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Global asymptotic stability in a two-species nonautonomous competition system

Hiroyuki USAMI, Gifu University
岐阜大学工学部数理デザイン工学科・宇佐美　広介

This talk is based on a joint work with Dr. Kunihiko Taniguchi, Hiroshima University. We will consider the following nonautonomous competition system

$$\begin{cases} x' = x\left(a_1(t) - b_1(t)\,F_1(x) - c_1(t)\,G_1(y)\right) \\ y' = y\left(a_2(t) - b_2(t)\,F_2(x) - c_2(t)\,G_2(y)\right) \end{cases} \tag{S}$$

The following conditions are always assumed through this talk:

(A_1) F_i, $G_i \in C([0,\infty);[0,\infty))$, $i=1,2$, are strictly increasing functions $F_i(0) = G_i(0) = 0$, and $F_i(\infty) = G_i(\infty) = \infty$;

(A_2) $a_i, b_i, c_i \in C([0,\infty);(0,\infty))$, $i = 1,2$;

(A_3) $\dfrac{a_1(t)}{b_1(t)}, \dfrac{c_1(t)}{b_1(t)}; \dfrac{a_2(t)}{c_2(t)}, \dfrac{b_2(t)}{c_2(t)}$ are all bounded and bounded away from 0 near $+\infty$;

(A_4) $\int^\infty b_1(t)dt = \int^\infty c_2(t)dt = \infty$.

Note that (A_3) and (A_4) imply that $\int^\infty b_i(t)dt = \int^\infty c_i(t)dt = \infty, i = 1,2$.

System (S) is a generalization of the classical Lotka-Volterra competition system with constant coefficients. We can show that if the initial values $x(0), y(0)$ are both positive, then the corresponding solutions of (S) exist globally on $[0,\infty)$, and remain positive there; see for example [3, 4].

It is an important problem to find conditions for the global asymptotic stability of (S). When (S) is an autonomous system, there are many contributions to this problem by means of phase plane analysis; see for example [3, 4]. For nonautonomous cases of (S), Ahmad & Lazer [1], Ahmad & Montes de Oca [2], and Taniguchi [5, 6] have proposed criterion for the global asymptotic stability of (S) by employing asymptotic behavior of time averages of coefficients functions. In this talk, we will consider this problem from a different point of view.

Before going ahead we must describe well-known results concerning the classical Lotka-Volterra competition systems

$$\begin{cases} x' = x\left(\alpha_1 - \beta_1 x - \gamma_1 y\right) \\ y' = y\left(\alpha_2 - \beta_2 x - \gamma_2 y\right) \end{cases} \tag{LV}$$

where α_i, β_i and $\gamma_i, i = 1,2$, are positive constants. For (LV) we can introduce the isoclines

$$x + \frac{\gamma_1}{\beta_1}y = \frac{\alpha_1}{\beta_1}, \tag{1}$$

$$\frac{\beta_2}{\gamma_2}x + y = \frac{\alpha_2}{\gamma_2}. \tag{2}$$

Theorem A. *If line* (1) *is located under line* (2)*, then every positive solution* (x, y) *of* (LV) *satisfies*

$$\lim_{t\to\infty}(x(t), y(t)) = (0, 0).$$

(Figure 1.)

Theorem B. *Suppose that lines* (1) *and* (2) *are located as in Figure 2. (Therefore they intersect only at one point* (ξ, η)*.) Then every positive solution* (x, y) *of* (LV) *satisfies*

$$\lim_{t\to\infty}(x(t), y(t)) = (\xi, \eta).$$

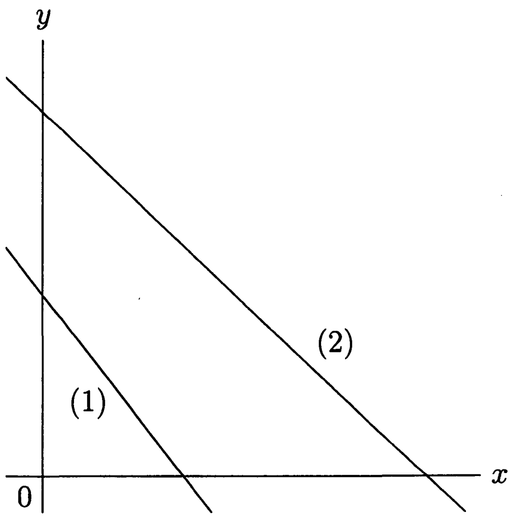

Figure 1

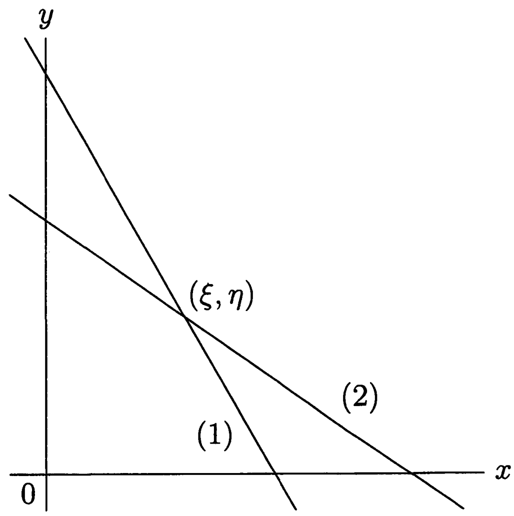

Figure 2

We will show that analogous results hold for our system (S). This is the aim of the talk. When (S) reduces to classical Lotka-Volterra competition system (LV), our results reduce to well-known classical ones.

To state our results, we introduce notation. For bounded function f defined near $+\infty$ we put

$$f_L = \liminf_{t\to\infty} f(t), \quad f_M = \limsup_{t\to\infty} f(t), \quad \text{and} \quad f_\infty = \lim_{t\to\infty} f(t).$$

We introduce the following continuous curves in the first quadrant of the xy-plane:

$$F_1(x) + \left(\frac{c_1}{b_1}\right)_\infty G_1(y) = \left(\frac{a_1}{b_1}\right)_\infty, \tag{$\mathrm{S}_1)_\infty$}$$

$$\left(\frac{b_2}{c_2}\right)_M F_2(x) + G_2(y) = \left(\frac{a_2}{c_2}\right)_L, \tag{$\mathrm{S}_2)_{ML}$}$$

and

$$\left(\frac{b_2}{c_2}\right)_\infty F_2(x) + G_2(y) = \left(\frac{a_2}{c_2}\right)_\infty. \tag{$\mathrm{S}_2)_\infty$}$$

By our assumption (A_1) these three curves are all downward-sloping continuous curves. When (S) reduces to classical Lotka-Volterra system (LV), these three curves reduce to isoclines of the system, namely (1) and (2). These three curves can be regarded as isoclines of limiting systems of (S), in some sense. We will show that asymptotic stability of system (S) can be determined from the relative positions of these curves. Theorems 1 and 2 below are, respectively, generalizations of Theorems A and B.

Theorem 1. *Let $(a_1/b_1)_\infty$ and $(c_1/b_1)_\infty$ exist. If curve $(S_1)_\infty$ is located under curve $(S_2)_{ML}$, then every solution (x, y) of* (S) *satisfies*

$$\lim_{t\to\infty} x(t) = 0$$

and

$$0 < G_2^{-1}\left(\left(\frac{a_2}{c_2}\right)_L\right) \le \liminf_{t\to\infty} y(t) \le \limsup_{t\to\infty} y(t) \le G_2^{-1}\left(\left(\frac{a_2}{c_2}\right)_M\right).$$

(Figure 3.)

Theorem 2. *Let $(a_1/b_1)_\infty, (c_1/b_1)_\infty, (a_2/c_2)_\infty$ and $(b_2/c_2)_\infty$ exist. Suppose that curves $(S_1)_\infty$ and $(S_2)_\infty$ are located as in Figure 4. (Therefore they intersect only at one point (ξ, η).) Then every solution (x, y) of* (S) *satisfies*

$$\lim_{t\to\infty} (x(t), y(t)) = (\xi, \eta).$$

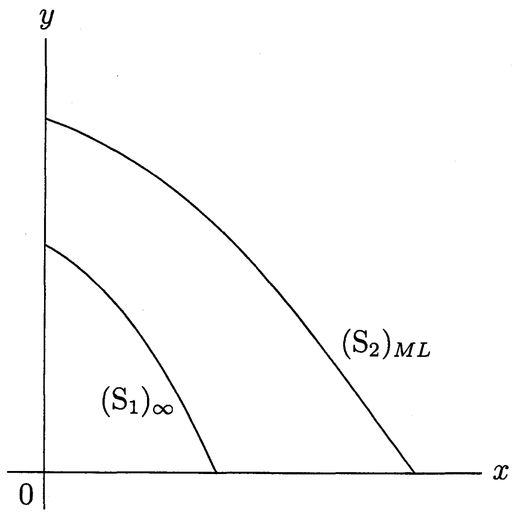

Figure 3

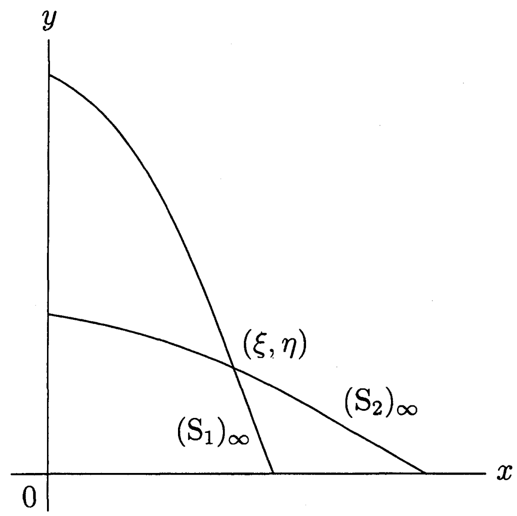

Figure 4

We need a simple lemma concerning ordinary differential inequalities, on which the proof of Theorems 1 and 2 are essentially based. Let us consider the following differential inequalities near ∞:

$$z' \le z\big(p(t) - q(t)H(z)\big) \tag{3}$$

and

$$w' \geq w\big(p(t) - q(t)H(z)\big). \tag{4}$$

For them we assume the following:

(L_1) $H \in C([0,\infty);[0,\infty))$ is a strictly increasing function, $H(0) = 0$, and $H(\infty) = \infty$;

(L_2) $p, q \in C([0,\infty);(0,\infty))$;

(L_3) $0 < (p/q)_L \leq (p/q)_M < \infty$;

(L_4) $\int^\infty q(t)dt = \infty$.

Lemma 3. (i) *Let z be a positive function satisfying inequality* (3) *near ∞. Then,*

$$\limsup_{t\to\infty} z(t) \leq H^{-1}((p/q)_M).$$

(ii) *Let w be a positive function satisfying inequality* (4) *near ∞. Then,*

$$\liminf_{t\to\infty} w(t) \geq H^{-1}((p/q)_L).$$

Proof of Lemma 3. (i) The proof is divided into three cases.

Suppose that $z(t) \geq H^{-1}(p(t)/q(t))$ near ∞. Since $p(t) - q(t)H(z(t)) \leq 0$, we find that $z(t)$ decreases, $\lim_{t\to\infty} z(t)$ exists and $\lim_{t\to\infty} z(t) \geq H^{-1}((p/q)_M)$. To prove $\lim_{t\to\infty} z(t) = H^{-1}((p/q)_M)$, suppose to the contrary that $\lim_{t\to\infty} z(t) > H^{-1}((p/q)_M)$. Then there are two positive constants δ_1 and δ_2 satisfying

$$H(z(t)) > H(\delta_1) > H(\delta_2) > \frac{p(t)}{q(t)} \quad \text{for all sufficiently large } t.$$

Then inequality (3) implies that

$$\begin{aligned} z'(t) \leq q(t)\left(\frac{p(t)}{q(t)} - H(z(t))\right)z(t) &< q(t)(H(\delta_2) - H(\delta_1))z(t) \\ &= -q(t)(H(\delta_1) - H(\delta_2))z(t). \end{aligned}$$

Since $\int^\infty q(t)dt = \infty$, this implies that $z(t) \to -\infty$ as $t \to \infty$. This is a contradiction.

Next suppose that $z(t) \leq H^{-1}(p(t)/q(t))$ near ∞. Obviously in this case there is nothing to prove.

Finally suppose that the function $z(t) - H^{-1}(p(t)/q(t))$ changes its sign in any neighborhood of ∞. Suppose that $\limsup_{t\to\infty} z(t) > H^{-1}((p/q)_M)$. Then, there are three sufficiently large numbers $t_1 < \tau < t_2$ satisfying $z'(\tau) = 0$, $z(t_i) = H^{-1}(p(t_i)/q(t_i)), i = 1, 2$, and $z(t) > H^{-1}(p(t)/q(t))$ for $t \in (t_1, t_2)$. It follows therefore that $z'(t) < 0$ for $t \in (t_1, t_2)$. This is a contradiction. The proof is complete.

(ii) As in the proof of (i), the proof is divided into several cases.

Suppose that $w(t) \leq H^{-1}(p(t)/q(t))$ near ∞. Since $p(t) - q(t)H(w(t)) \geq 0$, we find that $w(t)$ increases, $\lim_{t\to\infty} w(t) \in (0,\infty)$ exists and $\lim_{t\to\infty} w(t) \geq H^{-1}((p/q)_L)$. To prove

$\lim_{t\to\infty} w(t) = H^{-1}((p/q)_L)$, suppose to the contrary that $\lim_{t\to\infty} w(t) < H^{-1}((p/q)_L)$. Then there are two positive constants δ_1 and δ_2 satisfying

$$H(w(t)) < H(\delta_1) < H(\delta_2) < \frac{p(t)}{q(t)} \quad \text{for all sufficiently large } t.$$

Then inequality (4) implies that

$$w'(t) \geq q(t)\left(\frac{p(t)}{q(t)} - H(w(t))\right)w(t) > q(t)(H(\delta_2) - H(\delta_1))w(t).$$

Since $\int^{\infty} q(t)dt = \infty$, this implies that $w(t) \to \infty$ as $t \to \infty$. This is a contradiction.

Other cases can be treated similarly; so we omit them. The proof is complete. □

Sketch of the proof of Theorem 1. By Lemma 3, it suffices to show that $\lim_{t\to\infty} x(t) = 0$. From the first equation of (S) we get

$$x' \leq x\big(a_1(t) - b_1(t)F_1(x)\big)$$

near ∞. Lemma 3-(i) shows that

$$\limsup_{t\to\infty} x(t) \leq F_1^{-1}((a_1/b_1)_\infty) \overset{\text{put}}{=} X_1.$$

Then, by the second equation of (S) for every $\varepsilon > 0$ we have

$$y' \geq y\left[\big(a_2(t) - b_2(t)F_2(X_1+\varepsilon)\big) - c_2(t)G_2(y)\right]$$

near ∞. By Lemma 3-(ii) and our assumptions

$$\liminf_{t\to\infty} y(t) \geq G_2^{-1}\left(\left(\frac{a_2}{c_2}\right)_L - \left(\frac{b_2}{c_2}\right)_M F_2(X_1)\right) \overset{\text{put}}{=} Y_1.$$

Again returning to the first equation of (S), we have for every $\varepsilon > 0$

$$x' \leq b_1(t)\left[\left(\frac{a_1(t)}{b_1(t)} - \frac{c_1(t)}{b_1(t)}G_1(Y_1-\varepsilon)\right) - F_1(x)\right]$$

near ∞. So, if $(a_1/b_1)_\infty - (c_1/b_1)_\infty G_1(Y_1) \leq 0$, then assumptions ($A_3$) and ($A_4$) show that $\lim_{t\to\infty} x(t) = 0$; accordingly the proof is complete. So we may suppose that $(a_1/b_1)_\infty - (c_1/b_1)_\infty G_1(Y_1) \geq 0$.

Repeating the above consideration, we find that

$$\limsup_{t\to\infty} x(t) \leq F_1^{-1}\left(\left(\frac{a_1}{b_1}\right)_\infty - \left(\frac{c_1}{b_1}\right)_\infty G_1(Y_1)\right) \overset{\text{put}}{=} X_2,$$

and

$$\liminf_{t\to\infty} y(t) \geq G_2^{-1}\left(\left(\frac{a_2}{c_2}\right)_\infty - \left(\frac{b_2}{c_2}\right)_\infty F_2(X_2)\right) \overset{\text{put}}{=} Y_2.$$

Continuing this procedure, we can construct inductively two *infinite* sequences $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ satisfying

$$X_1 = F_1^{-1}((a_1/b_1)_\infty),$$

$$F_1(X_n) + \left(\frac{c_1}{b_1}\right)_\infty G_1(Y_{n-1}) = \left(\frac{a_1}{b_1}\right)_\infty, \quad n = 2, 3, 4, \dots, \tag{5}$$

$$\left(\frac{b_2}{c_2}\right)_M F_2(X_n) + G_2(Y_n) = \left(\frac{a_2}{c_2}\right)_L, \quad n = 1, 2, 3, \dots, \tag{6}$$

and

$$G_1(Y_n) < \frac{(a_1/b_1)_\infty}{(c_1/b_1)_\infty}, \quad n = 1, 2, 3, \dots,$$

unless

$$\left(\frac{a_1}{b_1}\right)_\infty - \left(\frac{c_1}{b_1}\right)_\infty G_1(Y_m) \le 0 \quad \text{for some } m \in \mathbf{N}.$$

By the assumption of Theorem 1, we can show inductively that

$$Y_1 < Y_2 < \dots < Y_n < Y_{n+1} < \dots < G_1^{-1}\left(\frac{(a_1/b_1)_\infty}{(c_1/b_1)_\infty}\right).$$

So $\lim_{n\to\infty} Y_n \overset{\text{put}}{=} \tilde{Y} > 0$ exists, which means that $\lim_{n\to\infty} X_n \overset{\text{put}}{=} \tilde{X} \ge 0$ also exists in turn. Let $n \to \infty$ in (5) and (6). We then find that

$$F_1(\tilde{X}) + \left(\frac{c_1}{b_1}\right)_\infty G_1(\tilde{Y}) = \left(\frac{a_1}{b_1}\right)_\infty,$$

and

$$\left(\frac{b_2}{c_2}\right)_M F_2(\tilde{X}) + G_2(\tilde{Y}) = \left(\frac{a_2}{c_2}\right)_L.$$

These means that two curves $(S_1)_\infty$ and $(S_2)_{ML}$ intersect at $(\tilde{X}, \tilde{Y}) \in [0, \infty) \times [0, \infty)$, a contradiction to the assumption of Theorem 1. Thus it must hold that

$$\left(\frac{a_1}{b_1}\right)_\infty - \left(\frac{c_1}{b_1}\right)_\infty G_1(Y_m) \le 0 \quad \text{for some } m \in \mathbf{N}.$$

Since for every $\varepsilon > 0$

$$x' \le b_1(t)\left[\left(\frac{a_1(t)}{b_1(t)} - \frac{c_1(t)}{b_1(t)} G_1(Y_m - \varepsilon)\right) - F_1(x)\right]$$

near ∞, we can show that $\lim_{t\to\infty} x(t) = 0$ by assumptions (A_3) and (A_4). This completes the proof. $\square$

Sketch of the proof of Theorem 2. Arguing as in the sketch of the proof of Theorem 1, we obtain sequences $\{\overline{X}_n\}_{n=1}^{\infty}$, $\{\underline{X}_n\}_{n=1}^{\infty}$, $\{\overline{Y}_n\}_{n=1}^{\infty}$, and $\{\underline{Y}_n\}_{n=1}^{\infty}$ such that

$$\overline{X}_1 = F_1^{-1}\left((a_1/b_1)_\infty\right), \quad \overline{Y}_1 = G_2^{-1}\left((a_2/c_2)_\infty\right);$$

$$0 < \underline{X}_n \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq \overline{X}_n, \quad n = 1, 2, 3, \ldots; \tag{7}$$

$$0 < \underline{Y}_n \leq \liminf_{t\to\infty} y(t) \leq \limsup_{t\to\infty} y(t) \leq \overline{Y}_n, \quad n = 1, 2, 3, \ldots; \tag{8}$$

and

$$F_1(\overline{X}_{n+1}) + \left(\frac{c_1}{b_1}\right)_\infty G_1(\underline{Y}_n) = \left(\frac{a_1}{b_1}\right)_\infty; \tag{9}$$

$$\left(\frac{b_2}{c_2}\right)_\infty F_2(\underline{X}_n) + G_2(\overline{Y}_{n+1}) = \left(\frac{a_2}{c_2}\right)_\infty; \tag{10}$$

$$F_1(\underline{X}_n) + \left(\frac{c_1}{b_1}\right)_\infty G_1(\overline{Y}_n) = \left(\frac{a_1}{b_1}\right)_\infty; \tag{11}$$

$$\left(\frac{b_2}{c_2}\right)_\infty F_2(\overline{X}_n) + G_2(\underline{Y}_n) = \left(\frac{a_2}{c_2}\right)_\infty. \tag{12}$$

These four formulas show that the points $(\overline{X}_{n+1}, \underline{Y}_n)$ and $(\underline{X}_n, \overline{Y}_n)$ exist on curve $(\mathrm{S}_1)_\infty$, and the points $(\underline{X}_n, \overline{Y}_{n+1})$ and $(\overline{X}_n, \underline{Y}_n)$ exist on curve $(\mathrm{S}_2)_\infty, n = 1, 2, 3, \ldots$. Therefore, we can find inductively that

$$\underline{X}_n \leq \underline{X}_{n+1} \leq \xi \leq \overline{X}_{n+1} \leq \overline{X}_n,$$

and

$$\underline{Y}_n \leq \underline{Y}_{n+1} \leq \eta \leq \overline{Y}_{n+1} \leq \overline{Y}_n.$$

So, these four sequences all have positive limits as $n \to \infty$:

$$\lim_{n\to\infty} \underline{X}_n = \underline{\xi}, \quad \lim_{n\to\infty} \overline{X}_n = \overline{\xi}, \quad \lim_{n\to\infty} \underline{Y}_n = \underline{\eta} \quad \text{and} \quad \lim_{n\to\infty} \overline{Y}_n = \overline{\eta}.$$

Letting $n \to \infty$ in (9), (10), (11) and (12), we have

$$F_1(\overline{\xi}) + \left(\frac{c_1}{b_1}\right)_\infty G_1(\underline{\eta}) = \left(\frac{a_1}{b_1}\right)_\infty; \tag{13}$$

$$\left(\frac{b_2}{c_2}\right)_\infty F_2(\underline{\xi}) + G_2(\overline{\eta}) = \left(\frac{a_2}{c_2}\right)_\infty; \tag{14}$$

$$F_1(\underline{\xi}) + \left(\frac{c_1}{b_1}\right)_\infty G_1(\overline{\eta}) = \left(\frac{a_1}{b_1}\right)_\infty; \tag{15}$$

$$\left(\frac{b_2}{c_2}\right)_\infty F_2(\overline{\xi}) + G_2(\underline{\eta}) = \left(\frac{a_2}{c_2}\right)_\infty. \tag{16}$$

The formulas (13) and (16) imply that two curves $(\mathrm{S}_1)_\infty$ and $(\mathrm{S}_2)_\infty$ intersect at $(\overline{\xi}, \underline{\eta})$; similarly curves $(\mathrm{S}_1)_\infty$ and $(\mathrm{S}_2)_\infty$ intersect at $(\underline{\xi}, \overline{\eta})$. By the assumption of Theorem 2, it must hold that $(\overline{\xi}, \underline{\eta}) = (\underline{\xi}, \overline{\eta}) = (\xi, \eta)$, that is,

$$\lim_{n\to\infty} \overline{X}_n = \lim_{n\to\infty} \underline{X}_n = \xi, \quad \text{and} \quad \lim_{n\to\infty} \overline{Y}_n = \lim_{n\to\infty} \underline{Y}_n = \eta.$$

Letting $n \to \infty$ in (7) and (8), we can get $\lim_{t\to\infty}(x(t), y(t)) = (\xi, \eta)$. This completes the proof. $\square$

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Hiroyuki Usami

Department of Mathematical and Design Engineering
Faculty of Engineering
Gifu University
Gifu City, 501-1193
Japan
husami@gifu-u.ac.jp