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A Forcing Axiom and Chang's Conjecture

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Abstract

It is shown by H. Sakai that Chang's Conjecture and a form of weak square hold in the two stage generic extensions constructed by first forcing with the Levy collapse which turns a measurable cardinal into the second uncountable cardinal and then forcing the weak square over the first extensions. We extract a form of forcing axiom which captures an important property of the first extensions and show that Chang's Conjecture and an instance of strong non-reflection hold by forcing over the ground model where the forcing axiom is assumed.

Introduction

We extract a forcing axiom from the construction in [S]. This type of forcing axiom is new in the sense that it sort of has a combined strength of Chang's Conjecture together with internal genericity. As an example, we show that Chang's Conjecture and a form of strong non-reflection (SNR) of [Sh] hold in the generic extensions by forcing SNR over the ground model where the forcing axiom is assumed. However there is a trade-off between the consistency strengths and the sizes of relevant partially ordered sets. We opt to relax requirements on the sizes of partially ordered sets. Namely, we use ordinary generic conditions rather than strong generic conditions ([S]) which typically function with respect to σ-closed partially ordered sets. We hope possible uses of this type of forcing axiom in contexts not only with σ-closed notions of forcing but also, say, with proper forcing. For further developments on this subject, we may consult [U].

§ 1. A New Forcing Axiom

We extract our new forcing axiom out of [S].

1.1 Definition. FA^*(*σ-closed*) holds, if for any σ-closed p.o. set P, any set $\mathcal{D}$ of dense subsets of P with $|\mathcal{D}| \leq \omega_1$, any (P,N)-generic condition q, where $(N,\in)$ is any countable elementary substructure of $(H_\theta,\in)$ with $P \in N$, θ is sufficiently large, there exists a directed subset F of P with $|F| \leq \omega_1$ such that

(1) $o(q) \cap F \neq \emptyset$.

(2) For all $D \in \mathcal{D}, D \cap F \neq \emptyset$.

(3) $N \prec_{\omega_1} N(F)$,

where $o(q) = \{q' \in P \mid q' \leq q\}$ and so F picks some (P,N)-generic condition below q. We denote

$$N(F) = \{g(F) \mid g \in N,\ g : \mathcal{P}(P) \longrightarrow H_\theta\}.$$

Then $(N(F),\in)$ is the $\subseteq$-least elementary substructure $(M,\in)$ of $(H_\theta,\in)$ with $N \cup \{F\} \subseteq M$. Note that $\mathcal{P}(P) \in N$, as $P \in N$ and θ is large. We write $N \prec H_\theta$ to mean that $(N,\in)$ is an elementary substructure of $(H_\theta,\in)$. It is that N are of size countable, unless $|N| = \omega_1$ gets specified. We also write $N \prec_{\omega_1} N(F)$ to mean that $N \subseteq N(F)$ and $N \cap \omega_1 = N(F) \cap \omega_1$. In particular, $N \prec N(F) \prec H_\theta$ holds.

1.2 Definition. ([Sh]) *SNR* holds, if there exists a function $f : \omega_2 \longrightarrow \omega_1$ such that for all $\alpha \in S^2_1 = \{\alpha' < \omega_2 \mid \mathrm{cf}(\alpha') = \omega_1\}$, there exists a closed and cofinal subset C of α such that the order-type of C is ω_1 and f restricted to C is strictly increasing.

We first show the following in this study of new forcing axiom.

1.3 Theorem. Let FA*(σ-closed) hold. Force SNR by the p.o set P of the initial segments. Then Chang's Conjecture and SNR hold in the generic extensions.

§ 2. Forcing SNR

We define the p.o. set of the initial segments to force SNR.

2.1 Definition. $p \in P$, if

(1) p is a sequence of countable ordinals of length less than ω_2.

(2) For all $\alpha \in S^2_1$ less than or equal to the length of p, there exists a closed and cofinal subset C of α such that the order-type of C is ω_1 and p restricted to C is strictly increasing.

We write α_p for the length of p and so $\alpha_p < \omega_2$.

For $p, q \in P$, we set $q \leq p$, if q end-extends p. We write $q < p$, if $q \leq p$ and $q \neq p$.

2.2 Lemma. P is σ-closed.

Proof. Let $\langle p_n \mid n < \omega \rangle$ be a descending sequence of conditions. Let $q = \cup\{p_n \mid n < \omega\}$. Then $q \in P$ and for all $n < \omega$, we have $q \leq p_n$, as there is no new $\alpha \in S^2_1$ to worry about.

□

2.3 Lemma. For any $p \in P$, any η with $\alpha_p < \eta < \omega_2$, there exists $q \in P$ such that $q < p$ and $\alpha_q = \eta$.

Proof. By induction on η. We give an account on the case cf$(\eta) = \omega_1$. Let $\langle \eta_i \mid i < \omega_1 \rangle$ be a continuously strictly increasing sequence of ordinals such that $\alpha_p = \eta_0$ and $\sup\{\eta_i \mid i < \omega_1\} = \eta$. Construct $\langle p_i \mid i < \omega_1 \rangle$ as follows. Let $p_0 = p$. Given p_i with $\alpha_{p_i} = \eta_i$. Consider $p_i \cup \{(\eta_i, i)\}$ and end-extend it to p_{i+1} with $\alpha_{p_{i+1}} = \eta_{i+1}$. Given a limit $i < \omega_1$ and $\langle p_j \mid j < i \rangle$, let $\langle j_n \mid n < \omega \rangle$ be a strictly increasing sequence of cofinal ordinals below i. Let $p_i = \bigcup\{p_{j_n} \mid n < \omega\}$. Then $p_i \in P$ with $\alpha_{p_i} = \eta_i$ and for all $j < i$, we have $p(\eta_j) = j$. This completes the construction. Let $q = \cup\{p_i \mid i < \omega_1\}$. Then $q \in P$ and $\alpha_q = \eta$, as q restricted to the club $\{\eta_i \mid i < \omega_1\}$ is strictly increasing.

□

The following suffices to show that ω_2 is preserved.

2.4 Lemma. P is ω_2-Baire.

Proof. Let $\langle D_i \mid i < \omega_1 \rangle$ be a sequence of open dense subsets of P. Let $p \in P$. Want to find $q \in P$ such that $q \leq p$ and for all $i < \omega_1$, we have $q \in D_i$. To this end, construct $\langle p_i \mid i < \omega_1 \rangle$ as follows. Let $p_0 = p$. Given $p_i \in P$, consider $p_i \cup \{(\alpha_{p_i}, i)\}$ and strictly end-extend it to p_{i+1}. We may assume $p_{i+1} \in D_i$. Let i be a limit ordinal with $i < \omega_1$ and have $\langle p_j \mid j < i \rangle$. Let $\langle j_n \mid n < \omega \rangle$ be a stricIty increasing cofinal ordinals below i. Let $p_i = \bigcup\{p_{j_n} \mid n < \omega\}$. Then $p_i \in P$ and $\alpha_{p_i} = \sup\{\alpha_{p_j} \mid j < i\}$ hold. This completes the construction. Let $q = \bigcup\{p_i \mid i < \omega_1\}$. Then $q \in P$, $q \leq p$ and $q \in \bigcap\{D_i \mid i < \omega_1\}$, as q restricted to the club $\{\alpha_{p_i} \mid i < \omega_1\}$ of α_q is strictly increasing.

□

This P does what we want.

2.5 Lemma. Let G be P-generic over the ground model V. Then $f_G = \bigcup G$ satisfies SNR.

Proof. Since ω_1, ω_2, S^2_0 and S^2_1 are all preserved, we are done.

□

This P forces a stationary subset of $[\omega_2]^\omega$.

2.6 Lemma. Let G be P-generic over V. Let

$$E = \{X \in [\omega_2]^\omega \mid f_G(\sup(X)) = X \cap \omega_1\}.$$

Then this E is a stationary subset of $[\omega_2]^\omega$ in the generic extension $V[G]$.

Proof. Let $p \Vdash_P$ "$\dot{g} : [\omega_2]^{<\omega} \longrightarrow \omega_2$". Let $N \prec H_\theta$ with $p, P, \dot{g} \in N$. Let $\langle p_n \mid n < \omega \rangle$ be a (P, N)-generic sequence. Let $q = \bigcup\{p_n \mid n < \omega\}$ and let

$$q^* = q \cup \{(\sup(N \cap \omega_2), N \cap \omega_1)\}.$$

Then this q^* is (P, N)-generic and q^* forces that $N \cap \omega_2 = N[\dot{G}] \cap \omega_2$ is $\dot{g}$-closed and $N \cap \omega_2 \in \dot{E}$, as $f_{\dot{G}}(\sup(N \cap \omega_2)) = N \cap \omega_1$.

□

The following is an extraction out of [S].

2.7 Lemma. Let FA*(σ-closed) hold. Let $P \in N \prec H_\theta$. Let q^* be (P, N)-generic with

$$q^*(\sup(N \cap \omega_2)) \geq N \cap \omega_1.$$

Then there exists a pair (p, N') such that

(1) $N \prec_{\omega_1} N' \prec H_\theta$.

(2) $p < q^*$ in P and $p \in N'$.

Proof. We consider a two stage iteration $P * \dot{Q}$ such that $\dot{Q}$ forces, over $V[G]$, a club $\dot{C}$ of ω_2, the order-type of $\dot{C}$ is ω_1 and f_G restricted to $\dot{C}$ is strictly increasing, where G is P-generic over V. More precisely, let us define $\dot{Q}$ in $V[G]$ as follows. Let $r \in Q = \dot{Q}_G$, if

(1) r is a countable closed subset of ω_2.

(2) f_G restricted to r is strictly increasing.

For $r, r' \in Q$, let $r' \leq r$, if r' end-extends r.

2.8 Lemma. In $V[G]$, for any $r \in Q$, any $\eta < \omega_2$ and any ξ with o.t.$(r) \leq \xi < \omega_1$, there exists $r' \in Q$ such that $r' < r$ in Q, $\eta < \max(r')$ and o.t.$(r') = \xi + 1$.

Proof. Given r, η and ξ, choose $\alpha \in S^2_1$ such that $\max(r), \eta < \alpha$. Then fix a club C in α such that f_G restricted to C is strictly increasing. Cut a sufficient amount out of C and append it to r to form r'.

□

2.9 Lemma. In $V[G]$, Q is E-complete. By this we mean that for any $N \prec H_\theta^{V[G]}$ such that $G \in N$ and $N \cap \omega_2 \in E$ and any (Q, N)-generic sequence $\langle r_n \mid n < \omega \rangle$, there exists a lower bound r of the r_n's in Q.

Proof. Since $\langle r_n \mid n < \omega \rangle$ is a (Q, N)-generic sequence, we have $\sup\{\max(r_n) \mid n < \omega\} = \sup(N \cap \omega_2)$, for all $\eta \in \bigcup\{r_n \mid n < \omega\}$, $f_G(\eta) < \sup\{f_G(\max(r_n)) \mid n < \omega\} = N \cap \omega_1 = f_G(\sup(N \cap \omega_2))$. Let $r = \bigcup\{r_n \mid n < \omega\} \cup \{\sup(N \cap \omega_2)\}$. Then this r works.

□

2.10 Lemma. In $V[G]$, Q is σ-Baire.

Proof. Since E is stationary and Q is E-complete, this holds.

□

2.11 Lemma. In $V[G]$, Q forces a closed cofinal subset C of $\omega_2^V = \omega_2^{V[G]}$ such that f_G restricted to C is strictly increasing.

Proof. Let H be Q-generic over $V[G]$. Let $C = \bigcup H$. Then this C works.

□

2.12 Lemma. Let $(p, \dot{r}) \in R$, if there exists pair (β, ξ) such that

- $(p, \dot{r}) \in P * \dot{Q}$.
- $\alpha_p = \beta + 1$ and $p(\beta) = \xi$.
- $p \Vdash_P$ "$\max(\dot{r}) = \beta$ and o.t.$(\dot{r}) = \xi + 1$".

Hence $p \in P$ decides two values associated with $\dot{r}$. Namely, $\max(\dot{r})$ to be β and the order-type of $\dot{r}$ to be $\xi + 1$ and furthermore the length of p is $\beta + 1$ and $p(\beta) = \xi$. It is not required that p decides the entire $\dot{r}$. The size of R would be still large, however we have

(1) R is a dense subset of $P * \dot{Q}$.
(2) R is σ-closed.

Proof. For (1): Let $(a, \dot{b}) \in P * \dot{Q}$. Let $N \prec H_\theta$ with $(a, \dot{b}), P * \dot{Q} \in N$. Fix $p \in P$ such that $p < a$ in P, p is (P, N)-generic, $\alpha_p = \sup(N \cap \omega_2) + 1$ and $p(\sup(N \cap \omega_2)) = N \cap \omega_1$. Then $p \Vdash_P$ "$N[\dot{G}] \prec H_\theta^{V[\dot{G}]}, \dot{G}, \dot{b}, \dot{Q} \in N[\dot{G}], N[\dot{G}] \cap \omega_2 = N \cap \omega_2 \in \dot{E}$". Hence we may fix a P-name $\dot{r}$ such that $(p, \dot{r}) \in P * \dot{Q}$, $p \Vdash_P$ "$\dot{r} < \dot{b}, \dot{r}$ is $(\dot{Q}, N[\dot{G}])$-generic, $\max(\dot{r}) = \sup(N[\dot{G}] \cap \omega_2) = \sup(N \cap \omega_2)$ and o.t.$(\dot{r}) = (N[\dot{G}] \cap \omega_1) + 1 = (N \cap \omega_1) + 1$". Hence $(p, \dot{r}) \in R$ with $(p, \dot{r}) \leq (a, \dot{b})$ in $P * \dot{Q}$.

For (2): Let $\langle (p_n, \dot{r}_n) \mid n < \omega \rangle$ be a descending sequence in R. Let $\beta_n + 1 = \alpha_{p_n}$ and $p_n(\beta_n) = \xi_n$. Let $\beta = \sup\{\beta_n \mid n < \omega\}$ and $\xi = \sup\{\xi_n \mid n < \omega\}$. Let $p = \bigcup\{p_n \mid n < \omega\} \cup \{(\beta, \xi)\}$. Then let $\dot{r}$ be a P-name such that $(p, \dot{r}) \in P * \dot{Q}$ and $p \Vdash_P$ "$\dot{r} = \bigcup\{\dot{r}_n \mid n < \omega\} \cup \{\beta\}$". Then this $(p, \dot{r})$ works.

□

Now go back to the proof of 2.7 Lemma. We have q^* which is (P, N)-generic with $q^*(\sup(N \cap \omega_2)) \geq N \cap \omega_1$. Take a P-name $\dot{r}$ such that $q^* \Vdash_P$ "$\dot{r}$ is $(\dot{Q}, N[\dot{G}])$-generic" as before. Then $(q^*, \dot{r})$ is $(P * \dot{Q}, N)$-generic. Since R is dense, we have $(q^{*\prime}, \dot{r}') \in R$ with $(q^{*\prime}, \dot{r}') \leq (q^*, \dot{r})$ in $P * \dot{Q}$. Then $(q^{*\prime}, \dot{r}')$ is (R, N)-generic. Apply FA*(σ-closed) to R, $(q^{*\prime}, \dot{r}')$ and $\mathcal{D}$ which specified later. We have a directed subset G of R and $(q^{**}, \dot{r}'')$ such that

(1) $(q^{**}, \dot{r}'') \leq (q^{*\prime}, \dot{r}')$ in R and $(q^{**}, \dot{r}'') \in G$.
(2) For all $D \in \mathcal{D}$, $D \cap G \neq \emptyset$.
(3) $N \prec_{\omega_1} N(G)$.

Let $p = \bigcup\{a \mid (a, \dot{b}) \in G\}$, $C = \{c_i \mid i < \omega_1\}$, where $\Vdash_{P * \dot{Q}}$ "$\dot{c}_i$ is the i-th element of $\dot{C}$" and there exists $w \in G$ such that $w \Vdash_{P * \dot{Q}}$ "$\dot{c}_i = c_i$". To make sense of this construction, let $D = \{w \in R \mid \exists\, c\ s.t.\ w \Vdash_{P * \dot{Q}}$ "$\dot{c}_i = c$"$\}$ and let $D \in \mathcal{D}$ for each $i < \omega_1$. Let $\alpha = \sup(C)$ and $F = \{a \mid (a, \dot{b}) \in G\}$.

Claim. (1) C is a club in α.
(2) The length of p is α.
(3) p restricted to C is strictly increasing and so $p \in P$ with $p < q^*$ in P.
(4) $N \prec_{\omega_1} N(F)$ and $p \in N(F)$.

Proof. For (1): (Strictly increasing): Let $i < j < \omega_1$. Take $w, w' \in G$ such that $w \Vdash_{P * \dot{Q}}$ "$\dot{c}_i = c_i$" and $w' \Vdash_{P * \dot{Q}}$ "$\dot{c}_j = c_j$". Since $\Vdash_{P * \dot{Q}}$ "$\dot{c}_i < \dot{c}_j$", any common extension of w, w' would establish $c_i < c_j$.

(Closed): To show the closedness of C, we take an approach told to us by Y. Yoshinobu which would work not only for ω_2 but also for bigger cardinal than ω_2. Let $i < \omega_1$ be a limit ordinal. Then $\Vdash_{P * \dot{Q}}$ "cf$(\dot{c}_i) = \omega$" and so for each $n < \omega$, there exists $j < i$ and $w \in G$ such that $w \Vdash_{P * \dot{Q}}$ "$\dot{D}_{\dot{c}_i}(n) < \dot{c}_j$", where $\langle \dot{D}_c(n) \mid n < \omega \rangle$

is a prefixed ladder at c for each $c \in S^2_0$. Hence by a common extension, we would have $D_{c_i}(n) < c_j$. To make sense of this construction, let $D = \{w \in R \mid \exists j < i \text{ s.t. } w \Vdash_{P*\dot{Q}} \text{“}D_{\dot{c}_i}(n) < \dot{c}_j\text{”}\}$ and let $D \in \mathcal{D}$ for all limit $i < \omega_1$ and all $n < \omega$.

For (2): Let $(a, \dot{b}) \in G$. Then there exits a pair (β, ξ) such that $\alpha_p = \beta + 1$, $p(\eta) = \xi$, $p \Vdash_{P*\dot{Q}} \text{“}\max(\dot{b}) = \beta, \text{o.t.}(\dot{b}) = \xi + 1\text{”}$. Hence $\beta = c_\xi < \alpha$ and so the length of p is less than or equal to α. Conversely, let $i < \omega_1$, $w \in G$ and $w \Vdash_{P*\dot{Q}} \text{“}\dot{c}_i = c_i\text{”}$. Since we may assume $D = \{w = (a, \dot{b}) \in R \mid \exists\, \xi \text{ s.t. } i < \xi < \omega_1\ w \Vdash_{P*\dot{Q}} \text{“o.t.}(\dot{b}) = \xi + 1\text{”}\ \} \in \mathcal{D}$ for all $i < \omega_1$, we may assume that $c_i < c_\xi < \alpha_a$ and so α is less than or equal to the length of p.

For (3): Let $j < i$. Want $p(c_i) < p(c_j)$. To this end take $w = (a, \dot{b}) \in G$ such that $c_j < c_i < \alpha_a$. We may further assume that $w \Vdash_{P*\dot{Q}} \text{“}a(c_j) = f_{\dot{G}}(\dot{c}_j) < f_{\dot{G}}(\dot{c}_i) = a(c_i)\text{”}$ and so $p(c_j) = a(c_j) < a(c_i) = p(c_i)$.

For (4): Since $p = \bigcup F$ and $N \prec N(F) \prec N(G) \prec H_\theta^{V[G]}$, we have $N \prec_{\omega_1} N(F)$ with $p \in N(F)$.

□

Now we are ready to get Chang's Conjecture.

2.13 Lemma. For any $p \in P$, there exists a pair (M, q) such that $M \prec H_\theta$, $M \cap \omega_1 < \omega_1$, $|\, M \cap \omega_2 \,| = \omega_1$, $q \leq p$ in P and q is (P, M)-generic and so $\Vdash_P$“Chang's Conjecture holds”.

Proof. Repeated use of the 2.7 Lemma. Construct $\langle (p_i, q^*_i, N_i) \mid i < \omega_1 \rangle$ as follows. Let N_0 be a countable elementary substructure of H_θ with $p, P \in N_0$. Let $p_0 = p$ and $q^*_0 \in P$ such that $q^*_0 \leq p_0$, q^*_0 is (P, N_0)-generic and $\alpha_{q^*_0} = \sup(N_0 \cap \omega_2) + 1$ and $q^*_0(\sup(N_0 \cap \omega_2)) \geq N_0 \cap \omega_1$. Suppose we have $q^*_i \in P$ such that q^*_i is (P, N_i)-generic and $q^*(\sup(N_i \cap \omega_2)) \geq N_i \cap \omega_1$. Then by 2.7 Lemma, we have a pair (p_{i+1}, N_{i+1}) such that $p_{i+1} < q^*_i, p_{i+1} \in P \cap N_{i+1}$.

For limit $i < \omega_1$, let $\langle j_n \mid n < \omega \rangle$ be strictly increasing and cofinal in i. Then let $N_i = \bigcup\{N_{j_n} \mid n < \omega\}$, $v = \sup\{q^*_{j_n}(\sup(N_{j_n} \cap \omega_2)) \mid n < \omega\}$, $p_i = \bigcup\{q^*_{j_n} \mid n < \omega\}$ and $q^*_i = p_i \cup \{(\sup(N_i \cap \omega_2), v)\}$. Then N_i is a countable elementray substructure of H_θ and q^*_i is (P, N_i)-generic with $q^*_i(\sup(N_i \cap \omega_2)) \geq N_i \cap \omega_1 = N_0 \cap \omega_1$.

This completes the construction. Let $M = \bigcup\{N_i \mid i < \omega_1\}$ and $q = \bigcup\{p_i \mid i < \omega_1\}$. Then this pair (M, q) works.

□

2.14 Corollary. It is consistent that PFA, Chang's Conjecture and SNR hold.

Proof. We out-line. Assume PFA^+ in the ground model. Then PFA^+ implies PFA and $\text{MA}^+(\sigma\text{-closed})$. It is shown by T. Usuba that $\text{MA}^+(\sigma\text{-closed})$ iff $\text{FA}^*(\sigma\text{-closed})$. It is also known that the p.o. set P preserves PFA, say, by Y. Yoshinobu. Hence we are done by forcing with the P over this ground model.

□

§ 3. A Variation

We turn our attention to an $*$-type forcing axiom for the partially ordered sets which have the countable chain condition (c.c.c.). We observe a relation to Chang's Conjecture.

3.1 Definition. Let ${}^{<\omega}2$ denote the notion of forcing which adds a new subset of ω by the initial segments. Hense it is a form of Cohen forcing adding a real. Now $FA^*({}^{<\omega}2)$ holds, if for any set $\mathcal{D}$ of dense subsets of ${}^{<\omega}2$ with $|\,\mathcal{D}\,| \leq \omega_1$ and any countable elementary substructure $(N, \in)$ of $(H_\theta, \in)$, where θ is a sufficiently large regular cardinal, there exists a directed subset F of ${}^{<\omega}2$ such that

(1) For all $D \in \mathcal{D}$, we have $F \cap D \neq \emptyset$.

(2) $N \prec_{\omega_1} N(F)$.

Hence $FA^*({}^{<\omega}2)$ implies $MA_{\omega_1}({}^{<\omega}2)$, an ordinary Martin's Axiom for ${}^{<\omega}2$ with at most ω_1 many dense subsets. In particular, $FA^*({}^{<\omega}2)$ entails that $\omega_2 \leq 2^\omega$. We introduce two types of Chang's Conjecture.

3.2 Definition. (1) CC^* holds, if for any countable elementary substructure $(N, \in)$ of $(H_\theta, \in)$, where θ is a sufficiently large regular cardinal, there exists $\beta \in \omega_2 \setminus N$ such that $N \prec_{\omega_1} N(\beta)$.

(2) CC^{**} holds, if for any countable elementary substructure $(N, \in)$ of $(H_\theta, \in)$, where θ is a sufficiently large regular cardinal, $\{\beta < \omega_2 \mid N \prec_{\omega_1} N(\beta)\}$ is cofinal in ω_2.

It is clear that CC** implies CC* and that CC* in turn implies Chang's Conjecture. We observe that there exist so many countable elemenatry substructures N of H_θ such that $\{\beta < \omega_2 \mid N \prec_{\omega_1} N(\beta)\}$ is stationary.

3.3 Lemma. Let θ be a regular cardinal with $\theta \geq \omega_3$. Let X be such that $\omega_1 \subset X \prec (H_\theta, \in)$, $|X| = \omega_1$ and Let $\langle X_i \mid i < \omega_1\rangle$ be a sequence of continuously increasing countable subsets of X with $\bigcup\{X_i \mid i < \omega_1\} = X$.

Then there exists a club $C \subset \omega_1$ such that

$$C \subset \{i < \omega_1 \mid \{\beta < \omega_2 \mid X_i \prec_{\omega_1} X_i(\beta)\} \text{ is stationary}\}.$$

Proof. By contradiction. Suppose $E = \{i < \omega_1 \mid \{\beta < \omega_2 \mid X_i \prec_{\omega_1} X_i(\beta)\}$ is not stationary$\}$ is stationary. For each $i \in E$, let $D(i)$ be a club in ω_2 such that $D(i) \cap \{\beta < \omega_2 \mid X_i \prec_{\omega_1} X_i(\beta)\} = \emptyset$. Let $\beta \in \bigcap\{D(i) \mid i \in E\}$. Let M be a countable elementary substructure of H_θ such that $\langle X_i \mid i < \omega_1\rangle, \beta \in M$ and $i = M \cap \omega_1 \in E$. Then $X_i \prec_{\omega_1} M$ and $\beta \in M$. Hence $X_i \prec_{\omega_1} X_i(\beta)$. Since $\beta \in D(i)$, this is a contradiction.

□

3.4 Proposition. If the following reflection (RF) holds, then CC** holds such that $\{\beta < \omega_2 \mid N \prec_{\omega_1} N(\beta)\}$ is not only cofinal but stationary for all countable elementary substructures N of $(H_\theta, \in)$, where θ is a regular cardinal with $H_{\omega_3} \in H_\theta$.

(RF) For any stationary $T \subseteq [H_{\omega_3}]^\omega$, there exists X such that $\omega_1 \subset X \prec (H_{\omega_3}, \in)$ and $|X| = \omega_1$ and $T \cap [X]^\omega$ is stationary in $[X]^\omega$.

Proof. It suffices to show that $S = \{N \in [H_{\omega_3}]^\omega \mid N \prec H_{\omega_3}, \{\beta < \omega_2 \mid N \prec_{\omega_1} N(\beta)\}$ is stationary$\}$ contains a club in $[H_{\omega_3}]^\omega$. Let $T = \{N \in [H_{\omega_3}]^\omega \mid N \prec H_{\omega_3}, \{\beta < \omega_2 \mid N \prec_{\omega_1} N(\beta)\}$ is not stationary$\}$. Suppose T were stationary. Then by the reflection assumed, we have X and $\langle X_i \mid i < \omega_1\rangle$ such that $E = \{i < \omega_1 \mid X_i \in T\}$ is stationary. But by 3.3 Lemma, there exists $i \in E$ such that $\{\beta < \omega_2 \mid X_i \prec_{\omega_1} X_i(\beta)\}$ is stationary. Since $i \in E$ and so $X_i \in T$, this would be a contradiction.

□

We observe that CC* and CC** are equivalent in some situation.

3.5 Proposition. Let $FA^*({}^{<\omega}2)$ hold. Then the following are equivalent.

(1) $2^\omega = \omega_2$.

(2) CC**.

(3) CC*.

Proof. (1) implies (2): Let $\langle r_\beta \mid \beta < \omega_2\rangle$ be a one-to-one enumeration of the functions from ω into 2. We may assume that $\langle r_\beta \mid \beta < \omega_2\rangle \in N$. Let $\eta < \omega_2$. Let $D(i) = \{p \in {}^{<\omega}2 \mid p \not\subset r_i\}$ for each $i < \eta + 1$. Then $D(i)$ is a dense subset of ${}^{<\omega}2$. Let $\mathcal{D} \supset \{D(i) \mid i < \eta + 1\}$ and apply $FA^*({}^{<\omega}2)$ to this $\mathcal{D}$ and N to get F. We may assume that $r = \bigcup F = r_\beta$ for some $\beta < \omega_2$ and so $\eta < \beta$ with $N \prec_{\omega_1} N(\beta)$ $\left(\prec_{\omega_1} N(F)\right)$.

(2) implies (3): Trivial.

(3) implies (1): It is known that CC* implies $2^\omega \leq \omega_2$ due to S. Todorcevic. Hence we have $2^\omega = \omega_2$.

□

3.6 Theorem. Let κ be a measurable cardinal. Let $P = \mathrm{Fn}(\kappa \times \omega, 2, \omega)$ denote the notion of forcing with the finite conditions which adds κ-many new functions from ω to 2. Then we have $\mathrm{FA}^*({}^{<\omega}2)$ and $2^\omega = \kappa$ in the generic extensions via P.

By first forcing $\Box_{\omega_1}$, we have

3.7 Corollary. It is consistent that $\mathrm{FA}^*({}^{<\omega}2)$ together with $\Box_{\omega_1}$ hold. Hence $\mathrm{FA}^*({}^{<\omega}2)$ may not imply Chang's Conjecture.

Since Chang's Conjecture is preserved by c.c.c. forcing, we also have

3.8 Corollary. It is consistent that $\mathrm{FA}^*({}^{<\omega}2)$, Chang's Conjecture and $2^\omega > \omega_2$.

Proof of theorem. Let $A = H_{\kappa^+}$ and θ be a regular cardinl with $A \in H_\theta$ in the ground model V. Let G be P-generic over V. It suffices to show that there exists a club C in $[H_\theta^{V[G]}]^\omega$ such that for any $\dot{N} \in C$ and any $\mathcal{D}$, there exists a filter F' in ${}^{<\omega}2$ such that for all $D \in \mathcal{D}$, $F' \cap D \neq \emptyset$ and $\dot{N} \prec_{\omega_1} \dot{N}(F')$ in $V[G]$.
To this end let $j : V \longrightarrow W$ be a nontrvial elementary embedding by κ. Then we have $j\lceil A = \{(a, j(a)) \mid a \in A\} \subset A \times j(A) \subset H_\theta^V$. Let F be $\mathrm{Fn}(\{\kappa\} \times \omega, 2, \omega)$-generic over $V[G]$ and let J be $\mathrm{Fn}\big((\kappa, j(\kappa)) \times \omega, 2, \omega\big)$-generic over $V[G][F]$. Let us again denote the elementary embedding by $j : V[G] \longrightarrow W[G][F][J]$ which naturally extends j. Let us consider a first order strutcure

$$\mathcal{A} = (H_\theta^{V[G]}, \in, H_\theta^V, H_\theta^{W[G]}, H_\theta^W, A, A[G], j\lceil A, G)$$

where G is viewed as a constant.

Claim (in $V[G]$). Let $\dot{N} \prec \mathcal{A}$ and $\mathcal{D}$ be any set of dense subsets of ${}^{<\omega}2$ with $|\mathcal{D}| \leq \omega_1$, then there exists a filter F' in ${}^{<\omega}2$ such that for all $D \in \mathcal{D}$, $F' \cap D \neq \emptyset$ and $\dot{N} \prec_{\omega_1} \dot{N}(F')$.

Proof. Let $N = \dot{N} \cap H_\theta^W[G]$. Then $\kappa, j(\kappa), G, \mathrm{Fn}(\{\kappa\} \times \omega, 2, \omega) \in N \in W[G]$ and $N \prec H_\theta^{W[G]}$. Hence we may form $N[F]$ in $W[G][F]$ so that $N[F] \prec (H_\theta^{W[G][F]}, \in)$ and $N[F] \cap \omega_1 = N \cap \omega_1$. Then $\mathrm{Fn}\big((\kappa, j(\kappa)), \times\omega, 2, \omega\big) \in N[F]$. Hence we may form $N[F][J]$ in $W[G][F][J]$ so that $N[F][J] \prec (H_\theta^{W[G][F][J]}, \in$) and $N[F][J] \cap \omega_1 = N \cap \omega_1$.

Subclaim. $\{j(a) \mid a \in \dot{N} \cap A\} \subset N$.

Proof. Let $a \in \dot{N} \cap A$. Then $j(a) \in j(A) = H^W_{(j(k)^+)^W} \subset H_\theta^W$. Hence $j(a) \in \dot{N} \cap H_\theta^W \subset \dot{N} \cap H_\theta^{W[G]} = N$.

□

Subclaim (in $W[G][F][J]$). $j(\dot{N} \cap A[G]) \prec_{\omega_1} j(\dot{N} \cap A[G])(F) \prec j(A[G])$. We also have for all $D \in j(\mathcal{D})$, $F \cap D \neq \emptyset$.

Proof. Since $\dot{N} \prec \mathcal{A}$, we have $\dot{N} \cap A[G] \prec (A[G], \in)$ and so $j(\dot{N} \cap A[G]) \prec \big(j(A[G]), \in\big)$. Since $\mathcal{P}^{V[G]}({}^{<\omega}2) \in \dot{N} \cap A[G]$, we have $F \in \mathcal{P}^{W[j(G)]}({}^{<\omega}2) \in j(\dot{N} \cap A[G])$. Hence $j(\dot{N} \cap A[G])(F) \prec j(A[G])$. We have

$$j(\dot{N} \cap A[G]) \cap \omega_1 = \{j(a) \mid a \in \dot{N} \cap A[G]\} \cap \omega_1 = \{j(\xi) \mid \xi \in \dot{N} \cap A[G] \cap \omega_1\} = \{\xi \mid \xi \in \dot{N} \cap A[G] \cap \omega_1\} = N \cap \omega_1.$$

$$j(\dot{N} \cap A[G])(F) \cap \omega_1 \subset \{j(f)(F) \mid f \in \dot{N} \cap A[G]\} \subset \{j(a)_{j(G)}(F) \mid a \in \dot{N} \cap A\} \subset N[F][J].$$

This is because $j(a) \in N$ and $j(G), F \in N[F][J]$. Hence

$$j(\dot{N} \cap A[G]) \cap \omega_1 = N \cap \omega_1 = N[F][J] \cap \omega_1 = j(\dot{N} \cap A[G])(F) \cap \omega_1.$$

Next, since $\mathcal{D} \in W[G]$, we have for all $D \in \mathcal{D}, F \cap D \neq \emptyset$. Hence in $W[G][F][J] = W[j(G)]$, we have for all $D \in \mathcal{D}$, $F \cap D \neq \emptyset$. But $j(\mathcal{D}) = \{j(D) \mid D \in \mathcal{D}\}$, as $|\mathcal{D}| \leq \omega_1 = j(\omega_1)$ in $V[G]$. Hence $F \cap D \neq \emptyset$ implies $F \cap j(D) \neq \emptyset$, as j is identity on ${}^{<\omega}2 \supset F$.

□

Now by elementarity,

Subclaim (in $V[G]$). $\dot{N} \cap A[G] \prec_{\omega_1} (\dot{N} \cap A[G])(F')$ for some directed subset F' of ${}^{<\omega}2$ such that for all $D \in \mathcal{D}, F' \cap D \neq \emptyset$. And so $\dot{N} \prec_{\omega_1} \dot{N}(F')$.

Proof. If $f : \mathcal{P}^{V[G]}({}^{<\omega}2) \longrightarrow \omega_1$ with $f \in \dot{N}$, then $f \in \dot{N} \cap A[G] = \dot{N} \cap (H^{V[G]}_{(k^+)^{V[G]}})$ holds. Hence we are done.

□

We consider a variation on $\mathrm{FA}^*({}^{<\omega}2)$ which entails that 2^ω is strictly greater than ω_2.

3.9 Definition. $FA^{\triangle}({}^{<\omega}2)$ holds, if for any set $\mathcal{D}$ of dense subsets of ${}^{<\omega}2$ with $|\mathcal{D}| \leq \omega_1$ and any non-empty $\mathcal{N}$ of countable elementary substructures of $(H_\theta, \in, \triangle)$, where θ is sufficiently large and $\triangle$ well-orders H_θ, with $|\mathcal{N}| \leq \omega$, there exists a directed subset F of ${}^{<\omega}2$ such that

(1) For all $D \in \mathcal{D}$, we have $F \cap D \neq \emptyset$.
(2) For all $N \in \mathcal{N}$, we have $N \prec_{\omega_1} N(F)$.

Hence F works simultaneously for countable many elementary substructures.

3.10 Theorem. $\mathrm{FA}^{\triangle}({}^{<\omega}2)$ implies $2^\omega > \omega_2$.

Proof. By contradiction. Suppose $2^\omega = \omega_2$. Since we have $\mathrm{FA}^*({}^{<\omega}2)$, we have CC**. But we also have

Claim. $\Vdash_{{}^{<\omega}2}$"CC**".

Proof. Let G be ${}^{<\omega}2$-generic over V. Let

$$C = \{\dot{N}[G] \mid \exists \langle N_n \mid n < \omega\rangle \in -\text{chain s.t.(1) } N_n \in V,\ N_n \prec H^V_\theta \text{ (for all } n) \text{ (2) } \dot{N} = \bigcup\{N_n \mid n < \omega\}\}.$$

Then this C is a club of $[H^{V[G]}_\theta]^\omega$ in $V[G]$. We show that for $\dot{N}[G] \in C$ and $\eta < \omega_2$, there exists $\beta < \omega_2$ such that $\eta < \beta$ and $\dot{N}[G] \prec_{\omega_1} \dot{N}[G](\beta) \in C$. To this end take a countable family $\mathcal{N} \in V$ of countable elementary substructures of H^V_θ such that $\{N_n \mid n < \omega\} \subset \mathcal{N}$. This is possible by c.c.c.

Apply $\mathrm{FA}^{\triangle}({}^{<\omega}2)$ to get $r = r_\beta$ with $\eta < \beta$ where $\langle r_i \mid i < \omega_2\rangle$ one-to-one enumerates the functions from ω to 2. We have $N_n \prec_{\omega_1} N_n(\beta)$ for all $n < \omega$. Since $N_n \in N_{n+1}$ and $N_n(\beta) = \{g(\beta) \mid g \in N_n\}$, we have $N_n(\beta) \in N_{n+1}(\beta)$. Hence $\langle N_n(\beta) \mid n < \omega\rangle$ is an $\in$-chain and $\dot{N}(\beta) = \bigcup\{N_n \mid n < \omega\}(\beta) = \bigcup\{N_n(\beta) \mid n < \omega\}$. Hence $\dot{N}(\beta)[G] \in C$ and

$$\dot{N}(\beta)[G] \cap \omega_1 = \bigcup\{N_n(\beta)[G] \cap \omega_1 \mid n < \omega\} = \bigcup\{N_n(\beta) \cap \omega_1 \mid n < \omega\} = \bigcup\{N_n \cap \omega_1 \mid n < \omega\}$$

$$= \bigcup\{N_n[G] \cap \omega_1 \mid n < \omega\} = (\bigcup\{N_n \mid n < \omega\})[G] \cap \omega_1 = \dot{N}[G] \cap \omega_1.$$

Hence $\dot{N}[G] \prec_{\omega_1} \dot{N}[G](\beta)$.

□

This contradicts to the following fact which is due to J. Baumgartner and possibly to others, too.

Claim. $\Vdash_{{}^{<\omega}2}$"$\neg$ CC*".

Proof. (Out-line) CC* entails that every stationary subset S of $[\omega_2]^\omega$ reflects to some α with $\omega_1 \leq \alpha < \omega_2$. This is due to S. Todorcevic. But in the generic extension $V[G]$, $S = ([\omega_2]^\omega)^{V[G]} \setminus ([\omega_2]^\omega)^V$ is stationary.

Namely, every club subset of $[\omega_2]^\omega$ in $V[G]$ contains elements not in V. Hence this S reflects to some α. But any $\langle F_i^\alpha \mid i < \omega_1 \rangle \in V$ remains a club of α in $V[G]$. This would be a contradiction.

□

3.11 Question. (1) Are CC* and CC** equivalent ?

(2) What is the exact consistency strength of, say, FA*$(^{<\omega}2)$?

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