数理解析研究所講究録
第 1790 巻 2012 年 72-82

SOME RESULTS IN THE EXTENSION WITH A COHERENT SUSLIN TREE

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ABSTRACT. We show that under PFA(S), the coherent Suslin tree S (which is a witness of the axiom PFA(S)) forces that there are no ω_2-Aronszajn trees. We also determine the values of cardinal invariants of the continuum in this extension.

1. INTRODUCTION

In [20], Stevo Todorčević introduced the forcing axiom PFA(S), which says that there exists a coherent Suslin tree S such that the forcing axiom holds for every proper forcing which preserves S to be Suslin, that is, for every proper forcing $\mathbb{P}$ which preserves S to be Suslin and $\aleph_1$-many dense subsets D_α, $\alpha \in \omega_1$, of $\mathbb{P}$, there exists a filter on $\mathbb{P}$ which intersects all the D_α. PFA(S)[S] denotes the forcing extension with the coherent Suslin tree S which is a witness of PFA(S). Since the preservation of a Suslin tree by the proper forcing is closed under countable support iteration (due to Tadatoshi Miyamoto [15]), it is consistent relative to some large cardinal assumption that PFA(S) holds.

The first appearance of such a forcing axiom is in the paper [13] due to Paul B. Larson and Todorčević. In this paper, they introduced the weak version of PFA(S), called Souslin's Axiom (in which the properness is replaced by the cccness), and under this axiom, the coherent Suslin tree S, which is a witness of the axiom, forces a weak fragment of Martin's Axiom. In [20], it is also proved that under PFA(S), S forces the open graph dichotomy ([1]) and the P-ideal dichotomy. Namely, many consequences of PFA are satisfied in the extension with S under

2000 *Mathematics Subject Classification.* 03E50, 03E05, 03E35.

Key words and phrases. A coherent Suslin tree, PFA(S)[S], cardinal invariants, Aronszajn trees.

First author is supported by Grants-in-Aid for Scientific Research for JSPS Fellows No. 23·01017.

Second author is supported by FY 2010 Researcher Exchange Program between JSPS and NSERC, Grant-in-Aid for Scientific Research (C) 22540124 Japan Society for the Promotion of Science, and Leading Young Researchers of Shizuoka University 2011.

[1]This is the so called open coloring axiom [18, §8].

$\mathsf{PFA}(S)$. On the other hand, many people proved that some consequences from $\diamondsuit$ are satisfied in the extension with a Suslin tree (e.g. [16, Theorem 6.15.]). In particular, the pseudo-intersection number $\mathfrak{p}$ is $\aleph_1$ in the extension with a Suslin tree. In fact, the extension with S under $\mathsf{PFA}(S)$ is designed as a universe which satisfied some consequences of $\diamondsuit$ and PFA simultaneously. By the use of this model, Larson and Todorčević proved that the affirmative answer to Katětov's problem is consistent [13].

In this note, we point out the values of cardinal invariants of the continuum (e.g. in [2, 6]) in the extension with S under $\mathsf{PFA}(S)$. And we show that under $\mathsf{PFA}(S)$, S forces that there are no ω_2-Aronszajn trees. In [19], Todorčević demonstrated that many consequences of PFA are deduced from PID plus $\mathfrak{p} > \aleph_1$. In [17], the first author proved that PID plus $\mathfrak{p} > \aleph_1$ implies the failure of $\square_{\kappa,\omega_1}$ whenever $\mathrm{cf}(\kappa) > \omega_1$. It is not yet known whether PID plus $\mathfrak{p} > \aleph_1$ implies the failure of $\square_{\omega_1,\omega_1}$. Since $\square_{\omega_1,\omega_1}$ is equivalent to the existence of a special ω_2-Aronszajn tree, our result concludes that it is consistent that PID holds, $\mathfrak{p} = \aleph_1$ and $\square_{\omega_1,\omega_1}$ fails.

At last in the introduction, we introduce a coherent Suslin tree. A coherent Suslin tree S consists of functions in $\omega^{<\omega_1}$ and is closed under finite modifications. That is,

- for any s and t in S, $s \leq_S t$ iff $s \subseteq t$,
- S is closed under taking initial segments,
- for any s and t in S, the set
$$\{\alpha \in \min\{\mathsf{lv}(s), \mathsf{lv}(t)\}; s(\alpha) \neq t(\alpha)\}$$
is finite (here, $\mathsf{lv}(s)$ is the length of s, that is, the size of s), and
- for any $s \in S$ and $t \in \omega^{\mathsf{lv}(s)}$, if the set $\{\alpha \in \mathsf{lv}(s); s(\alpha) \neq t(\alpha)\}$ is finite, then $t \in S$ also.

For a countable ordinal α, let S_α be the set of the α-th level nodes, that is, the set of all members of S of domain α, and let $S_{\leq\alpha} := \bigcup_{\beta\leq\alpha} S_\beta$. For $s \in S$, we let
$$S{\restriction}s := \{u \in S; s \leq_S u\}.$$

We note that $\diamondsuit$, or adding a Cohen real, builds a coherent Suslin tree. A coherent Suslin tree has canonical commutative isomorphisms. Let s and t be nodes in S with the same level. Then we define a function $\psi_{s,t}$ from $S{\restriction}s$ into $S{\restriction}t$ such that for each $v \in S{\restriction}s$,
$$\psi_{s,t}(v) := t \cup (v{\restriction}[\mathsf{lv}(s), \mathsf{lv}(v)))$$
(here, $v{\restriction}[\mathsf{lv}(s), \mathsf{lv}(v))$ is the function v restricted to the domain $[\mathsf{lv}(s), \mathsf{lv}(v))$). We note that $\psi_{s,t}$ is an isomorphism, and if s, t, u are nodes in S of

the same level, then $\psi_{s,t}$, $\psi_{t,u}$ and $\psi_{s,u}$ commute. (On a coherent Suslin tree, see e.g. [10, 12].)

2. Cardinal invariants

Proposition 2.1 ([20, 4.3 Theorem]). $\mathsf{PFA}(S)$ *implies that* $\mathfrak{p} = \mathrm{add}(\mathcal{N}) = \mathfrak{c} = \aleph_2$ *holds.*

Proof. A forcing with property K doesn't destroy a Suslin tree ([14, Theorem 11.]). So, since a σ-centered forcing satisfies property K and $\mathfrak{p} = \mathfrak{m}(\sigma\text{-centered})$ (due to Bell, see e.g. in [6, 7.12 Theorem]), $\mathsf{PFA}(S)$ implies $\mathfrak{p} > \aleph_1$.

To see that $\mathsf{PFA}(S)$ implies $\mathrm{add}(\mathcal{N}) > \aleph_1$, here we consider the characterization of the additivity of the null ideal by the amoeba forcing $\mathbb{A}$ as follows (see [2, 6.1 Theorem] or [3, Theorem 3.4.17]).

$$\mathrm{add}(\mathcal{N}) = \min\Big\{|\mathcal{D}| : \mathcal{D} \text{ is a set of dense subsets of } \mathbb{A} \text{ such that there are no filters of } \mathbb{A} \text{ which meet every member of } \mathcal{D}\Big\}.$$

Since the amoeba forcing is σ-linked (so satisfies property K), $\mathsf{PFA}(S)$ implies $\mathrm{add}(\mathcal{N}) > \aleph_1$.

A proof that $\mathsf{PFA}(S)$ implies $\mathfrak{c} = \aleph_2$ is same to one for PFA due to Todorčević [5, 3.16 Theorem] (see also [9, Theorem 31.25]). We note that $\mathsf{PFA}(S)$ implies OCA ([8, Lemma 4]), so $\mathfrak{b} = \aleph_2$ holds ([18, 8.6 Theorem], also [9, Theorem 29.8]). In a proof that $\mathfrak{b} = \mathfrak{c}$ holds under PFA, an iteration of a σ-closed forcing and a ccc forcing which is defined by an unbounded family in ω^ω is used. A σ-closed forcing doesn't destroy a Suslin tree (see e.g. [15]). Since the cccness of the second iterand comes from the unboundedness of a family in ω^ω, this preserves a Suslin tree because a Suslin tree doesn't add new reals. So this iteration doesn't destroy a Suslin tree. Therefore $\mathfrak{b} = \mathfrak{c}$ holds under $\mathsf{PFA}(S)$. □

Proposition 2.2 ([8, Lemma 2.]). $\mathfrak{t} = \aleph_1$ *holds in the extension with a Suslin tree.*

Proof. Suppose that T is a Suslin tree, and let π be an order preserving function from T into the order structure $([\omega]^{\aleph_0}, \supseteq^*)$ such that if members s and t of T are incomparable in T, then $\pi(s) \cap \pi(t)$ is finite. Then for a generic branch G through T, the set $\{\pi(s) : s \in G\}$ is a $\subseteq^*$-decreasing sequence which doesn't have its lower bound in $[\omega]^{\aleph_0}$ (because T doesn't add new reals). □

Proposition 2.3. *Under* $\mathsf{PFA}(S)$, S *forces that* $\mathrm{add}(\mathcal{N}) = \mathfrak{c} = \aleph_2$.

Proof. Since S doesn't add new reals and preserves all cardinals, by Proposition 2.1, S forces that $\mathfrak{c} = \aleph_2$ ([20, 4.4 Corollary.]).

To see that S forces $\mathrm{add}(\mathcal{N}) > \aleph_1$, here we consider another characterization of the additivity of the null ideal (see [1], also [2, 3]). A function in the set $\prod_{n\in\omega}\left([\omega]^{\leq n+1} \setminus \{\emptyset\}\right)$ is called a slalom, and for a function f in ω^ω and a slalom φ, we say that φ captures f (denoted by $f \sqsubseteq \varphi$) if for all but finitely many $n \in \omega$, $f(n) \in \varphi(n)$. Then

$$\mathrm{add}(\mathcal{N}) = \min\Bigg\{ |F| : F \subseteq \omega^\omega \\ \& \ \forall \varphi \in \prod_{n\in\omega}\left([\omega]^{\leq n+1} \setminus \{\emptyset\}\right) \exists f \in F\, (f \not\sqsubseteq \varphi) \Bigg\}.$$

Let $\dot{X}$ be an S-name for a set of $\aleph_1$-many functions in ω^ω. For each $s \in S$, let

$$Y_s := \left\{ f \in \omega^\omega : s \Vdash_S \text{“} f \in \dot{X} \text{”} \right\}.$$

Since $\dot{X}$ is an S-name for a set of size $\aleph_1$, Y_s is of size at most $\aleph_1$ for each $s \in S$, so is the set $\bigcup_{s\in S} Y_s$. And we note that

$$\Vdash_S \text{“} \dot{X} \subseteq \bigcup_{s\in S} Y_s \text{”}.$$

By $\mathrm{add}(\mathcal{N}) > \aleph_1$ (Proposition 2.1), there exists a slalom φ which captures all functions in the set $\bigcup_{s\in S} Y_s$. Then

$$\Vdash_S \text{“} \varphi \text{ captures all functions in } \dot{X} \text{”},$$

which finishes the proof. □

Proposition 2.4. *Under* $\mathsf{PFA}(S)$, *S forces that* $\mathfrak{h} = \aleph_2$.

Proof. By Proposition 2.1, $\mathfrak{h} = \aleph_2$ holds in the ground model because of the inequality $\mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{c}$ (see e.g. [6, §6]).

Let $\dot{X}_\alpha$, for each $\alpha \in \omega_1$, be an S-name for a dense open subset of $[\omega]^{\aleph_0}$. For $\alpha \in \omega$ and $s \in S$, let

$$Y_{\alpha,s} := \left\{ x \in [\omega]^{\aleph_0} : \exists t \in S \left(s \leq_S t \ \& \ t \Vdash_S \text{“} x \in \dot{X}_\alpha \text{”} \right) \right\}.$$

Then we note that each $Y_{\alpha,s}$ is a dense open subset of $[\omega]^{\aleph_0}$, and

$$\Vdash_S \text{“} \bigcap_{s\in S} Y_{\alpha,s} \subseteq \dot{X}_\alpha \text{”}.$$

Since $\mathfrak{h} > \aleph_1$, for each $\alpha \in \omega_1$, the set $\bigcap_{\alpha\in\omega_1}\bigcap_{s\in S} Y_{\alpha,s}$ is a dense open subset of $[\omega]^{\aleph_0}$, in particular, it is nonempty. Therefore

$$\Vdash_S \text{“} \bigcap_{\alpha\in\omega_1} \dot{X}_\alpha \neq \emptyset \text{”},$$

which finishes the proof. □

We note that $\mathfrak{h}$ is less than or equal to many standard cardinal invariants, like $\mathfrak{a}$, $\mathfrak{s}$, etc. See e.g. [3, 6, 7].

3. ω_2-ARONSZAJN TREES

Theorem 3.1. *Under* PFA(S)*, S forces that there are no ω_2-Aronszajn trees.*

Proof. An outline of the proof is same to the proof due to Baumgartner in [4] (see also [9, Theorem 31.32.]). So this theorem follows from the following two claims.

Claim 3.2. *Let $\mathbb{P}$ be a σ-closed forcing notion, and let $\dot{T}$ be an S-name for an ω_2-Aronszajn tree. Then $\mathbb{P}$ adds no S-names for cofinal chains through $\dot{T}$ whenever $\mathfrak{c} > \aleph_1$ holds.*

Proof of **Claim 3.2**. At first, we see an easy proof by the result of product forcing ([9, Lemma 15.9] or [11, Ch.VIII, 1.4.Theorem]). We note that the two step iteration $\mathbb{P} * S$ is equal to the two step iteration $S * \mathbb{P}^V$ ([2]). In the extension with S, since $\mathfrak{c} > \aleph_1$, a σ-closed forcing $\mathbb{P}^V$ doesn't add a cofinal branch through the value of $\dot{T}$ by the generic of S, which is an ω_2-Aronszajn tree (this can be proved as in [9, Lemma 27.10]). Therefore $\mathbb{P}$ doesn't add an S-name for a cofinal chain through $\dot{T}$.

At last, we see a direct proof. In fact, we show that if $\mathbb{P}$ is σ-closed and $\dot{T}$ is an S-name for an ω_2-tree, then $\mathbb{P}$ adds no new S-names for cofinal chains through $\dot{T}$ whenever $\mathfrak{c} > \aleph_1$ holds.

Suppose that $\mathbb{P}$ adds a new S-name for a cofinal chain through $\dot{T}$, that is, there exists a sequence $\langle \dot{z}_\alpha; \alpha \in \omega_2\rangle$ of $\mathbb{P}$-names for S-names for members of $\dot{T}$ such that

$$\Vdash_{\mathbb{P}} \text{“} \Vdash_S \text{“}\forall\alpha < \beta < \omega_2,\ \dot{z}_\alpha <_{\dot{T}} \dot{z}_\beta \text{” ”}$$

and for every S-name $\dot{B}$ for a subset of $\dot{T}$ (in the ground model),

$$\Vdash_{\mathbb{P}} \text{“} \Vdash_S \text{“} \dot{B} \neq \{\dot{z}_\alpha; \alpha \in \omega_2\} \text{” ”}.$$

[2]In fact, in the first argument, we use a σ-forcing $\mathrm{Fn}(\omega_1, \omega_2, \aleph_1)$, which collapses ω_2 to ω_1 by countable approximations. S doesn't add new countable sets, so $\mathrm{Fn}(\omega_1, \omega_2, \aleph_1)$ doesn't change in the extension with S.

We note that we look at $\dot{T}$ as an object in the ground model even in the extension with $\mathbb{P}$. So for any $\mathbb{P}$-name $\dot{t}$ for an S-name for a member of $\dot{T}$ and $p \in \mathbb{P}$, densely many extensions of p in $\mathbb{P}$ decides the value of $\dot{t}$ as an S-name for a member of $\dot{T}$. By induction on $\sigma \in 2^{<\omega}$, we choose a condition p_σ in $\mathbb{P}$, an S-name $\dot{x}_\sigma$ for a member of $\dot{T}$ and countable ordinals $\alpha_{|\sigma|}$ and $\beta_{|\sigma|}$ such that

- for σ and τ in $2^{<\omega}$ with $\sigma \subseteq \tau$, $p_\tau \leq_{\mathbb{P}} p_\sigma$,
- $\Vdash_{\mathbb{P}}$ " $\Vdash_S$ "$\dot{x}_\sigma \in \{\dot{z}_\alpha; \alpha \in \omega_2\}$ " " for each $\sigma \in 2^{<\omega}$,
- $\Vdash_S$ "both $\dot{x}_{\sigma^\frown\langle 0\rangle}$ and $\dot{x}_{\sigma^\frown\langle 1\rangle}$ are above $\dot{x}_\sigma$ in $\dot{T}$" for each $\sigma \in 2^{<\omega}$,
- $\Vdash_S$ "$\dot{x}_{\sigma^\frown\langle 0\rangle}$ and $\dot{x}_{\sigma^\frown\langle 1\rangle}$ are incomparable in $\dot{T}$" for each $\sigma \in 2^{<\omega}$,
- for each $n \in \omega$ and $\sigma \in 2^n$, every α_n-th level node of S decides the value of $\dot{x}_\sigma$ which is of level less than β_n in $\dot{T}$.

This can be done because of the property of the sequence $\langle \dot{z}_\alpha; \alpha \in \omega_2\rangle$ and the cccness of S as a forcing notion.

Since $\mathbb{P}$ is σ-closed, for any $f \in 2^\omega$, there is $p_f \in \mathbb{P}$ such that $p_f \leq_{\mathbb{P}} p_{f\restriction n}$ holds for every $n \in \omega$. Since it is forced with $\mathbb{P}$ that $\langle \dot{z}_\alpha; \alpha \in \omega_2\rangle$ is a cofinal chain through $\dot{T}$, there exists an S-name $\dot{x}_f$ for a member of $\dot{T}$ which is of level $\sup_{n\in\omega} \beta_n$ such that

$$p_f \Vdash_{\mathbb{P}} \text{" } \Vdash_S \text{"} \dot{x}_f \in \{\dot{z}_\alpha; \alpha \in \omega_2\} \text{ " "}.$$

Then it holds that

$$p_f \Vdash_{\mathbb{P}} \text{" } \Vdash_S \text{"}\dot{x}_f \text{ is above } \dot{x}_{f\restriction n} \text{ in } \dot{T} \text{ for every } n \in \omega \text{" "}.$$

We note that the phrase $\Vdash_S$ "$\dot{x}_f$ is above $\dot{x}_{f\restriction n}$ in $\dot{T}$ for every $n \in \omega$" is also true in the ground model, so we conclude that

$$\Vdash_S \text{"}\{\dot{x}_f : f \in 2^\omega\} \text{ is a subset of the set of the members of } \dot{T} \text{ whose levels are } \sup_{n\in\omega} \beta_n, \text{ and is of size } \mathfrak{c} > \aleph_1\text{"},$$

which contradicts to the assumption that $\dot{T}$ is an S-name for an ω_2-tree.
$\dashv$ **Claim 3.2**

Claim 3.3. *Let $\dot{T}$ be an S-name for a tree of size $\aleph_1$ and of height ω_1 which doesn't have uncountable (i.e. cofinal) chains through $\dot{T}$. Then there exists a ccc forcing notion which preserves S to be Suslin and forces $\dot{T}$ to be special (i.e. to be a union of countably many antichains through $\dot{T}$).*

We note that this claim has been known if $\dot{T}$ is an S-name for an ω_1-Aronszajn tree.

Proof of **Claim 3.3**. For simplicity, we assume that $\dot{T}$ is an S-name for an order structure on ω_1, that is, $\dot{<}_{\dot{T}}$ is an S-name such that

$$\Vdash_S \text{“}\dot{T} = \langle \omega_1, \dot{<}_{\dot{T}} \rangle\text{”},$$

and that for any $s \in S$ and α, β in ω_1, if $s \Vdash_S$ “$\alpha \not\dot{<}_{\dot{T}} \beta$” and $\alpha < \beta$, then $s \Vdash_S$ “$\alpha \dot{<}_{\dot{T}} \beta$”. Since S is a ccc forcing notion, there exists a club C on ω_1 such that for every $\delta \in C$, every node of S of level δ decides $\dot{<}_{\dot{T}} \cap (\delta \times \delta)$.

We define the forcing notion $\mathbb{Q}(\dot{T}, C) = \mathbb{Q}$ which consists of finite partial functions p from S into the set $\bigcup_{\sigma \in [\omega]^{<\aleph_0}} \left([\omega_1]^{<\aleph_0}\right)^{\sigma}$ such that

- for every $s \in \mathrm{dom}(p)$ and $n \in \mathrm{dom}(p(s))$,
$$p(s)(n) \subseteq \sup(C \cap \mathrm{lv}(s))$$
and
$$s \Vdash_S \text{“}p(s)(n) \text{ is an antichain in } \dot{T}\text{”},$$
- for every s and t in $\mathrm{dom}(p)$, if $s <_S t$, then for every $n \in \mathrm{dom}(p(s)) \cap \mathrm{dom}(p(t))$,
$$t \Vdash_S \text{“}p(s)(n) \cup p(t)(n) \text{ is an antichain in } \dot{T}\text{”},$$

ordered by extensions, that is, for each p and q in $\mathbb{Q}$,

$$p \leq_{\mathbb{Q}} q :\iff p \supseteq q.$$

We note that $\mathbb{Q}$ adds an S-name which witnesses that $\dot{T}$ is special in the extension with S. We will show that if $\mathbb{Q} \times S$ has an uncountable antichain, then some node of S forces that $\dot{T}$ has an uncountable chain, which finishes the proof of the claim.

Suppose that a family $\{\langle p_\xi, s_\xi \rangle : \xi \in \omega_1\}$ is an uncountable antichain in $\mathbb{Q} \times S$. By shrinking it and extending each member of the family if necessary, we may assume that

- for each $\xi \in \omega_1$, $\mathrm{dom}(p_\xi) \subseteq S_{\leq \delta_\xi}$ for some $\delta_\xi \in \omega_1$,
- the sequence $\langle \delta_\xi ; \xi \in \omega_1 \rangle$ is strictly increasing,
- for each $\xi \in \omega_1$ and $s \in \mathrm{dom}(p_\xi)$, there exists $t \in \mathrm{dom}(p_\xi) \cap S_{\delta_\xi}$ such that $s \leq_S t$,
- for each $\xi \in \omega_1$, $s \in \mathrm{dom}(p_\xi)$ and $t \in \mathrm{dom}(p_\xi) \cap S_{\delta_\xi}$, if $s \leq_S t$, then $p_\xi(s) \subseteq p_\xi(t)$,
- all sets $\mathrm{dom}(p_\xi) \cap S_{\delta_\xi}$ are of size n, and say $\mathrm{dom}(p_\xi) \cap S_{\delta_\xi} = \left\{t_i^\xi : i \in n\right\}$,

- for each $i \in n$, all $\mathrm{dom}(p_\xi(t_i^\xi))$ are same, call it σ_i, and for each $k \in \sigma_i$, the size of each $p_\xi(t_i^\xi)(k)$ is constant, call it $m_{i,k}$ and say $p_\xi(t_i^\xi)(k) = \left\{\alpha_{i,k}^\xi(j) : j \in m_{i,k}\right\}$,
- for each $\xi \in \omega_1$, $\mathsf{lv}(s_\xi) > \delta_\xi$,
- there exists $\gamma \in \omega_1$ such that
  - for each ξ and η in ω_1, $s_\xi \restriction \gamma = s_\eta \restriction \gamma =: u_{-1}$,
  - for each $\xi \in \omega_1$ and $t \in \mathrm{dom}(p_\xi)$, $t \restriction [\gamma, \mathsf{lv}(t)) = s_\xi \restriction [\gamma, \mathsf{lv}(t))$,
  - for each ξ and η in ω_1 and $i \in n$, $t_i^\xi \restriction \gamma = t_i^\eta \restriction \gamma =: u_i$

  (this can be done because of the coherency of S),
- for each $i \in n$ and $k \in \sigma_i$, the set $\left\{p_\xi(t_i^\xi)(k) : \xi \in \omega_1\right\}$ is pairwise disjoint (by ignoring the root of the Δ-system), and
- the set $\{s_\xi : \xi \in \omega_1\}$ is dense above u_{-1} in S.

We note that for each distinct ξ and η in ω_1, since $\langle p_\xi, s_\xi\rangle \perp_{\mathbb{Q}\times S} \langle p_\eta, s_\eta\rangle$, $s_\xi \perp_S s_\eta$ or there are $i \in n$, $k \in \sigma_i$ and j_0 and j_1 in $m_{i,k}$ such that $t_i^\xi \not\perp_S t_i^\eta$ and

$$t_i^\xi \cup t_i^\eta \Vdash_S \text{“} \alpha_{i,k}^\xi(j_0) \not\perp_{\dot{T}} \alpha_{i,k}^\eta(j_1) \text{”}$$

(where $t_i^\xi \cup t_i^\eta$ is the longer one of t_i^ξ and t_i^η).

Let

$$u_{-1} \Vdash_S \text{“} \dot{I}_{-1} := \left\{\xi \in \omega_1 : s_\xi \in \dot{G}\right\}, \text{ which is uncountable”},$$

and $\dot{\mathcal{U}}$ an S-name for a uniform ultrafilter on $\dot{I}_{-1}$. We note that u_0 forces that the S-name

$$\psi_{u_{-1},u_0}(\dot{I}_{-1}) := \left\{\xi \in \omega_1 : u_0 \cup (s_\xi \restriction [\gamma, \mathsf{lv}(s_\xi)) \in \dot{G}\right\}$$

is an uncountable subset of ω_1. For each $\xi \in \omega_1$, $k \in \sigma_0$, l and j in $m_{0,k}$, we define

$$u_0 \Vdash_S \text{“whenever } \xi \in \psi_{u_{-1},u_0}(\dot{I}_{-1}),$$
$$\dot{Y}_{0,k,j}^{\xi,l} := \left\{\eta \in \psi_{u_{-1},u_0}(\dot{I}_{-1}) : t_0^\xi \cup t_0^\eta \Vdash_S \text{“} \alpha_{0,k}^\xi(l) \not\perp_{\dot{T}} \alpha_{0,k}^\eta(j) \text{”}\right\} \text{”}$$

([3]) and define

$$u_0 \Vdash_S \text{“}\dot{I}_0 := \begin{cases} \left\{\xi \in \psi_{u_{-1},u_0}(\dot{I}_{-1}) : \bigcup_{\substack{k\in\sigma_0\\ l,j\in m_{0,k}}} \dot{Y}^{\xi,l}_{0,k,j} \notin \psi_{u_{-1},u_0}(\dot{\mathcal{U}})\right\} & \\ \qquad \text{if it is in } \psi_{u_{-1},u_0}(\dot{\mathcal{U}}) \cdots \text{ case 1} & \\ \left\{\xi \in \psi_{u_{-1},u_0}(\dot{I}_{-1}) : \dot{Y}^{\xi,\dot{l}_0}_{0,\dot{k}_0,\dot{j}_0} \in \psi_{u_{-1},u_0}(\dot{\mathcal{U}})\right\} & \\ \qquad \text{which is in } \psi_{u_{-1},u_0}(\dot{\mathcal{U}}) \text{ for some } \dot{l}_0,\ \dot{k}_0 \text{ and } \dot{j}_0 & \\ \qquad \text{otherwise} \cdots \text{ case 2} & \end{cases}\text{”}.$$

If the case 2 happens, then we can make an S-name for a cofinal chain through $\dot{T}$ (which is forced by some node above u_0 in S), so we are done. Whenever the case 1 happens, we repeat this procedure, that is, given $\dot{I}_i$ for some $i \in n-1$, we define, for each $\xi \in \omega_1$, $k \in \sigma_{i+1}$, l and j in $m_{i+1,k}$,

$$u_{i+1} \Vdash_S \text{“whenever } \xi \in \psi_{u_i,u_{i+1}}(\dot{I}_i),$$
$$\dot{Y}^{\xi,l}_{i+1,k,j} := \left\{\eta \in \psi_{u_i,u_{i+1}}(\dot{I}_i) : t^{\xi}_{i+1} \cup t^{\eta}_{i+1} \Vdash_S \text{“}\alpha^{\xi}_{i+1,k}(l) \not\perp_{\dot{T}} \alpha^{\eta}_{i+1,k}(j)\text{”}\right\}\text{”}$$

and

$$u_{i+1} \Vdash_S \text{“}\dot{I}_{i+1} := \begin{cases} \left\{\xi \in \psi_{u_i,u_{i+1}}(\dot{I}_i) : \bigcup_{\substack{k\in\sigma_{i+1}\\ l,j\in m_{i+1,k}}} \dot{Y}^{\xi,l}_{i+1,k,j} \notin \psi_{u_{-1},u_{i+1}}(\dot{\mathcal{U}})\right\} & \\ \qquad \text{if it is in } \psi_{u_{-1},u_{i+1}}(\dot{\mathcal{U}}) \cdots \text{ case 1} & \\ \left\{\xi \in \psi_{u_i,u_{i+1}}(\dot{I}_i) : \dot{Y}^{\xi,\dot{l}_{i+1}}_{i+1,\dot{k}_{i+1},\dot{j}_{i+1}} \in \psi_{u_{-1},u_{i+1}}(\dot{\mathcal{U}})\right\} & \\ \qquad \text{which is in } \psi_{u_{-1},u_{i+1}}(\dot{\mathcal{U}}) \text{ for some } \dot{l}_{i+1},\ \dot{k}_{i+1} \text{ and } \dot{j}_{i+1} & \\ \qquad \text{otherwise} \cdots \text{ case 2} & \end{cases}\text{”}.$$

We show that for some $i \in n-1$, the case 2 happens in the construction of $\dot{I}_{i+1}$, which finishes the proof. Suppose that the case 1 happens in the construction of all the $\dot{I}_{i+1}$. We take $v \in S$ and $\xi \in \omega_1$ such that $u_{n-1} \leq_S v$ and

$$v \Vdash_S \text{“}\xi \in \dot{I}_{n-1} \text{ (which is in the set } \psi_{u_{-1},u_{n-1}}(\dot{\mathcal{U}}))\text{”}$$

[3]We note that by the property of the club C, for each ξ and η in ω_1, if $t^{\xi}_0 \cup t^{\eta}_0 \in S$, then this decides whether $\alpha^{\xi}_{0,k}(l) \perp_{\dot{T}} \alpha^{\eta}_{0,k}(j)$ or not.

Then it follows that

$$v \geq_S u_{n-1} \cup (s_\xi \restriction [\gamma, \mathsf{lv}(s_\xi))) \geq_S t^\xi_{n-1}.$$

We take $v' \in S$ and $\eta \in \omega_1$ such that $v' \geq_S v$ and

$$v' \Vdash_S \text{``} \eta \in \psi_{u_{-1},u_{n-1}}(\dot{I}_{-1}) \setminus \left(\bigcup_{i \in n} \psi_{u_i,u_{n-1}}\Big(\bigcup_{\substack{k \in \sigma_i \\ l,j \in m_{i,k}}} \dot{Y}^{\xi,l}_{i,k,j} \Big) \right)$$

$$\text{(which is in the set } \psi_{u_{-1},u_{n-1}}(\dot{\mathcal{U}})\text{)''}.$$

Then for every $i \in n$, $u_i \cup (v' \restriction [\gamma, \mathsf{lv}(v')))$ is above both t^ξ_i, t^η_i, $u_i \cup (s_\xi \restriction [\gamma, \mathsf{lv}(s_\xi)))$ and $u_i \cup (s_\eta \restriction [\gamma, \mathsf{lv}(s_\eta)))$. Then it follows that $s_\xi \not\perp_S s_\eta$, and by the property of the club set C, for every $i \in n$ and $k \in \sigma_i$,

$$t^\xi_i \cup t^\eta_i \Vdash_S \text{``} p_\xi(t^\xi_i)(k) \cup p_\eta(t^\eta_i)(k) \text{ is an antichain in } \dot{T}\text{''}.$$

Therefore $\langle p_\xi, s_\xi \rangle$ and $\langle p_\eta, s_\eta \rangle$ are compatible in $\mathbb{Q} \times S$, which is a contradiction. $\dashv$ **Claim 3.3** □

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