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NOTES ON MIYAMOTO'S FORCING AXIOMS

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ABSTRACT. In this short paper, we prove that Miyamoto's forcing axiom $\mathrm{FA}^*(\sigma\text{-closed})$ is equivalent to $\mathrm{MA}^+(\sigma\text{-closed})$. We also study some variants of $\mathrm{FA}^*(\sigma\text{-closed})$.

1. INTRODUCTION

Throughout this note, θ will denote a sufficiently large regular cardinal. For submodels $M, N \prec H_\theta$, $M \prec_{\omega_1} N$ means that $M \subseteq N$ and $M \cap \omega_1 = N \cap \omega_1$.

Miyamoto ([1]) introduced the following forcing axiom $\mathrm{FA}^*(\sigma\text{-closed})$:

Definition 1.1. $\mathrm{FA}^*(\sigma\text{-closed})$ is the assertion that for every σ-closed poset $\mathbb{P}$, every countable $M \prec H_\theta$ with $\mathbb{P} \in M$, every $(M, \mathbb{P})$-generic condition p, and every dense subsets $\langle D_i : i < \omega_1 \rangle$ in $\mathbb{P}$, there exists a directed set $F \subseteq \mathbb{P}$ such that:

(1) $|F| \leq \omega_1$,
(2) $F \cap D_i \neq \emptyset$ for every $i < \omega_1$,
(3) $q \in F$ for some $q \leq p$,
(4) $M \prec_{\omega_1} M(F)$.

Where $M(F) = \{h(F) : h : \mathcal{P}(\mathbb{P}) \to H_\theta, h \in M\} \prec H_\theta$. $M(F)$ is the minimal elementary submodel of H_θ containing $M \cup \{F\}$.

He showed that $\mathrm{FA}^*(\sigma\text{-closed})$ implies $\mathrm{MA}^+(\sigma\text{-closed})$, where:

Definition 1.2. $\mathrm{MA}^+(\sigma\text{-closed})$ is the assertion that for every σ-closed poset $\mathbb{P}$, every $\mathbb{P}$-name $\dot{S}$ of a stationary subset of ω_1, and every dense subsets $\langle D_i : i < \omega_1 \rangle$ in $\mathbb{P}$, there exists a filter $F \subseteq \mathbb{P}$ such that $F \cap D_i \neq \emptyset$ for $i < \omega_1$ and the set $S = \{\alpha < \omega_1 : \exists p \in F\, (p \Vdash \text{“}\alpha \in \dot{S}\text{”})\}$ is stationary in ω_1.

We show that the converse direction is also true, hence we have:

Theorem 1.3. $\mathrm{FA}^*(\sigma\text{-closed}) \iff \mathrm{MA}^+(\sigma\text{-closed})$.

On the other hand, we shall consider some variants of $\mathrm{FA}^*(\sigma\text{-closed})$, which are also suggested by Miyamoto.

Definition 1.4. $\mathrm{FA}'(\sigma\text{-closed})$ is the assertion that for every σ-closed poset $\mathbb{P}$, every countable $M \prec H_\theta$ with $\mathbb{P} \in M$, and every dense subsets $\langle D_i : i < \omega_1 \rangle$ in $\mathbb{P}$, there exists a directed set $F \subseteq \mathbb{P}$ such that

(1) $|F| \leq \omega_1$,
(2) $F \cap D_i \neq \emptyset$ for every $i < \omega_1$,
(3) F contains an $(M, \mathbb{P})$-generic condition,
(4) $M \prec_{\omega_1} M(F)$.

Definition 1.5. $\mathrm{FA}^0(\sigma\text{-closed})$ is the assertion that for every σ-closed poset $\mathbb{P}$, every countable $M \prec H_\theta$ with $\mathbb{P} \in M$, and every dense subsets $\langle D_i : i < \omega_1 \rangle$ in $\mathbb{P}$, there exists a directed set $F \subseteq \mathbb{P}$ such that:

(1) $|F| \leq \omega_1$,
(2) $F \cap D_i \neq \emptyset$ for every $i < \omega_1$,
(3) $M \prec_{\omega_1} M(F)$.

Obviously $\mathrm{FA}^*(\sigma\text{-closed}) \Rightarrow \mathrm{FA}'(\sigma\text{-closed}) \Rightarrow \mathrm{FA}^0(\sigma\text{-closed})$.

We prove that $\mathrm{FA}'(\sigma\text{-closed})$ and $\mathrm{FA}^0(\sigma\text{-closed})$ can be characterized by known reflection principles.

Definition 1.6 (Shelah [2])**.** For $\kappa > \omega_1$ and $S \subseteq [\kappa]^\omega$, let $\hat{S} = \{x \in [\kappa]^\omega : \exists a \in S\,(a \subseteq x \text{ and } a \cap \omega_1 = x \cap \omega_1)\}$. SSR (Semi-Stationary Reflection principle) is the assertion that for every cardinal $\kappa > \omega_1$ and every stationary $S \subseteq [\kappa]^\omega$, there is $X \subseteq \kappa$ such that $|X| = \omega_1 \subseteq X$ and $\hat{S} \cap [X]^\omega$ is stationary in $[X]^\omega$.

Definition 1.7. IRP (Internally approachable Reflection Principle) is the assertion that for every θ and every stationary $S \subseteq [H_\theta]^\omega$, there is an internally approachable continuous sequence $\langle M_i : i < \omega_1 \rangle$ of countable submodels of H_θ such that $\{i < \omega_1 : M_i \in S\}$ is stationary in ω_1.

We will show the following equivalences:

Theorem 1.8. *(1)* $\mathrm{FA}'(\sigma\text{-closed}) \iff \mathrm{IRP}$.
(2) $\mathrm{FA}^0(\sigma\text{-closed}) \iff \mathrm{SSR}$.

2. Proofs

Proof of Theorem 1.3. The direction $\mathrm{FA}^*(\sigma\text{-closed}) \Rightarrow \mathrm{MA}^+(\sigma\text{-closed})$ is known, so we show only the converse direction. Suppose $\mathrm{MA}^+(\sigma\text{-closed})$. Let $\mathbb{P}$ be a σ-closed poset. Let S be the set of all countable $M \prec H_\theta$ such that $\mathbb{P} \in M$ but there are an $(M, \mathbb{P})$-generic condition p_M and dense subsets $\langle D_i^M : i < \omega_1 \rangle$ in $\mathbb{P}$ such that there is no directed set $F \subseteq \mathbb{P}$ satisfying (1)–(4) in the definition of $\mathrm{FA}^*(\sigma\text{-closed})$. We see that S is non-stationary in $[H_\theta]^\omega$, this suffices to prove the theorem.

Suppose otherwise. Let $\dot{T}$ be a $\mathbb{P}$-name such that $\Vdash$"$\dot{T} = \{M \in S : p_M \in \dot{G}\}$".

Claim 2.1. *There is $p \in \mathbb{P}$ such that $p \Vdash$ "$\dot{T}$ is stationary in $[H_\theta^V]^\omega$".*

Proof. Suppose otherwise. Then there is a $\mathbb{P}$-name $\dot{f}$ such that $\Vdash$"$\dot{f} : [H_\theta^V]^{<\omega} \to H_\theta^V$ such that there is no $x \in \dot{T}$ closed under $\dot{f}$". Take a sufficiently large another regular cardinal $\chi > \theta$. Take a countable $N \prec H_\chi$ such that $N \cap H_\theta \in S$ and $\dot{f} \in N$. Then p_M is also an $(N, \mathbb{P})$-generic condition. Thus $p_M \Vdash$"$N \cap H_\theta \in \dot{T}$ but is closed under $\dot{f}$". This is a contradiction. □

Pick $p \in \mathbb{P}$ such that $p \Vdash$"$\dot{T}$ is stationary in $[H_\theta^V]^\omega$". By replacing $\mathbb{P}$ by the suborder $\{q \in \mathbb{P} : q \leq p\}$ and p_M by $p_M \wedge p$ for $M \in S$, we may assume that $\Vdash$"$\dot{T}$ is stationary in $[H_\theta^V]^\omega$". Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name of a σ-closed poset which adds a bijection from ω_1 to H_θ^V. Take a $(V, \mathbb{P} * \dot{\mathbb{Q}})$-generic $G * H$ and work in $V[G * H]$. Let T be the interpretation of $\dot{T}$ by $G * H$. Since $\mathbb{Q}$ is σ-closed, T remains stationary in $[H_\theta^V]^\omega$. We know $|H_\theta^V| = \omega_1$, thus we can find a club $\langle M_i : i < \omega_1 \rangle$ in $[H_\theta^V]^\omega$ such that $E = \{i < \omega_1 : i = M_i \cap \omega_1, M_i \in T\}$ is stationary in ω_1. Let $\dot{E}$ be a name of E, and $\dot{M}_i$ of M_i.

Return to V. Notice that for $\langle p, q \rangle \in \mathbb{P} * \dot{\mathbb{Q}}$, $i < \omega_1$, and $M \in S$, if $\langle p, q \rangle \Vdash$"$i \in \dot{E}$ and $M = \dot{M}_i$", then p is compatible with p_M. For $i < \omega_1$, let $D_i = \{\langle p, q \rangle \in \mathbb{P} * \dot{\mathbb{Q}} :$ $\langle p, q \rangle$ decides "$i \in \dot{E}$" and if $\langle p, q \rangle \Vdash$"$i \in \dot{E}$" then $\exists M \in S\,(\langle p, q \rangle \Vdash$"$\dot{M}_i = M$" and $p \leq p_M)\}$. D_i is dense in $\mathbb{P} * \dot{\mathbb{Q}}$. For $i, j < \omega_1$, let $D_{ij} = \{\langle p, q \rangle \in D_i :$ if $\exists M \in S\,(\langle p, q \rangle \Vdash$"$i \in \dot{E} \wedge M_i = M$"), then $p \in D_j^M\}$. Each D_{ij} is also dense in $\mathbb{P} * \dot{\mathbb{Q}}$. By $\mathrm{MA}^+(\sigma\text{-closed})$, we can find a directed set $F' \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ such that $|F'| = \omega_1$, $D_i \cap F' \neq \emptyset$, $D_{ij} \cap F' \neq \emptyset$ for each $i, j < \omega_1$, and $E^* = \{i < \omega_1 :$ $\exists \langle p, q \rangle \in F'\,(\langle p, q \rangle \Vdash$"$i \in \dot{E}$")$\}$ is stationary in ω_1. Let F be the projection of F' into $\mathbb{P}$. F is a directed set in $\mathbb{P}$ with $|F| = \omega_1$. For $i \in E^*$, there is $M_i \in S$ and $\langle p, q \rangle \in F'$ such that $\langle p, q \rangle \Vdash$"$i \in \dot{E}$ and $\dot{M}_i = M_i$" and $p \leq p_{M_i}$. Thus we may assume that $p_{M_i} \in F$ for $i \in E^*$. Moreover, for $i \in \lim(E^*) \cap E^*$, it is easy to check that $M_i = \bigcup_{j \in E^* \cap i} M_j$, and for each $i \in E^*$ and $j < \omega_1$, we have $D_j^{M_i} \cap F \neq \emptyset$. Now take a countable $N \prec H_\theta$ such that $N \cap \omega_1 \in E^*$ and $\langle M_i : i \in E^* \rangle, F \in N$. Let $i^* = N \cap \omega_1$. Then $M_{i^*} \prec_{\omega_1} N$, thus we have $M_{i^*} \prec_{\omega_1} M_{i^*}(F)$. This is a contradiction. □

Proof of (1) of Theorem 1.8. First we prove $\mathrm{FA}'(\sigma\text{-closed}) \Rightarrow \mathrm{IRP}$. Fix a stationary $S \subseteq [H_\theta]^\omega$. Let $\mathbb{P}$ be a σ-closed poset adding a bijection from ω_1 to H_θ. Fix $\mathbb{P}$-names $\langle \dot{M}_i : i < \omega_1 \rangle$ such that $\Vdash$"$\langle \dot{M}_i : i < \omega_1 \rangle$ is an internally approachable club in $[H_\theta^V]^\omega$".

Fix another large regular cardinal $\chi > \theta$ and $N \prec H_\chi$ with $\theta, S, \mathbb{P}, \langle \dot{N}_i : i < \omega \rangle \in N$ and $N \cap H_\theta \in S$. Let $M = N \cap H_\theta$ and $i^* = M \cap \omega_1$. Note that if $p \in \mathbb{P}$ is an $(N, \mathbb{P})$-generic condition, then $p \Vdash$"$M = \dot{M}_{i^*}$". For $i < \omega_1$, let $D_i = \{p \in \mathbb{P}$: $\exists M'\,(p \Vdash$"$M' = \dot{M}_i$")$\}$. D_i is dense in $\mathbb{P}$. Applying $\mathrm{FA}'(\sigma\text{-closed})$, we can find a directed set $F \subseteq \mathbb{P}$ such that $F \cap D_i \neq \emptyset$, F contains an $(N, \mathbb{P})$-generic condition, and $N \prec_{\omega_1} N(F)$. For each $i < \omega_1$ define M_i as $\exists p \in F\,(p \Vdash$"$\dot{M}_i = M_i$"). Then

$\langle M_i : i < \omega_1 \rangle$ is an internally approachable continuous sequence and $\langle M_i : i < \omega_1 \rangle \in N(F)$. Moreover, since F contains an $(N, \mathbb{P})$-generic condition, we have $M_{i^*} = M$. Finally we see that $\{i < \omega_1 : M_i \in S\}$ is stationary, but this follows from that $M_{i^*} \cap \omega_1 = M \cap \omega_1 = N \cap \omega_1$ and $N \prec_{\omega_1} N(F)$.

Next we prove the converse direction IRP $\Rightarrow$ FA$'$(σ-closed). Fix a σ-closed poset $\mathbb{P}$. Fix a well-ordering Δ on H_θ. Let S be the set of all countable $M \prec \langle H_\theta, \in, \Delta \rangle$ such that: there are dense open subsets $\langle D_i^M : i < \omega_1 \rangle$ of $\mathbb{P}$ such that there is no directed set F satisfying (1)–(4) in the definition of FA$'$(σ-closed). We claim that S is non-stationary, which completes our proof. Suppose to contrary that S is stationary. By IRP, we can find an internally approachable continuous sequence $\langle N_i : i < \omega_1 \rangle$ with $E = \{i < \omega_1 : N_i \in S\}$ stationary. Since it is internally approachable, we may assume that for each $i < j < \omega_1$, $\langle D_k^{N_i} : k < \omega_1 \rangle \in N_j$, where we are letting $D_k^{N_i} = \mathbb{P}$ for $i \notin E$. By induction on $i < \omega_1$, we would define a descending sequence $\langle p_i : i < \omega_1 \rangle$ such that p_i is $(N_i, \mathbb{P})$-generic and $p_{i+1} \in D_k^{N_j}$ for $k, j \leq i$.

First, take the Δ-least $(N_0, \mathbb{P})$-generic condition p_0. Since $N_0 \in N_1$, we have $p_0 \in N_1$. Moreover, since $\langle D_k^{N_0} : k < \omega_1 \rangle \in N_1$, there is the Δ-least condition $p_1 \leq p_0$ with $p_1 \in D_0^{N_0}$. Repeating this procedure ω_1-times; When $i < \omega_1$ is limit and $\langle p_j : j < i \rangle$ was defined, then p_i is the Δ-least lower bound of the p_j's. Then we know that p_i is $(N_i, \mathbb{P})$-generic and $p_i \in N_{i+1}$. Thus we can take $p_{i+1} \in N_{i+1}$ as intended.

Now we have a descending sequence $\langle p_i : i < \omega_1 \rangle$ such that p_i is $(N_i, \mathbb{P})$-generic and for every $i, k < \omega_1$, there is some j with $p_j \in D_k^{N_i}$. Let $F = \{p_i : i < \omega_1\}$. F is a directed set with size $\leq \omega_1$. Take a countable $M \prec \langle H_\theta, \in, \Delta \rangle$ such that $F, \langle N_i : i < \omega_1 \rangle \in M$ and $M \cap \omega_1 \in E$. Let $i^* = M \cap \omega_1$. Then $N_{i^*} \prec_{\omega_1} M$, in particular we have $N_{i^*} \prec_{\omega_1} N_{i^*}(F)$. However F contradicts $N_{i^*} \in S$. $\square$

Proof of (2) of Theorem 1.8. First we prove FA^0(σ-closed) $\Rightarrow$ SSR. Suppose FA^0(σ-closed). Fix $\kappa \geq \omega_2$ and σ-closed poset $\mathbb{P}$ which adds a bijection from ω_1 to κ. Fix a $\mathbb{P}$-name $\dot{\pi}$ such that $\Vdash$"$\dot{\pi}$ is a surjection from ω_1 to κ". To show that SSR, take a stationary $S \subseteq [\kappa]^\omega$. We will find $X \subseteq \kappa$ such that $|X| = \omega_1 \subseteq X$ and $\hat{S} \cap [X]^\omega$ is stationary in $[X]^\omega$.

Take a countable $M \prec H_\theta$ with $\mathbb{P}, \dot{\pi}, S \in M$ and $M \cap \kappa \in S$. For $i < \omega_1$, let $D_i = \{p \in \mathbb{P} : \exists j < \omega_1 \, (p \Vdash \text{“}\dot{\pi}(j) = i\text{”})\}$, and for $\alpha \in M \cap \kappa$, let $E_\alpha = \{p \in \mathbb{P} : \exists i < \omega_1 \, (p \Vdash \text{“}\dot{\pi}(i) = \alpha\text{”})\}$. Each D_i and E_α are dense in $\mathbb{P}$. By FA^0(σ-closed), we can find a directed set $F \subseteq \mathbb{P}$ such that $|F| = \omega_1$, $M \prec_{\omega_1} M(F)$, and F intersects each D_i and E_α. Let $X = \{\alpha < \kappa : \exists p \in F \exists i < \omega_1 \, (p \Vdash \text{“}\dot{\pi}(i) = \alpha\text{”})\}$. Then $X \in M(F)$, and by the choice of the D_i's and the E_α's, we have $|X| = \omega_1 \subseteq X$ and

$M \cap \kappa \subseteq X$. Then $M(F) \cap X \in \hat{S} \cap [X]^\omega$. By the elementarity of $M(F)$, $\hat{S} \cap [X]^\omega$ is stationary in $[X]^\omega$. This completes the proof of one direction.

For the converse, let $\mathbb{P}$ be a σ-closed poset. Let S be the set of all countable $M \prec H_\theta$ such that $\mathbb{P} \in M$ but there are dense subsets $\langle D_i : i < \omega_1 \rangle$ in $\mathbb{P}$ such that there is no directed set F satisfying (1)–(3) in the definition of $\mathrm{FA}^0(\sigma\text{-closed})$. We see that S is non-stationary in $[H_\theta]^\omega$, which completes the proof.

Suppose to contrary that S is stationary. By SSR, we can find $X \subseteq H_\theta$ such that $|X| = \omega_1 \subseteq X$ and $\hat{S} \cap [X]^\omega$ is stationary in $[X]^\omega$. Let N be the Skolem hull of X under $\langle H_\theta, \in, \Delta \rangle$. Then we can take a club $\langle M_i : i < \omega_1 \rangle$ in $[N]^\omega$ such that $M_i \prec H_\theta$ for $i < \omega_1$. Since $\hat{S} \cap [X]^\omega$ is stationary, we have $E = \{i < \omega_1 : M_i \cap X \in \hat{S}\}$ is stationary in ω_1. Notice that for each $i \in E$, M_i belongs to S; Since $M_i \cap X \in \hat{S}$, there is $M' \prec H_\theta$ with $M' \prec_{\omega_1} M_i$. If $\langle D_j : j < \omega_1 \rangle$ are dense subsets in $\mathbb{P}$ witnessing $M' \in S$, then it also witnesses that $M_i \in S$. For each $i \in E$, fix dense subsets $\langle D^i_j : j < \omega_1 \rangle$ witnessing $M_i \in S$. Now take a countable $N \prec H_\theta$ such that $\langle M_i : i < \omega_1 \rangle, E, \langle D^i_j : i \in E, j < \omega_1 \rangle \in N$ and $N \cap \omega_1 \in E$. Let $i^* = N \cap \omega_1$. Then $M_{i^*} \prec_{\omega_1} N$. Since $\mathbb{P}$ is σ-closed, we can find a directed set $F \in N$ such that $|F| = \omega_1$ and $F \cap D^i_j \neq \emptyset$ for $i \in E$ and $j < \omega_1$. Then, since $M_{i^*} \prec_{\omega_1} N$ and $F \in N$, we have that $M_i \prec_{\omega_1} M_i(F)$. This is a contradiction. □

References

[1] T. Miyamoto, *A Forcing Axiom and Chang's Conjecture*, RIMS 2011.

[2] S. Shelah, *Proper and improper forcing*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1998.

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