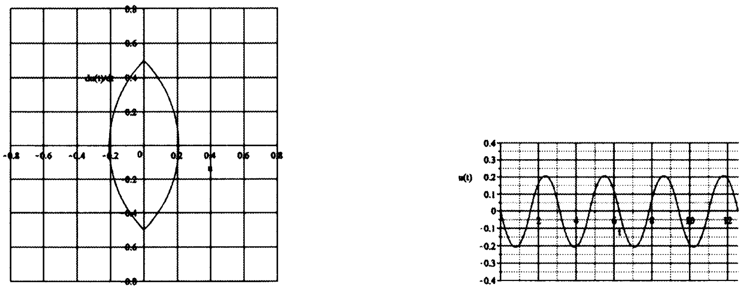


Figure 3.1: (left):The phase portrait of the $\omega(=\pi)$-periodic solution in the initial condition: $u(0) = 0, \dot{u}(0) = -\frac{1}{2}$ in Example 3.1. (right): The time history of the $\omega(=\pi)$-periodic solution. The time is shown up to four periods.

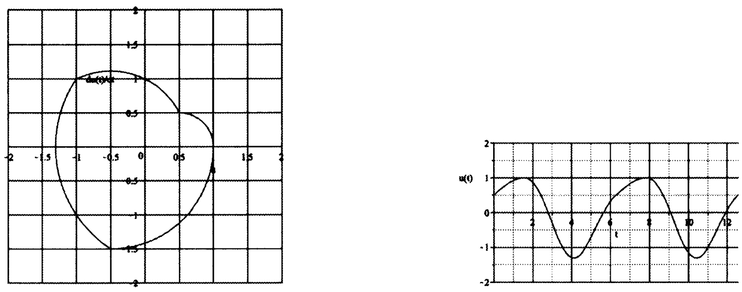


Figure 3.2: (left):The phase portrait of the hidden(2π) periodic solution in the initial condition: $u(0) = \frac{1}{2}, \dot{u}(0) = \frac{1}{2}$ in Example 3.1. (right): The time history of the hidden(2π) periodic solution. The time is shown up to two periods.

Example 3.2. *Let $p = 1, \omega = 1, e = 1$ in Eq.(3.21), then we find $\frac{\omega}{2\pi/\sqrt{p}} = \frac{1}{2\pi}$ so that this example corresponds to Theorem 3.2. We obtain the following, $\omega(=1)$-periodic solution by Eq.(3.3):*

$$u^*(t) = \begin{cases} \frac{1}{2\sin\frac{1}{2}}\left(\sin\left(t - (\nu + \frac{1}{2})\right) - \sin(t - \nu) + \sin\frac{1}{2}\right), & \nu \le t < \nu + \frac{1}{2} \\ \frac{1}{2\sin\frac{1}{2}}\left(\sin\left(t - (\nu + \frac{1}{2})\right) - \sin\left(t - (\nu + 1)\right) - \sin\frac{1}{2}\right), & \nu + \frac{1}{2} \le t < \nu + 1 \end{cases}$$

The initial conditions except $u(0) = 0, \dot{u}(0) = \dfrac{\cos\frac{1}{2} - 1}{2\sin\frac{1}{2}} = -0.1276709606\ldots$ *create quasi-periodic solutions. Fig.3.3 shows the* $\omega(=1)$*-periodic solution's orbit with the initial condition:* $u(0) = 0, \dot{u}(0) = \dfrac{\cos\frac{1}{2} - 1}{2\sin\frac{1}{2}}$ *and the time histories shown up to four periods. Fig.3.4 shows the quasi-periodic orbit with the initial condition:* $u(0) = 0, \dot{u}(0) = -0.12$.

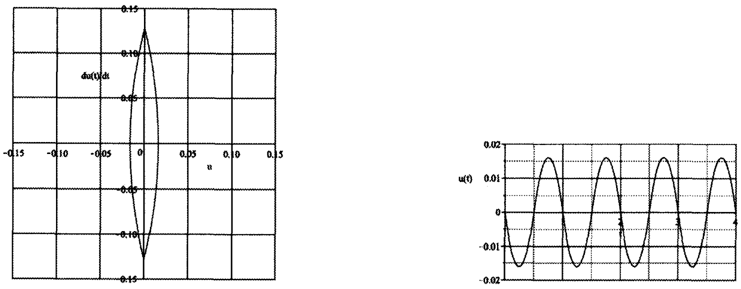


Figure 3.3: (left):The $\omega(=1)$-periodic orbit with the initial condition: $u(0) = 0, \dot{u}(0) = \dfrac{\cos\frac{1}{2} - 1}{2\sin\frac{1}{2}} = -0.1276709606\ldots$ in Example 3.2. (right):The time histories shown up to four periods.

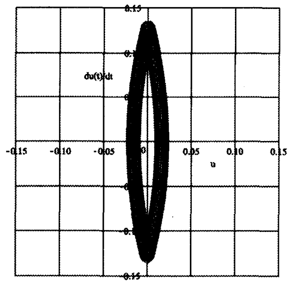
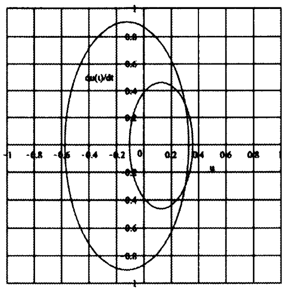


Figure 3.4: The quasi-periodic orbit with the initial condition: $u(0) = 0, \dot{u}(0) = -0.12$ in Example 3.2. The orbit is shown up to $t = 30$.

Example 3.3. *Let* $p = 4, \omega = 2\pi, e = 1$ *in Eq.(3.21), then we find* $\omega\sqrt{p} = 0(\mathrm{mod}\ 2\pi)$ *so that this example corresponds to Theorem 3.3. We obtain the following,* $\omega(=2\pi)$*-periodic solution by Eq.(3.20):*

$$u^*(t) = \begin{cases} \left(u(0) - \dfrac{1}{8}\right)\cos(2t) + \dfrac{\dot{u}(0)}{2}\sin(2t) + \dfrac{1}{8}, & \nu\pi \le t < (\nu+1)\pi \\ \left(u(0) + \dfrac{1}{8}\right)\cos(2t) + \dfrac{\dot{u}(0)}{2}\sin(2t) - \dfrac{1}{8}, & (\nu+1)\pi \le t < (\nu+2)\pi \end{cases}$$

All solutions become the $\omega(=2\pi)$*-periodic solution the initial condition. Fig.3.5 shows the* $\omega(=2\pi)$*-periodic orbit with the initial condition:* $u(0) = \dfrac{3}{10}, \dot{u}(0) = \dfrac{3}{10}$ *and the time histories shown up to two periods.*



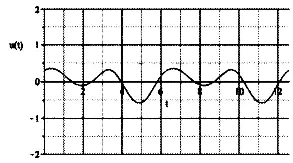


Figure 3.5: (left):The $\omega(=2\pi)$-periodic solution's orbit with the initial condition: $u(0)=\frac{3}{10}, \dot{u}(0)=\frac{3}{10}$ in Example 3.3. (right):The time histories shown up to two periods.

4 Periodic solutions of the Duffing equation with the square wave external force

In this section, we obtain the solution of the nonlinear differential equation

$$\ddot{u}+pu+2qu^3=\begin{cases}\frac{e}{2}, & \nu\omega\le t<\left(\nu+\frac{1}{2}\right)\omega,\ \nu=0,1,2,\dots\\ -\frac{e}{2}, & \left(\nu+\frac{1}{2}\right)\omega\le t<(\nu+1)\omega\end{cases}$$

First, we treat the following nonlinear differential equation modified from the above equaiton:

$$\ddot{u}+pu+2qu^3=\frac{e}{2},\ 0\le t,\ p,q,e>0, \tag{4.1}$$

with the initial condition: $u(0)=u_0, \dot{u}(0)=u_{00}$. From the first integral of motion in Eq.(4.1), we let

$$f(u)=c+eu-pu^2-qu^4, \tag{4.2}$$

where $c=u_{00}^2-eu_0+pu_0^2+qu_0^4$. Here let $c>0$.

Now let α_i $(i=1,2,3,4)$ be the roots of $f(u)=0$ and we define p_1 and p_2 as

$$f(u)=-qp_1(u)p_2(u), \tag{4.3}$$

$$p_1(u)=(u-\alpha_1)(u-\alpha_2), \tag{4.4}$$

$$p_2(u)=(u-\alpha_3)(u-\alpha_4). \tag{4.5}$$

We easily find two complex roots, which are named α_1,α_2, and two real ones, one is positive and the other negative, named α_3,α_4 for $f(u)=0$. So we have $\bar{\alpha_1}=\alpha_2$ and $\alpha_4<0<\alpha_3$. The elementary symmetric polynomials of Eq.(4.3) are

$$\begin{aligned}\sigma_1&=\alpha_1+\alpha_2+\alpha_3+\alpha_4,\\ \sigma_2&=\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4,\\ \sigma_3&=\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4,\\ \sigma_4&=\alpha_1\alpha_2\alpha_3\alpha_4,\end{aligned}$$

then we have the following relations using Eq.(4.2):

$$\sigma_1=0,\ \sigma_2=\frac{p}{q},\ \sigma_3=\frac{e}{q},\ \sigma_4=-\frac{c}{q}. \tag{4.6}$$

We obtain

$$\alpha_1+\alpha_2=-(\alpha_3+\alpha_4) \tag{4.7}$$

from $\sigma_1=0$.

Lemma 4.1. $\alpha_3 + \alpha_4 > 0$.

Proof. Using Eq.(4.7), we have

$$\begin{aligned}\sigma_3 &= \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 = \\ &= \alpha_1\alpha_2(\alpha_3+\alpha_4) + (\alpha_1+\alpha_2)\alpha_3\alpha_4 = \\ &= (\alpha_1\alpha_2 - \alpha_3\alpha_4)(\alpha_3+\alpha_4), \end{aligned} \tag{4.8}$$

namely

$$\frac{e}{q} = (\alpha_1\alpha_2 - \alpha_3\alpha_4)(\alpha_3+\alpha_4). \tag{4.9}$$

In Eq.(4.9), the facts that $\frac{e}{q} > 0, \alpha_1\alpha_2 = |\alpha_1|^2 > 0$ and $\alpha_3\alpha_4 < 0$ imply $\alpha_3 + \alpha_4 > 0$. ∎

By the variable transform

$$v = u - \alpha_4, \tag{4.10}$$

$p_1(u), p_2(u)$ are transformed to

$$p_1^*(v) = (v-N)(v-\overline{N}) = v^2 - (N+\overline{N})v + |N|^2, \tag{4.11}$$
$$p_2^*(v) = v(v-M) = v^2 - Mv, \tag{4.12}$$

where $M = \alpha_3 - \alpha_4, N = \alpha_1 - \alpha_4$. Here we construct the quadratic equation[2] using the coefficients of $p_1^*(v), p_2^*(v)$ as follows:

$$(M - (N+\overline{N}))x^2 + 2|N|^2 x - M|N|^2 = 0. \tag{4.13}$$

Let m, n be the roots of Eq.(4.13), then we have

$$m = \frac{\sqrt{|N|^4 + M|N|^2(M-(N+\overline{N}))} - |N|^2}{M-(N+\overline{N})}, \tag{4.14}$$
$$n = \frac{-\sqrt{|N|^4 + M|N|^2(M-(N+\overline{N}))} - |N|^2}{M-(N+\overline{N})}. \tag{4.15}$$

Since $M - (N+\overline{N}) = 2(\alpha_3+\alpha_4) > 0$ from Lemma 4.1 and $M = \alpha_3 - \alpha_4 > 0$, we find $m > 0$ and $n < 0$.

Here we check the signs of $p_1^*(m), p_1^*(n), p_2^*(m)$ and $p_2^*(n)$.

Lemma 4.2.

$$p_1^*(m) > 0, p_1^*(n) > 0, p_2^*(m) < 0, p_2^*(n) > 0. \tag{4.16}$$

Proof. $p_1^*(m) > 0$ and $p_1^*(n) > 0$ are clear since p_1^* has no real roots. Also $p_2^*(n) = n(n-M) > 0$ since $n < 0, M > 0$. On the other hand, since

$$\sqrt{|N|^4 + M|N|^2(M-(N+\overline{N}))} < |N|^2 + M(M-(N+\overline{N})), \tag{4.17}$$

then

$$\begin{aligned} m - M &= \frac{\sqrt{|N|^4 + M|N|^2(M-(N+\overline{N}))} - |N|^2}{M-(N+\overline{N})} - M < \\ &< \frac{|N|^2 + M(M-(N+\overline{N})) - |N|^2}{M-(N+\overline{N})} - M = 0. \end{aligned} \tag{4.18}$$

So we have $p_2^*(m) = m(m-M) < 0$. ∎

Theorem 4.1. *Let the initial condition be $u(0) = u_0, \dot{u}(0) = u_{00}$ and suppose $c = u_{00}^2 - eu_0 + pu_0^2 + qu_0^4 > 0$. The nonlinear differential equation*

$$\ddot{u} + pu + 2qu^3 = \frac{e}{2},\ 0 \le t,\ p, q, e > 0,$$

has the solution

$$u(t) = \frac{m-n}{1+|B|\mathrm{cn}\big(\Omega(t-t_0), k\big)} + n + \alpha_4, \tag{4.19}$$

where t_0 is determined by the initial condition and

$$\Omega = \frac{\sqrt{A^2+B^2}\sqrt{qp_1^*(n)p_2^*(n)}}{m-n},\ A^2 = \frac{p_1^*(m)}{p_1^*(n)}, B^2 = -\frac{p_2^*(m)}{p_2^*(n)}, k = \frac{|B|}{\sqrt{A^2+B^2}}. \tag{4.20}$$

Proof. The first integral of Eq.(4.1) is $\dot{u}^2 = f(u)$, that is,

$$\dot{u}^2 = c + eu - pu^2 - qu^4. \tag{4.21}$$

Using the variable transform Eq.(4.10), Eq.(4.21) becomes

$$\dot{v}^2 = -qp_1^*(v)p_2^*(v). \tag{4.22}$$

Here let Eq.(4.22) rewrite using the linear transform

$$v = \frac{m+nw}{1+w}. \tag{4.23}$$

First

$$\begin{aligned} p_1^*(v) = p_1^*\Big(\frac{m+nw}{1+w}\Big) &= \Big(\frac{m+nw}{1+w}\Big)^2 - (N+\overline{N})\frac{m+nw}{1+w} + |N|^2 = \\ &= \frac{1}{(1+w)^2}\Big(\big(m^2 - m(N+\overline{N}) + |N|^2\big) + \big(2mn - (N+\overline{N})(m+n) + 2|N|^2\big)w + \\ &\qquad + \big(n^2 - n(N+\overline{N}) + |N|^2\big)w^2\Big) \end{aligned} \tag{4.24}$$

Here m and n are the roots of Eq.(4.13) so we have $m+n = -\dfrac{2|N|^2}{M-(N+\overline{N})}, mn = -\dfrac{M|N|^2}{M-(N+\overline{N})}$. Then we easily find the second term in Eq.(4.24) is zero, that is,

$$2mn - (N+\overline{N})(m+n) + 2|N|^2 = 0. \tag{4.25}$$

Therefore $p_1^*(v)$ is rewritten as

$$p_1^*(v) = \frac{1}{(1+w)^2}\big(p_1^*(m) + p_1^*(n)w^2\big). \tag{4.26}$$

Similarly,

$$p_2^*(v) = \frac{1}{(1+w)^2}\big(p_2^*(m) + p_2^*(n)w^2\big). \tag{4.27}$$

Consequently, the linear transform Eq.(4.23) to Eq.(4.22) yields

$$\dot{w}^2 = \frac{-q}{(m-n)^2}\big(p_1^*(m) + p_1^*(n)w^2\big)\big(p_2^*(m) + p_2^*(n)w^2\big). \tag{4.28}$$

Taking account Lemma 4.2 in Eq.(4.28), we have

$$\frac{dw}{\sqrt{\left(\frac{p_1^*(m)}{p_1^*(n)}+w^2\right)\left(-\frac{p_2^*(m)}{p_2^*(n)}-w^2\right)}}=\pm\frac{\sqrt{qp_1^*(n)p_2^*(n)}}{m-n}dt. \tag{4.29}$$

Let $A^2=\frac{p_1^*(m)}{p_1^*(n)}, B^2=-\frac{p_2^*(m)}{p_2^*(n)}$, then we obtain

$$\frac{1}{\sqrt{A^2+B^2}}\mathrm{cn}^{-1}\left(\frac{w}{|B|},\frac{|B|}{\sqrt{A^2+B^2}}\right)=\pm\frac{\sqrt{qp_1^*(n)p_2^*(n)}}{m-n}(t-t_0). \tag{4.30}$$

From this, we directly have

$$w(t)=|B|\mathrm{cn}\left(\frac{\sqrt{A^2+B^2}\sqrt{qp_1^*(n)p_2^*(n)}}{m-n}(t-t_0),\frac{|B|}{\sqrt{A^2+B^2}}\right). \tag{4.31}$$

Inversing the linear transform Eq.(4.23) and variable transform Eq.(4.10) leads to Eq.(4.19). ■

Note that the denominator of Eq.(4.19) does not become zero, that is, $1+|B|\mathrm{cn}(\Omega(t-t_0),k)\neq 0$. Because $0<m<-n$ and $0<M-m<M-n$ lead to $0<m(M-m)<n(n-M)$. Then $B^2=\frac{m(M-m)}{n(n-M)}<1$. So we have $|B|<1$.

Corollary 4.1. *In Theorem 4.1, we let $u(0)=0, \dot{u}(0)=\sqrt{c}$. t_0 in Eq.(4.19) must satisfy*

$$\mathrm{cn}(\Omega t_0)=-\frac{m+\alpha_4}{|B|(n+\alpha_4)},$$
$$\mathrm{sn}(\Omega t_0)<0.$$

Proof. First, if $u(0)=0$, then $u_{00}^2=c(>0)$. We treat $u_{00}=\sqrt{c}$ in this Corollary. From $u(0)=0$, we directly have $\mathrm{cn}(\Omega t_0)=-\frac{m+\alpha_4}{|B|(n+\alpha_4)}$ in Eq.(4.19). Also we obtain

$$\dot{u}(t)=\frac{(m-n)|B|\Omega\ \mathrm{sn}(\Omega(t-t_0),k)\mathrm{dn}(\Omega(t-t_0),k)}{\left(1+|B|\mathrm{cn}(\Omega(t-t_0),k)\right)^2}. \tag{4.32}$$

Since $m-n>0$ and the assumption: $\dot{u}(0)=\sqrt{c}>0$, $\mathrm{sn}(\Omega t_0)<0$ must be satisfied. ■

Lemma 4.3. *Let the right-hand side of Eq.(4.19) be $h_1(t)$. Then $-h_1(t-\frac{\omega}{2})$ satisfies*

$$\ddot{u}+pu+2qu^3=-\frac{e}{2},\ \frac{\omega}{2}\le t,\ p,q,e>0. \tag{4.33}$$

Proof. Substituing $-h_1(t-\frac{\omega}{2})$ to the equation: the (left side) - (right side) of Eq.(4.33) yields

$$\begin{aligned}
&-\ddot{h}_1(t-\tfrac{\omega}{2})-ph_1(t-\tfrac{\omega}{2})-2qh_1^3(t-\tfrac{\omega}{2})+\tfrac{e}{2}=\\
&=-\left(\ddot{h}_1(t-\tfrac{\omega}{2})+ph_1(t-\tfrac{\omega}{2})+2qh_1^3(t-\tfrac{\omega}{2})-\tfrac{e}{2}\right)= \qquad (\text{for } \tfrac{\omega}{2}\le t)\\
&=-\left(\ddot{h}_1(\tau)+ph_1(\tau)+2qh_1^3(\tau)-\tfrac{e}{2}\right)= \qquad (\text{for } 0\le\tau)\\
&=0
\end{aligned}$$

since $h_1(t)$ satisfies Eq.(4.1). ■

Lemma 4.4. $|B| = -\dfrac{m}{n}$.

Proof. We remember that m and n are the roots of the quadratic equation (4.13). Then we have

$$m+n = -\frac{-2|N|^2}{M-(N+\overline{N})}, \quad mn = -\frac{-M|N|^2}{M-(N+\overline{N})}.$$

Using these relations, we find that the following trivial equality

$$m(M-n)+n(M-m) = (m+n)M - 2nm \tag{4.34}$$

equals zero. Therefore, $-\dfrac{m}{n} = \dfrac{M-m}{M-n}$. This follows the lemma. ■

Lemma 4.5. *Let* $H(\tau) = \dfrac{m-n}{1+|B|\mathrm{cn}(\tau,k)} + n + \alpha_4$. *Then* $\max_\tau H(\tau) = \alpha_3, \ \min_\tau H(\tau) = \alpha_4$.

Proof. First, we show $\alpha_4 \le H(\tau) \le \alpha_3$. The first integral of Eq.(4.1) is $\dot{u}^2 = c+eu-pu^2-qu^4$ and we find that $f(u) = c+eu-pu^2-qu^4 = 0$ has two real roots: α_3, α_4, which have the relation $\alpha_4 < 0 < \alpha_3$, and the others are complex ones. So we have $\alpha_4 \le u(t) \le \alpha_3$. Also we obtain Eq.(4.19), therefore it follows $\alpha_4 \le H(\tau) \le \alpha_3$. Using Lemma 4.4, $H(\tau) = \dfrac{n(m-n)}{n-m\,\mathrm{cn}(\tau,k)} + n + \alpha_4$. Therefore,

$$\begin{aligned}\max_\tau H(\tau) = H(\tau)\big|_{\mathrm{cn}(\tau,k)=-1} &= \frac{n(m-n)}{m+n} + n + \alpha_4 = \\ &= \alpha_4 + \frac{2mn}{m+n} = \alpha_4 + M = \alpha_4 + (\alpha_3 - \alpha_4) = \alpha_3,\end{aligned}$$

$$\min_\tau H(\tau) = H(\tau)\big|_{\mathrm{cn}(\tau,k)=1} = \frac{n(m-n)}{n-m} + n + \alpha_4 = \alpha_4.$$

■

From now on, we pay for Eq.(4.35) in the sence of Farkas[5]. That is, we let a period ω of the external force be able to be chosen appropriately. Also we fix the initial condition $\big(u(0), \dot{u}(0)\big) = (0, \sqrt{c}), c > 0$ in order to show the existence of the ω-periodic solutions.

Theorem 4.2. *Suppose that* $c > 0$. *Let* $T = \mathrm{cn}^{-1}\Big(-\dfrac{m+\alpha_4}{|B|(n+\alpha_4)}\Big)$, $0 < T < 2K(k)$ *and* $\omega = \dfrac{4}{\Omega}\big(2(1+2l)K(k) - T\big)$ *for some non-negative integers* l. *The nonlinear differential equation*

$$\ddot{u} + pu + 2qu^3 = \begin{cases} \dfrac{e}{2}, & \nu\omega \le t < \Big(\nu + \dfrac{1}{2}\Big)\omega, \quad \nu = 0,1,2,\dots \\ -\dfrac{e}{2}, & \Big(\nu+\dfrac{1}{2}\Big)\omega \le t < (\nu+1)\omega \end{cases} \tag{4.35}$$

with the initial condition $\big(u(0), \dot{u}(0)\big) = (0, \sqrt{c})$ *has* C^1 ω*-periodic solutions*

$$u^*(t) = \begin{cases} h_+^o(t-\nu\omega), & \nu\omega \le t < (\nu+\frac{1}{2})\omega, \\ -h_+^o(t-(\nu+\frac{1}{2})\omega), & (\nu+\frac{1}{2})\omega \le t < (\nu+1)\omega, \end{cases} \tag{4.36}$$

where

$$h_+^o(t) = \frac{m-n}{1-|B|\mathrm{cn}\big(\Omega(t-\frac{\omega}{4}), k\big)} + n + \alpha_4. \tag{4.37}$$

We use the suffix "o" in Eq.(4.36). This means "odd"-time oscillations, that is, the solution has the odd number of oscillations in a period like single-time, triple-time, quintic-time oscillations and so on. Also the suffix "e" means "even"-time oscillations, that is, the solution has the even number of oscillations in a period like double-time, quadruple-time, sextic-time oscillations and so on. We will claim in Theorem 4.3, the solution given in Thereom 4.2 has "odd"-time oscillations.

Proof. We formally construct the ω-periodic solution for Eq.(4.35) using Theorem 4.1 and Lemma 4.3 as follows:

$$u^*(t) = \begin{cases} h_1(t-\nu\omega), & \nu\omega \le t < (\nu+\frac{1}{2})\omega, \\ -h_1(t-(\nu+\frac{1}{2})\omega), & (\nu+\frac{1}{2})\omega \le t < (\nu+1)\omega, \end{cases} \tag{4.38}$$

where

$$h_1(t) = \frac{m-n}{1+|B|\mathrm{cn}\big(\Omega(t-t_0),k\big)} + n + \alpha_4. \tag{4.39}$$

Basically, the form of Eq. (4.38) is ω-periodic so that the following conditions must hold in order that the solution is guaranteed as C^1 in $t \ge 0$:

$$\lim_{\varepsilon\to 0} u^*\Big((\nu+\frac{1}{2})\omega-\varepsilon\Big) = \lim_{\varepsilon\to 0} u^*\Big((\nu+\frac{1}{2})\omega+\varepsilon\Big),\ \forall\nu \tag{4.40}$$

$$\lim_{\varepsilon\to 0} u^*(\nu\omega-\varepsilon) = \lim_{\varepsilon\to 0} u^*(\nu\omega+\varepsilon),\ \forall\nu \tag{4.41}$$

$$\lim_{\varepsilon\to 0} \dot{u}^*\Big((\nu+\frac{1}{2})\omega-\varepsilon\Big) = \lim_{\varepsilon\to 0} \dot{u}^*\Big((\nu+\frac{1}{2})\omega+\varepsilon\Big),\ \forall\nu \tag{4.42}$$

$$\lim_{\varepsilon\to 0} \dot{u}^*(\nu\omega-\varepsilon) = \lim_{\varepsilon\to 0} \dot{u}^*(\nu\omega+\varepsilon),\ \forall\nu \tag{4.43}$$

From Eqs.(4.40) and (4.41), we obtain $h_1(\frac{\omega}{2}) = -h_1(0)$. Also from Eqs.(4.42) and (4.43), we obtain $\dot{h}_1(\frac{\omega}{2}) = -\dot{h}_1(0)$. Therefore Eqs. (4.40) to (4.43) are rewritten as

$$\frac{m-n}{1+|B|\mathrm{cn}\big(\Omega(\frac{\omega}{2}-t_0),k\big)} + n + \alpha_4 = -\Big(\frac{m-n}{1+|B|\mathrm{cn}\big(\Omega t_0,k\big)} + n + \alpha_4\Big), \tag{4.44}$$

$$\frac{(m-n)|B|\Omega\,\mathrm{sn}\big(\Omega(\frac{\omega}{2}-t_0),k\big)\mathrm{dn}\big(\Omega(\frac{\omega}{2}-t_0),k\big)}{\Big(1+|B|\mathrm{cn}\big(\Omega(\frac{\omega}{2}-t_0),k\big)\Big)^2} = \frac{(m-n)|B|\Omega\,\mathrm{sn}(\Omega t_0,k)\mathrm{dn}(\Omega t_0,k)}{\Big(1+|B|\mathrm{cn}(\Omega t_0,k)\Big)^2}. \tag{4.45}$$

On the other hand, since the initial condition: $u(0)=0$, we directly have

$$\mathrm{cn}(\Omega t_0,k) = -\frac{m+\alpha_4}{|B|(n+\alpha_4)} \tag{4.46}$$

from Eq.(4.39). Applying Eq.(4.46) to Eq.(4.44) leads to

$$\mathrm{cn}\big(\Omega(\frac{\omega}{2}-t_0),k\big) = -\frac{m+\alpha_4}{|B|(n+\alpha_4)},$$

and from these two equations we obtain the relation

$$\mathrm{cn}\big(\Omega(\frac{\omega}{2}-t_0),k\big) = \mathrm{cn}(\Omega t_0,k).$$

Therefore we obtain the following necessary condition:

$$\Omega(\frac{\omega}{2}-t_0) = \Omega t_0 + 4\ell K(k),\ \exists\ell\in\mathbb{Z}, \tag{4.47}$$

for the C^1 ω-periodic solution which satisfies $u(0) = 0$. Note that Eq.(4.45) is automatically satisfied by the condition Eq.(4.47). So t_0 can be written by

$$t_0 = \frac{\omega}{4} + 2\ell\frac{K(k)}{\Omega}. \tag{4.48}$$

Next we consider such condition that the solution also must satisfies the other initial condition: $\dot{u}(0) = \sqrt{c}$. Substituting Eq.(4.48) to $\dot{h}_1(0)$ yields

$$\begin{aligned}\dot{h}_1(0) &= -\frac{(m-n)|B|\Omega\,\mathrm{sn}(\Omega t_0,k)\mathrm{dn}(\Omega t_0,k)}{\Big(1+|B|\mathrm{cn}(\Omega t_0,k)\Big)^2} = \\ &= -\frac{(m-n)|B|\Omega\,\mathrm{sn}\big(\frac{\omega}{4}\Omega+2\ell K(k),k\big)\mathrm{dn}\big(\frac{\omega}{4}\Omega+2\ell K(k),k\big)}{\Big(1+|B|\mathrm{cn}\big(\frac{\omega}{4}\Omega+2\ell K(k),k\big)\Big)^2}.\end{aligned} \tag{4.49}$$

From $\omega = \dfrac{4}{\Omega}\big(2(1+2l)K(k)-T\big)$, it follows $4lK(k) < \frac{\omega}{4}\Omega = 2(1+2l)K(k)-T < 2(1+2l)K(k)$. Then $\mathrm{sn}(\dfrac{\omega}{4}\Omega, k) > 0$. Therefore in order to $\dot{u}(0) = \dot{h}_1(0) = \sqrt{c} > 0$, ℓ must be odd since $m-n > 0$. Therefore substituting Eq.(4.48), that is, $t_0 = \dfrac{\omega}{4} + 2\ell\dfrac{K(k)}{\Omega}$ (ℓ:odd) to Eq.(4.39) yields Eq.(4.37).

The rest of the proof is to examine the suitability of $0 < T < 2K(k)$. T is defined by

$$\mathrm{cn}(T,k) = -\frac{m+\alpha_4}{|B|(n+\alpha_4)}. \tag{4.50}$$

Since $H(T) = \dfrac{m-n}{1+|B|\mathrm{cn}(T,k)} + n + \alpha_4$ has a period $4K(k)$ and $H(0) = \alpha_4 < 0$, $H(2K(k)) = \alpha_3 > 0$(from Lemma 4.5), we can select T as $0 < T < 2K(k)$ from the intermediate value theorem. ■

Theorem 4.3. *The C^1 ω-periodic solution in Theorem 4.2 has $(1+4l)$-oscillations$(l = 0, 1, 2, \ldots)$ in a period.*

Proof. The number of oscillations in a period can be found by accounting the number of zeros of $\dot{u}^*(t) = 0, \nu\omega \le t < (\nu+1)\omega$. Let $N_0 = \sharp\{t|\dot{u}^*(t) = 0, \nu\omega \le t < (\nu+1)\omega\}$. From the structure of Eq.(4.36), it is sufficient to account $n_0 = \sharp\{t|\dot{h}^o_+(t) = 0, 0 \le t < \frac{\omega}{2}\}$. That is, $n_0 = \frac{N_0}{2}$. Consequently, the number of oscillations in a period equals n_0. Now, we easily have

$$\dot{h}^o_+(t) = \frac{(m-n)|B|\Omega\,\mathrm{sn}\big(\Omega(t-\frac{\omega}{4}),k\big)\mathrm{dn}\big(\Omega(t-\frac{\omega}{4}),k\big)}{\Big(1-|B|\mathrm{cn}\big(\Omega(t-\frac{\omega}{4}),k\big)\Big)^2} \tag{4.51}$$

then we only need to account the number of zeros, say n^s_0, of $\mathrm{sn}\big(\Omega(t-\frac{\omega}{4}),k\big)|_{\omega=\frac{4}{\Omega}(2(1+2l)K(k)-T)} = 0, 0 \le t < \frac{\omega}{2}$. From this we find that $\mathrm{sn}(-2(1+2l)K(k)+T) < 0$ at $t = 0$ since $-2(1+2l)K(k) < -2(1+2l)K(k)+T < -4lK(k)$ ($\because 0 < T < 2K(k)$). Also we have $\mathrm{sn}(2(1+2l)K(k)-T) > 0$ at $t = \frac{\omega}{2}$. Therefore the argument of the sn function lies in $8lK(k) < 8lK(k) + 2(2K(k)-T) < 4(1+2l)K(k)$. From this fact n^s_0 equals the number of zeros of the sn function during over $2l$ periods and less than $1+2l$ periods. Moreover, since $\mathrm{sn}(-2(1+2l)K(k)+T) < 0$ and $\mathrm{sn}(2(1+2l)K(k)-T) > 0$, we obtain $n^s_0 = 1+4l$. Of cource, $n_0 = n^s_0$ then the theorem is proven. ■

We only state the other theorems since we can prove them similarly.

Theorem 4.4. *Suppose that* $c > 0$. *Let* $T = \mathrm{cn}^{-1}\Big(-\frac{m+\alpha_4}{|B|(n+\alpha_4)}\Big)$, $0 < T < 2K(k)$ *and* $\omega = \frac{4}{\Omega}\big(4(1+l)K(k) - T\big)$ *for some non-negative integers* l. *The nonlinear differential equation*

$$\ddot{u} + pu + 2qu^3 = \begin{cases} \frac{e}{2}, & \nu\omega \le t < \Big(\nu + \frac{1}{2}\Big)\omega, \quad \nu = 0, 1, 2, \ldots \\ -\frac{e}{2}, & \Big(\nu + \frac{1}{2}\Big)\omega \le t < (\nu+1)\omega \end{cases}$$

with the initial condition $\big(u(0), \dot{u}(0)\big) = (0, \sqrt{c})$ *has* C^1 ω*-periodic solutions*

$$u^*(t) = \begin{cases} h_-^o(t - \nu\omega), & \nu\omega \le t < (\nu + \frac{1}{2})\omega, \\ -h_-^o(t - (\nu + \frac{1}{2})\omega), & (\nu + \frac{1}{2})\omega \le t < (\nu+1)\omega, \end{cases} \tag{4.52}$$

where

$$h_-^o(t) = \frac{m-n}{1 + |B|\mathrm{cn}\big(\Omega(t - \frac{\omega}{4}), k\big)} + n + \alpha_4. \tag{4.53}$$

Theorem 4.5. *The* C^1 ω*-periodic solution in Theorem 4.4 has* $(3+4l)$*-oscillations*$(l = 0, 1, 2, \ldots)$ *in a period.*

Next we state the theorem regarding "even"-time oscillations.

Theorem 4.6. *Suppose that* $c > 0$. *Let* $T = \mathrm{cn}^{-1}\Big(-\frac{m+\alpha_4}{|B|(n+\alpha_4)}\Big)$, $0 < T < 2K(k)$ *and* $\omega = 8(1+l)\frac{K(k)}{\Omega}$ *for some non-negative integers* l. *The nonlinear differential equation (4.35) with the initial condition* $\big(u(0), \dot{u}(0)\big) = (0, \sqrt{c})$ *has* C^1 ω*-periodic solutions*

$$u^*(t) = \begin{cases} h_+^e(t - \nu\omega), & \nu\omega \le t < (\nu + \frac{1}{2})\omega, \\ -h_+^e((\nu+1)\omega - t), & (\nu + \frac{1}{2})\omega \le t < (\nu+1)\omega, \end{cases} \tag{4.54}$$

where

$$h_+^e(t) = \frac{m-n}{1 + |B|\mathrm{cn}\big(\Omega t + T), k\big)} + n + \alpha_4. \tag{4.55}$$

Proof. We construct the ω-periodic solution for Eq.(4.35) as the same manner of Theorem 4.2 as follows:

$$u^*(t) = \begin{cases} h_1^e(t - \nu\omega), & \nu\omega \le t < (\nu + \frac{1}{2})\omega, \\ -h_1^e((\nu+1)\omega - t), & (\nu + \frac{1}{2})\omega \le t < (\nu+1)\omega, \end{cases} \tag{4.56}$$

where

$$h_1^e(t) = \frac{m-n}{1 + |B|\mathrm{cn}\big(\Omega(t - t_0), k\big)} + n + \alpha_4. \tag{4.57}$$

The form of Eq. (4.56) is ω-periodic so that Eqs.(4.40) to (4.43) must hold in order that the solution is guaranteed as C^1 in $t \ge 0$. Eqs.(4.40) and (4.41) are written as

$$\frac{m-n}{1 + |B|\mathrm{cn}\big(\Omega(\frac{\omega}{2} - t_0), k\big)} + n + \alpha_4 = -\frac{m-n}{1 + |B|\mathrm{cn}\big(\Omega(\frac{\omega}{2} - t_0), k\big)} - (n + \alpha_4), \tag{4.58}$$

$$\frac{m-n}{1 + |B|\mathrm{cn}\big(\Omega t_0, k\big)} + n + \alpha_4 = -\frac{m-n}{1 + |B|\mathrm{cn}\big(\Omega t_0, k\big)} - (n + \alpha_4). \tag{4.59}$$

Note that Eqs.(4.42) and (4.43) are automatically satisfied. Eq.(4.59) is identical with $u(0) = 0$. Also Eq.(4.58) becomes to be identical with Eq.(4.59) using $\omega = 8(1+l)\frac{K(k)}{\Omega}$. Therefore

Eq.(4.56) is guaranteed as the C^1 ω-periodic solution if $\omega = 8(1+l)\dfrac{K(k)}{\Omega}$ and $u(0) = 0$. From Eq.(4.56) the condition $u(0) = 0$ indicates $\mathrm{cn}(\Omega t_0, k) = -\dfrac{m+\alpha_4}{|B|(n+\alpha_4)}$.

Moreover, we will check the another initial condition: $\dot{u}(0) = \sqrt{c}$, that is,

$$-\frac{(m-n)|B|\Omega\ \mathrm{sn}(\Omega t_0, k)\mathrm{dn}(\Omega t_0, k)}{\Big(1+|B|\mathrm{cn}(\Omega t_0, k)\Big)^2} = \sqrt{c}.$$

This leads to the condition $\mathrm{sn}(\Omega t_0, k) < 0$ must hold. From the relations $\mathrm{sn}(\Omega t_0, k) < 0$ and $\mathrm{cn}(\Omega t_0, k) = -\dfrac{m+\alpha_4}{|B|(n+\alpha_4)}$, we can select such $T_+ = \Omega t_0$ as $2K(k) < T_+ < 4K(k)$. Because there exists a root of $H(T) = \dfrac{m-n}{1+|B|\mathrm{cn}(T,k)} + n + \alpha_4 = 0$ in $(0, 2K(k))$ and $(2K(k), 4K(k))$, respectively. ($\because$ $H(T)$ has a period $4K(k)$ and $H(0) = \alpha_4 < 0$, $H(2K(k)) = \alpha_3 > 0$, from Lemma 4.5). Here let $\mathrm{cn}(T,k) = -\dfrac{m+\alpha_4}{|B|(n+\alpha_4)}$, $0 < T < 2K(k)$, we find easily $T_+ = 4K(k) - T$. Substituting $t_0 = \dfrac{T_+}{\Omega} = \dfrac{4K(k)-T}{\Omega}$ to Eq.(4.57) yields Eq.(4.55). ■

Theorem 4.7. *The C^1 ω-periodic solutions in Theorem 4.6 create $2(1+l)$-oscillations($l = 0, 1, 2\dots$) in a period.*

Proof. The proof can be performed by the same manner in Theorem 4.3. ■

We can analize the case of $(u(0), \dot{u}(0)) = (0, -\sqrt{c})$ similarly. We summarize the result of this section including the case of $(u(0), \dot{u}(0)) = (0, -\sqrt{c})$ in Tables 1 and 2. That is, the nonlinear differential equation:

$$\ddot{u} + pu + 2qu^3 = \begin{cases} \dfrac{e}{2}, & \nu\omega \le t < \Big(\nu + \dfrac{1}{2}\Big)\omega, \quad \nu = 0, 1, 2, \dots \\ -\dfrac{e}{2}, & \Big(\nu + \dfrac{1}{2}\Big)\omega \le t < (\nu+1)\omega \end{cases}$$

has the C^1 ω-periodic solutions shown in Tables 1 and 2.

Finally, we give an example of Theorems.

Example 4.1. *We consider the following example:*

$$\ddot{u} + 3u + 2u^3 = \begin{cases} 13, \ t \in \Big(\nu\omega, (\nu + \dfrac{1}{2})\omega\Big), \\ -13, \ t \in \Big((\nu + \dfrac{1}{2})\omega, (\nu+1)\omega\Big). \end{cases} \tag{4.60}$$

We obtain the C^1 ω-periodic solution for the "odd"-time oscillations with the initial condition $(0, -\sqrt{30})$ as

$$h_-^o(t) = \frac{30}{4 + \mathrm{cn}\big(\sqrt{15}(t - \frac{\omega}{4}), \frac{1}{\sqrt{5}}\big)} - 7,$$

and with the initial condition $(0, \sqrt{30})$ as

$$h_+^o(t) = \frac{30}{4 - \mathrm{cn}\big(\sqrt{15}(t - \frac{\omega}{4}), \frac{1}{\sqrt{5}}\big)} - 7.$$

	"odd"-time oscillations
C^1 ω-periodic solutions for IC=$(0,-\sqrt{c})$	$(1+4l)$-oscillations $\omega=\frac{4}{\Omega}(4lK(k)+T)$ $u^*(t)=\begin{cases} h^o_-(t-\nu\omega), & \nu\omega\le t<(\nu+\frac{1}{2})\omega,\\ -h^o_-(t-(\nu+\frac{1}{2})\omega), & (\nu+\frac{1}{2})\omega\le t<(\nu+1)\omega,\end{cases}$ where $h^o_-(t)=\dfrac{m-n}{1+\lvert B\rvert\mathrm{cn}(\Omega(t-\frac{\omega}{4}),k)}+n+\alpha_4.$
	$(3+4l)$-oscillations $\omega=\frac{4}{\Omega}(2(1+2l)K(k)+T)$ $u^*(t)=\begin{cases} h^o_+(t-\nu\omega), & \nu\omega\le t<(\nu+\frac{1}{2})\omega,\\ -h^o_+(t-(\nu+\frac{1}{2})\omega), & (\nu+\frac{1}{2})\omega\le t<(\nu+1)\omega,\end{cases}$ where $h^o_+(t)=\dfrac{m-n}{1-\lvert B\rvert\mathrm{cn}(\Omega(t-\frac{\omega}{4}),k)}+n+\alpha_4.$
C^1 ω-periodic solutions for IC=$(0,\sqrt{c})$	$(1+4l)$-oscillations $\omega=\frac{4}{\Omega}(2(1+2l)K(k)-T)$ $u^*(t)=\begin{cases} h^o_+(t-\nu\omega), & \nu\omega\le t<(\nu+\frac{1}{2})\omega,\\ -h^o_+(t-(\nu+\frac{1}{2})\omega), & (\nu+\frac{1}{2})\omega\le t<(\nu+1)\omega,\end{cases}$ where $h^o_+(t)=\dfrac{m-n}{1-\lvert B\rvert\mathrm{cn}(\Omega(t-\frac{\omega}{4}),k)}+n+\alpha_4.$
	$(3+4l)$-oscillations $\omega=\frac{4}{\Omega}(4(1+l)K(k)-T)$ $u^*(t)=\begin{cases} h^o_-(t-\nu\omega), & \nu\omega\le t<(\nu+\frac{1}{2})\omega,\\ -h^o_-(t-(\nu+\frac{1}{2})\omega), & (\nu+\frac{1}{2})\omega\le t<(\nu+1)\omega,\end{cases}$ where $h^o_-(t)=\dfrac{m-n}{1+\lvert B\rvert\mathrm{cn}(\Omega(t-\frac{\omega}{4}),k)}+n+\alpha_4.$

Table 1: The C^1 ω-periodic solutions for "odd"-time oscillations. In this table, $l=0,1,2,\ldots,\nu=0,1,2,\ldots$ and $T=\mathrm{cn}^{-1}\Big(-\dfrac{m+\alpha_4}{\lvert B\rvert(n+\alpha_4)}\Big),\ \ 0<T<2K(k)$.

	"even"-time oscillations
C^1 ω-periodic solutions for IC=$(0,-\sqrt{c})$	$2(1+l)$-oscillations $\omega=8(1+l)\dfrac{K(k)}{\Omega}$ $u^*(t)=\begin{cases} h^e_-(t-\nu\omega), & \nu\omega\le t<(\nu+\frac{1}{2})\omega,\\ -h^e_-((\nu+1)\omega-t), & (\nu+\frac{1}{2})\omega\le t<(\nu+1)\omega,\end{cases}$ where $h^e_-(t)=\dfrac{m-n}{1+\lvert B\rvert\mathrm{cn}(\Omega t-T,k)}+n+\alpha_4,$
C^1 ω-periodic solutions for IC=$(0,\sqrt{c})$	$2(1+l)$-oscillations $\omega=8(1+l)\dfrac{K(k)}{\Omega}$ $u^*(t)=\begin{cases} h^e_+(t-\nu\omega), & \nu\omega\le t<(\nu+\frac{1}{2})\omega,\\ -h^e_+((\nu+1)\omega-t), & (\nu+\frac{1}{2})\omega\le t<(\nu+1)\omega,\end{cases}$ where $h^e_+(t)=\dfrac{m-n}{1+\lvert B\rvert\mathrm{cn}(\Omega t+T),k)}+n+\alpha_4,$

Table 2: The C^1 ω-periodic solutions for "even"-time oscillations. In this table, $l=0,1,2,\ldots,\nu=0,1,2,\ldots$ and $T=\mathrm{cn}^{-1}\Big(-\dfrac{m+\alpha_4}{\lvert B\rvert(n+\alpha_4)}\Big),\ \ 0<T<2K(k)$.

Also we have the C^1 ω-periodic solution for the "even"-time oscillations with the initial condition $(0, -\sqrt{30})$ *as*

$$h_-^e(t) = \frac{30}{4 + \mathrm{cn}\left(\sqrt{15}t - T, \frac{1}{\sqrt{5}}\right)} - 7,$$

where $T \risingdotseq 1.337$ *and with the initial condition* $(0, \sqrt{30})$ *as*

$$h_+^e(t) = \frac{30}{4 + \mathrm{cn}\left(\sqrt{15}t + T, \frac{1}{\sqrt{5}}\right)} - 7.$$

We find ω concretely as follows: For the "odd"-time oscillations with the initial condition $(0, -\sqrt{30})$

$$\omega \risingdotseq 1.380,\ 4.808,\ 8.236, \ldots$$

and for the "odd"-time oscillations with the initial condition $(0, \sqrt{30})$

$$\omega \risingdotseq 2.047,\ 5.475,\ 8.903, \ldots$$

moreover the "even"-time oscillations with the initial condition $(0, \pm\sqrt{30})$

$$\omega \risingdotseq 3.428,\ 6.856,\ 10.284, \ldots$$

The numerical computations are shown in Figures 4.1 to 4.8. Figures 4.1 to 4.3 show "odd"-time oscillations orbits with the initial condition: $u^*(0) = 0, \dot{u}^*(0) = -\sqrt{30}$. *Figures 4.4 and 4.5 also show "odd"-time oscillations orbits with the initial condition:* $u^*(0) = 0, \dot{u}^*(0) = \sqrt{30}$. *Moreover Figures 4.6 to 4.8 show "even"-time oscillations with the initial condition:* $u^*(0) = 0, \dot{u}^*(0) = \sqrt{30}$ *or* $\dot{u}^*(0) = -\sqrt{30}$.

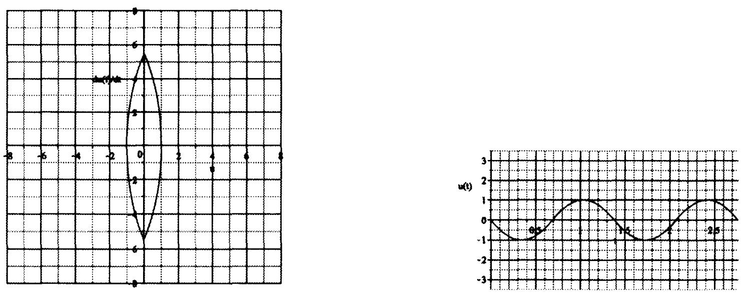


Figure 4.1: (left):The "single-time oscillation" orbit with the initial condition: $u(0) = 0, \dot{u}(0) = -\sqrt{30}$ in Example 4.1. $\omega \risingdotseq 1.380$. (right): The time history of the "single-time oscillation" orbit. The time is shown up to two periods.

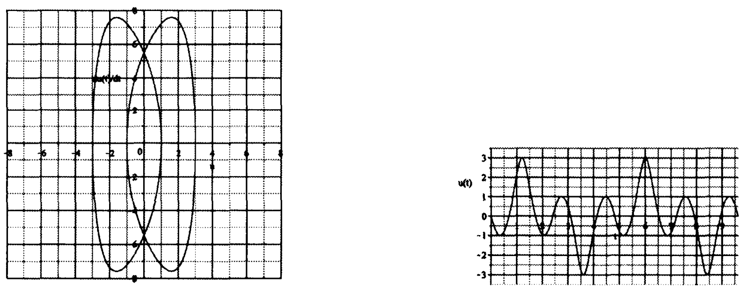


Figure 4.2: (left):The "triple-time oscillations" orbit with the initial condition: $u(0) = 0, \dot{u}(0) = -\sqrt{30}$ in Example 4.1. $\omega \risingdotseq 4.808$. (right): The time history of the "triple-time oscillations" orbit. The time is shown up to two periods.

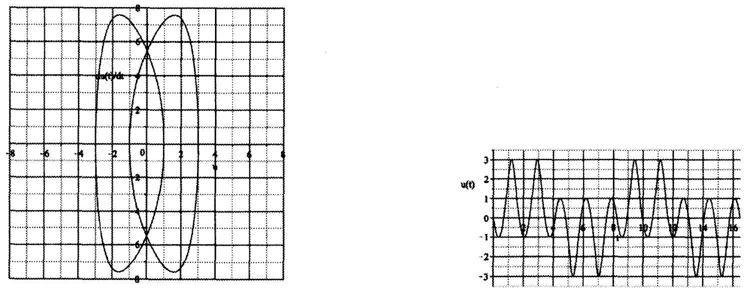


Figure 4.3: (left):The "quintic-time oscillations" orbit with the initial condition: $u(0) = 0, \dot{u}(0) = -\sqrt{30}$ in Example 4.1. $\omega \fallingdotseq 8.236$. The orbit is the same of "quintic-time oscillations" one. (right): The time history of the "quintic-time oscillations" orbit. The time is shown up to two periods.

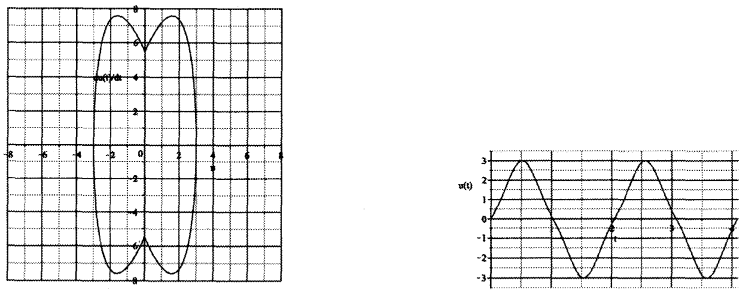


Figure 4.4: (left):The "single-time oscillation" orbit with the initial condition: $u(0) = 0, \dot{u}(0) = \sqrt{30}$ in Example 4.1. $\omega \fallingdotseq 2.047$. (right): The time history of the "single-time oscillation" orbit. The time is shown up to two periods.

Figure 4.5: (left):The time history of the "triple-time oscillations" orbit with the initial condition: $u(0) = 0, \dot{u}(0) = \sqrt{30}$. The time is shown up to two periods. $\omega \fallingdotseq 5.4755$. (right): The time history of the "quintic-time oscillations" orbit with the initial condition: $u(0) = 0, \dot{u}(0) = \sqrt{30}$. The time is also shown up to two periods. $\omega \fallingdotseq 8.9036$. The orbits of both figures are the same of Figure 4.2 and/or 4.3.

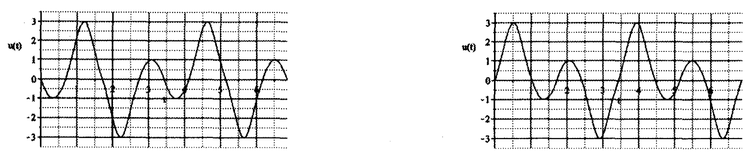


Figure 4.6: (left):The time history of the "double-time oscillations" orbit with the initial condition: $u(0) = 0, \dot{u}(0) = -\sqrt{30}$. The time is shown up to two periods. $\omega \fallingdotseq 3.428$. (right): The time history of the "double-time oscillations" orbit with the initial condition: $u(0) = 0, \dot{u}(0) = \sqrt{30}$. The time is also shown up to two periods. The orbits of both figures are the same of Figure 4.2 and/or 4.3.

Figure 4.7: (left):The time history of the "quadruple-time oscillations" orbit with the initial condition: $u(0) = 0, \dot{u}(0) = -\sqrt{30}$. The time is shown up to two periods. $\omega \fallingdotseq 6.856$. (right): The time history of the "quadruple-time oscillations" orbit with the initial condition: $u(0) = 0, \dot{u}(0) = \sqrt{30}$. The time is also shown up to two periods. The orbits of both figures are the same of Figure 4.2 and/or 4.3.

Figure 4.8: (left):The time history of the "sextic-time oscillations" orbit with the initial condition: $u(0) = 0, \dot{u}(0) = -\sqrt{30}$. The time is shown up to two periods. $\omega \fallingdotseq 10.284$. (right): The time history of the "sextic-time oscillations" orbit with the initial condition: $u(0) = 0, \dot{u}(0) = \sqrt{30}$. The time is also shown up to two periods. The orbits of both figures are the same of Figure 4.2 and/or 4.3.

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