<table>
<thead>
<tr>
<th>Title</th>
<th>Shape optimization for partial differential equations/system with mixed boundary conditions (The latest developments in theory and application on scientific computation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Ohtsuka, Kohji</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2012), 1791: 172-181</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2012-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/172826">http://hdl.handle.net/2433/172826</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
</tbody>
</table>
1 Theory of GJ-integral

Let $X$ and $M$ be real Banach spaces and let $X'$ and $M'$ be their dual spaces, respectively. For $\mathcal{U}_0 \subset X$ ($\mathcal{U}_0 \neq \emptyset$) and an open subset $\mathcal{O}_0 \subset M$ ($\mathcal{O}_0 \neq \emptyset$), we consider a real valued functional $J : \mathcal{U}_0 \times \mathcal{O}_0 \to \mathbb{R}$. In general, for $u \in \mathcal{U}_0$ and $w \in X$, the Gâteaux derivative $\delta_X J(u, \mu)[w] \in \mathbb{R}$ is defined as

$$\Delta J(u, \mu)[w] = \frac{d}{dt} J(u + tw, \mu) \bigg|_{t=0},$$

when it exists. If $\delta_X J(u, \mu)[w]$ exists, from the linearity of the Gâteaux derivative, $\delta_X J(u, \mu)[\alpha w]$ exists for arbitrary $\alpha \in \mathbb{R}$ and it satisfies

$$\delta_X J(u, \mu)[\alpha w] = \alpha \delta_X J(u, \mu)[w].$$

We use the symbols $\partial_X$ and $\partial_M$ to denote the partial Fréchet derivative operators for $J(u, \mu)$ with respect to $u \in X$ and $\mu \in M$, respectively, and assume the following.

(H1) $[\mu \mapsto J(w, \mu)] \in C^1(\mathcal{O}_0)$ for all $w \in \mathcal{U}_0$, and $\partial_M J : \mathcal{U}_0 \times \mathcal{O}_0 \to M'$ is continuous at $(u(\mu_0), \mu_0)$.

(H2) The Banach space $X$ is reflexive and $\mathcal{U}_0$ is closed and convex in $X$.

(H3) For the functional $[v \mapsto J(v, \mu_0)]$, $u_0$ is a unique minimizer over $\mathcal{U}_0$.

(H4) The functional $[v \mapsto J(v, \mu_0)]$ is sequentially lower semicontinuous with respect to the weak topology of $X$.

(H5) There is a monotone nondecreasing function $\beta_0$ defined on $[0, \infty)$ with $\lim_{s \to \infty} \beta_0(s) = \infty$ such that

$$\beta_0(||v||_X) \leq J(v, \mu) \quad (v \in \mathcal{U}_0, \mu \in \mathcal{O}_0).$$

*email: ohtsuka@hkg.ac.jp
(H6) For any $\varepsilon > 0$ and $R > 0$, there exists $\delta > 0$ such that

$$|J(v, \mu) - J(v, \mu_0)| \leq \varepsilon \quad (v \in \mathcal{U}_0, \|v\|_X \leq R, \mu \in \mathcal{O}_0, \|\mu - \mu_0\|_M \leq \delta).$$

(H7) For $v \in \mathcal{U}_0$, the function $[t \mapsto J(u_0 + t(v - u_0), \mu_0)]$ belongs to $C^1((0, 1])$. Moreover, for a sequence $\{u_n\}_n \subset \mathcal{U}_0$ which weakly converges to $u_0$ as $n \to \infty$, the condition

$$\lim_{n \to \infty} \delta_X J(u_n, \mu_0)[u_n - u_0] \leq 0$$

implies that $u_n \to u_0$ strongly in $X$ as $n \to \infty$.

In particular, under the condition (H7),

$$\delta_X J(v, \mu_0)[v - u_0] = \left. \frac{d}{dt} J(u_0 + t(v - u_0), \mu_0) \right|_{t=1}$$

exists for all $v \in \mathcal{U}_0$. The condition (H7) is often called the $(S_+)$-property.

**Theorem 1** Under the conditions (H1)-(H7), $[\mu \mapsto J_*(\mu)]$ is Fréchet differentiable at $\mu = \mu_0$ and the following holds.

$$D_{\mu}[J(u(\mu_0), \mu_0)] = \partial_M J(u(\mu_0), \mu_0)$$

where the $D_{\mu}$ denotes the Fréchet differential operator with respect to $\mu \in M$.

See [9] for the proof. Theorem 1 will play an important role in design sensitivity analysis by considering $\mu u$ to be a design variable. We shall show that Theorem 1 derive an important result in the shape sensitivity analysis of energy.

### 1.1 Boundary value problems and its Lipschitz perturbation

Let $\Omega$ be a bounded domain in $\mathbb{R}^d (d \geq 2)$ and $L^p(\Omega, \mathbb{R}^m)$ Lebesgue space of all measurable functions $v : \Omega \to \mathbb{R}^m$ (a real number $1 < p < \infty$ and an integer $m \geq 1$) with $\|v\|_{p, \Omega}$

$$\|v\|_{p, \Omega} = \left( \sum_{i=1}^{m} \int |v_i|^p \right)^{1/p}, v = (v_1, \cdots, v_m)$$

$P(f, V(\Omega, \Gamma_D))$: For a given function $f \in L^{p'}(\Omega, \mathbb{R}^m), p' = p/(p-1)$, find $u$ minimizing the following functional

$$\mathcal{E}(v; f, \Omega) = \int_{\Omega} \left\{ \hat{W}(x, v, \nabla v) - f \cdot v \right\} dx$$

over the space

$$V(\Omega, \Gamma_D) = \{ v \in W^{1,p}(\Omega, \mathbb{R}^m) : v = 0 \text{ on } \Gamma_D \}$$

where $\Gamma_D$ stands for the part of $\partial \Omega$ and a scalar function $\hat{W}(\xi, z, \zeta) : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d \times m} \to \mathbb{R}$ is in $C^1(\mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d \times m})$ and $W^{1,p}(\Omega, \mathbb{R}^m)$ denote Sobolev space of functions $v \in L^p(\Omega, \mathbb{R}^m)$ with $\|v\|_{1,p, \Omega}$.

We now give a condition of the existence of minimizers for $P(f, V(\Omega, \Gamma_D))$. 
Theorem 2 If $\hat{W}$ satisfies the coercivity condition
\[
\hat{W}(\xi, z, \zeta) \geq c_{1}|\xi|^{p} + c_{2}|z|^{q} + \alpha_{1}(\xi)
\]
for almost every $\xi \in \Omega$ and for every $(z, \zeta) \in \mathbb{R}^{m} \times \mathbb{R}^{m \times d}$ and for some $\alpha_{1} \in L^{1}(\Omega)$, $c_{2} \in \mathbb{R}$, $c_{1} > 0$ and $p > q \geq 1$. Assume that $\zeta \rightarrow \hat{W}(\xi, z, \zeta)$ is convex and $\mathcal{E}(0; f, \Omega) < \infty$, then there is a minimizer $u$
\[
E(u; f, \Omega) = \min_{v \in \mathcal{V}(\Omega, \Gamma_{D})} E(v; f, \Omega)
\]
attains its minimum.

Furthermore, if $(z, \zeta) \rightarrow \hat{W}(\xi, z, \zeta)$ is strictly convex for almost every $\xi \in \Omega$, then the minimizer is unique.

See [2][Theorem 3.30] for the proof.

We assume the growth conditions of $\hat{W}$, that is, for almost every $\xi \in \Omega$, for every $(z, \zeta) \in \mathbb{R}^{m} \times \mathbb{R}^{m \times d}$
\[
|\hat{W}(\xi, z, \zeta)| \leq \alpha_{1}(\xi) + c(|z|^{p} + |\zeta|^{p})
\]
\[
|D_{z}\hat{W}(\xi, z, \zeta)| \leq \alpha_{2}(\xi) + c(|z|^{p-1} + |\zeta|^{p-1})
\]
\[
|D_{\zeta}\hat{W}(\xi, z, \zeta)| \leq \alpha_{3}(\xi) + c(|z|^{p-1} + |\zeta|^{p-1})
\]
where $\alpha_{1} \in L^{1}(\Omega_{0})$, $\alpha_{2}, \alpha_{3} \in L^{p(p-1)}(\Omega_{0})$ and $c \geq 0$,
\[
D_{\zeta}\hat{W}(u) = \begin{pmatrix}
\frac{\partial}{\partial\zeta_{11}}\hat{W}(\xi,z,\zeta) & \cdots & \frac{\partial}{\partial\zeta_{m1}}\hat{W}(\xi,z,\zeta) \\
\vdots & \ddots & \vdots \\
\frac{\partial}{\partial\zeta_{md}}\hat{W}(\xi,z,\zeta) & \cdots & \frac{\partial}{\partial\zeta_{md}}\hat{W}(\xi,z,\zeta)
\end{pmatrix}_{(\xi,z,\zeta)=(x,v(x),\nabla v(x))}
\]
\[
D_{z}\hat{W}(u) = \begin{pmatrix}
\frac{\partial}{\partial z_{1}}\hat{W}(\xi,z,\zeta) & \cdots & \frac{\partial}{\partial z_{d}}\hat{W}(\xi,z,\zeta)
\end{pmatrix}^{T}_{(\xi,z,\zeta)=(x,v(x),\nabla v(x))}
\]
Then, we have the following proposition.

Proposition 3 If $u$ is the solution of $P(f, V(\Omega, \Gamma_{D}))$ and $\hat{W}$ satisfy Condition (3), then
\[
\int_{\Omega} \left\{ \frac{\partial}{\partial x} \hat{W}(x, u, \nabla u) : \nabla v + D_{z}\hat{W}(x, u, \nabla u)v - fv \right\} \, dx = 0
\]
for all $v \in W_{0}^{1,p}(\Omega, \mathbb{R}^{m})$, where $A : B = A_{ij}B_{ij}$ for two matrices $A$ and $B$.

See [2][Theorem 3.37] for the proof.

Putting
\[
F(v) = \int_{\Omega} \hat{W}(x, v, \nabla v) \, dx
\]
we have for $X = W^{1,p}(\Omega, \mathbb{R}^m)$,
\[
\langle \delta_X F(v), w \rangle_X = \lim_{\epsilon \to 0} \epsilon^{-1} [F(v + \epsilon w) - F(v)]
\]
\[
= \int_{\Omega} \left\{ D_{\xi} \hat{W}(x, v, \nabla v) : \nabla w + D_{z} \hat{W}(x, v, \nabla v)w \right\} dx
\]

The operator $[v \mapsto \delta_X F(v)]$ is called uniformly monotone, if there is a strictly monotone increasing continuous function $a : \mathbb{R}_+ \to \mathbb{R}_+$ with $a(0) = 0$ and $\lim_{t \to \infty} a(t) = +\infty$ such that
\[
\langle \delta_X F(v) - \delta_X F(w), v - w \rangle_X \geq a(\|v - w\|_X)\|v - w\|_X
\]

If $[v \mapsto \delta_X F(v)]$ is uniformly monotone, then the condition (H7) is satisfied. Indeed, taking a sequence $u_n \to u$ weakly in $X$, we have
\[
\langle \delta_X F(u_n) - f, u_n - u \rangle_X = \langle \delta_X F(u_n) - \delta_X F(u), u_n - u \rangle_X \geq a(\|u_n - u\|_X)\|u_n - u\|_X
\]

Since $X$ is reflexive, the strong convergence $u_n \to u$ follows from
\[
a(\|u - v\|_X) \leq \|\delta_X F(u_n) - f\|_{X'}
\]

1.1.1 Perturbation

We choose a bounded convex domain $\Omega$ with $\overline{\Omega} \subset \Omega$, and define $M = W^{1,\infty}(\Omega, \mathbb{R}^d)$ and
\[
\mathcal{O}_0 = \left\{ \varphi \in M : |\varphi - \varphi_0|_{\text{Lip}, \Omega} < a_0 < 1, \varphi(\overline{\Omega}) \subset \Omega \right\},
\]
where $a_0 \in (0, 1)$ is a fixed number and we denote by $\varphi_0$ the identity map on $\mathbb{R}^d$, i.e., $\varphi_0(x) = x (x \in \mathbb{R}^d)$. Then $\varphi \in \mathcal{O}_0$ becomes a bi-Lipschitz transform from $\Omega$ onto $\varphi(\Omega)$.

For the domain $\varphi(\Omega)$, $\varphi \in \mathcal{O}_0$, we consider the problem $P(f, V(\varphi(\Omega), \varphi(\Gamma_D)))$: Find $u(t)$ minimizing the following functional
\[
\mathcal{E}(v; f, \varphi(\Omega)) = \int_{\varphi(\Omega)} \left\{ \hat{W}(x, v, \nabla v) - f \cdot v \right\} dx
\]
over the space
\[
V(\varphi(\Omega), \varphi(\Gamma_D)) = \{ w \in W^{1,p}(\varphi(\Omega), \mathbb{R}^m) : w = 0 \text{ on } \varphi(\Gamma_D) \}
\]

We define a pushforward operator $\varphi_*$ which transforms a function $v$ on $\Omega$ to a function $\varphi_* v = v \circ \varphi^{-1}$ on $\varphi(\Omega)$. For $q \in [1, \infty]$, $\varphi_*$ is a linear topological isomorphism from $L^q(\Omega)$ onto $L^q(\varphi(\Omega))$, and a linear topological isomorphism from $W^{1,q}(\Omega)$ onto $W^{1,q}(\varphi(\Omega))$. For $v \in V(\Omega, \Gamma_D)$, we get the equivalence,
\[
\mathcal{E}(\varphi_* v, f, \varphi(\Omega)) = \int_{\Omega} \left\{ \hat{W}(\varphi(x), v(x), [A(\varphi)(x)]\nabla v(x)) - f \circ \varphi(x)v(x) \right\} \kappa(\varphi)(x) dx
\]
where
\[
A(\varphi) = (\nabla \varphi^T)^{-1} \in L^\infty(\Omega_0, \mathbb{R}^{d \times d}), \quad \kappa(\varphi) = \det \nabla \varphi^T \in L^\infty(\Omega_0, \mathbb{R}).
\]
We denote the right-hand side of (5) by $J(v, \varphi)$, and apply Theorem 1 to $J(v, \varphi)$.

If $\tilde{W}$ satisfy the growth condition (3) and

$$|D_{\xi}W(\xi, z, \zeta)| \leq \alpha_1(\xi) + c(|z|^p + |\zeta|^p)$$

then we have the following proposition.

**Proposition 4** Suppose that $f \in W^{1,p/(p-1)}(\Omega_0)$. Then $J \in C^1(X \times \mathcal{O}_0)$ and

$$\partial_M J(u, \varphi_0)[\mu] = \frac{d}{dt} \int_{\Omega} \tilde{W}(x + t\mu(x), u(x), [A(\varphi_0 + t\mu(x)]\nabla u(x)) \kappa(\varphi_0 + t\mu(x))dx \bigg|_{t=0}$$

$$- \int_{\Omega} f \circ (\varphi_0 + t\mu)(x)v(x)\kappa(\varphi_0 + t\mu(x))dx \bigg|_{t=0}$$

$$= \int_{\Omega} (D_{\xi}W(u) \cdot \mu - (D_{\zeta}W(u))^T(\nabla \mu^T)\nabla u + W(u)\text{div} \mu - \text{div}(f \cdot \mu)v) \ dx.$$ (6)

where $(D_{\zeta}W(u))^T(\nabla \mu^T)\nabla u = \sum_{i,j,k} \partial_{\zeta_{ij}}\tilde{W}(x, u, \nabla u)(\partial_{j}\mu_k)(\partial_{k}u_i)$.

1.1.2 **Definition of GJ-integral**

For an open subset $\omega$ in $\mathbb{R}^d$ and $\rho \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, GJ-integral [5, 6, 7, 8]

$$\mathcal{J}_{\omega}(u, \rho) = P_{\omega}(u, \rho) + R_{\omega}(u, \rho)$$

is defined by

$$P_{\omega}(u, \rho) = \int_{\partial(\omega \cap \Omega)} \left\{ \tilde{W}(u)(\rho \cdot n) - \tilde{T}(u) \cdot (\nabla u \cdot \rho) \right\} ds$$

$$R_{\omega}(u, \rho) = -\int_{\omega \cap \Omega} \left\{ \nabla_\xi \tilde{W}(u) \cdot \rho + f \cdot (\nabla u \cdot \rho) - \left( \nabla_\zeta \tilde{W}(u) \right)^T(\nabla \rho^T) \nabla u + \tilde{W}(u)\text{div} \rho \right\} dx$$

where $n = (n_1, \cdots, n_d)^T$ is the outward unit normal of $\partial(\omega \cap \Omega)$, $i$-th component of $\tilde{T}(u)$ is $n_j \nabla_\zeta_{ij}\tilde{W}(x, u, \nabla u)$ and $ds$ the surface(line) element of $\partial(\omega \cap \Omega)$.

**Proposition 5** If $u|_{\omega \cap \Omega} \in W^{2,p}(\omega \cap \Omega, \mathbb{R}^m)$ and the divergence formula

$$\int_{\omega \cap \Omega} \nabla \tilde{W}(u) \cdot \rho \ dx = \int_{\partial(\omega \cap \Omega)} \tilde{W}(u)(\rho \cdot n)ds - \int_{\omega \cap \Omega} \tilde{W}(u)\text{div} \rho \ dx$$ (7)

holds, then

$$\mathcal{J}_{\omega}(u, \rho) = 0 \text{ for all } \rho \in W^{1,\infty}(\Omega_0, \mathbb{R}^d)$$ (8)

Consider the perturbation $\Omega(t), 0 \leq t < \epsilon$ of $\Omega$ and the problems $P(f, V(\Omega(t), \Gamma_D(t)))$ that are given by $\varphi_t$ such as $\Omega(t) = \varphi_t(\Omega)$ and $\Gamma_D(t) = \varphi_t(\Gamma_D)$.

[M1] For each $t \in [0, \epsilon)$, $\varphi_t$ is 1-1 mapping and has the inverse $\varphi_t^{-1}$. 

\[
[t \mapsto \varphi_t] \in C^1 ([0, \epsilon), W^{1,\infty} (\Omega_0, \mathbb{R}^d))
\]

**Theorem 6** If \( \bar{E}(v, \varphi) = \mathcal{E}(\varphi_\ast v, f, \varphi(\Omega)) \) satisfies the conditions [H1] - [H7], then it follows for all \( \varphi_t \) satisfying [M1] and [M2] that

\[
\frac{d}{dt} \mathcal{E}(u(t); f, \Omega(t))_{t=0} = -R_{\Omega}(u, \mu_{\varphi}) - \int_{\partial\Omega} f \cdot u (\mu_{\varphi} \cdot n) ds
\]

(9)

where \( \mu_{\varphi} = d\varphi_t/dt|_{t=0} \).

2 Application to shape optimization (Energy)

In Theorem 6, we get the shape sensitivity analysis of the potential energy \( \Omega \mapsto \mathcal{E}(u; f, \Omega) \). We introduce Azegami's method\(^1\)\(^4\) to find optimum shape \( \Omega^o \) assuming that the cost function is the energy, that is, find \( u^o \) and \( \Omega^o \) such that

\[
\mathcal{E}(f; u^o, \Omega^o) \leq \mathcal{E}(f; u(\bar{\Omega}), \bar{\Omega})
\]

for all domain \( \bar{\Omega} \) under some restrictions, where \( u(\bar{\Omega}) \) is the solution of \( P(f, V(\bar{\Omega}, \Gamma_D)) \).

2.1 Azegami's method

Let \( V(\Omega) \) be the subspace of \( W^{1,2}(\Omega, \mathbb{R}^d) \) and let \( b_{\Omega}(V, \mu) \) be a bilinear defined on \( V(\Omega) \times V(\Omega) \) satisfying the following conditions.

[A1] \( b_{\Omega}(V, \mu) \leq \alpha_8 \| V \|_{1,2,\Omega} \| \mu \|_{1,2,\Omega} \) for all \( V, \mu \in V(\Omega) \) with a constant \( \alpha_8 > 0 \).

[A2] \( b_{\Omega}(V, V) \geq \alpha_9 \| V \|_{1,2,\Omega}^2 \) for all \( V \in V(\Omega) \) with a constant \( \alpha_9 > 0 \).

Consider the variational problem \( \Pi(u, f, \Omega) \): Under the condition of Theorem 6, find \( V^o \in V(\Omega) \) such that

\[
b_{\Omega}(V^o, \mu) = R_{\Omega}(u, \mu) + \int_{\partial\Omega} f \cdot u (\mu \cdot n) ds \quad \text{for all } \mu \in V(\Omega)
\]

(10)

The mapping \( \varphi_t(x) = x + tV^o(x) \) from \( \Omega \) to \( \mathbb{R}^d \) is 1-1 if \( t \) is near 0. Unfortunately, \( [\mu \mapsto R_{\Omega}(u, \mu)] \) is linear functional on \( W^{1,\infty}(\Omega, \mathbb{R}^d) \), and is not on \( W^{1,2}(\Omega, \mathbb{R}^d) \). To extend it on \( W^{1,2}(\Omega, \mathbb{R}^d) \), we need slightly smoothness of \( u \).

**Proposition 7** If \( d = 2 \) or \( 3 \), and the solution \( u \) of \( P(f, V(\Omega, \Gamma_D)) \) is in \( W^{1,2p}(\Omega, \mathbb{R}^m) \), then there is a constant \( \alpha_{10} > 0 \) such that

\[
R_{\Omega}(u, \mu) + \int_{\partial\Omega} f \cdot u (\mu \cdot n) ds \leq \alpha_{10} \| \mu \|_{1,2,\Omega} \quad \text{for all } \mu \in W^{1,\infty}(\Omega, \mathbb{R}^d)
\]
If $u \in W^{1,2p}(\Omega, \mathbb{R}^m)$, then there is a unique solution $V^o \in W^{1,2}(\Omega, \mathbb{R}^d)$ of $\Pi(u, f, \Omega)$. If $V^o \in W^{1,\infty}(\Omega, \mathbb{R}^d)$, then $[t \mapsto \varphi_t(x) = x + tV^o(x)] \in C^1([0, \epsilon), W^{1,\infty}(\Omega, \mathbb{R}^d))$. So we can apply Theorem 6

$$\mathcal{E}(u(t); f, \Omega(t)) = \mathcal{E}(u; f, \Omega) - t \left\{ R_{\Omega}(u, V^o) + \int_{\Omega} f \cdot u (V^o \cdot u) \, ds \right\} + o(t)$$

$$\leq -t \alpha_9 \| V^o \|_{1,2,\Omega}^2 + o(t)$$

### 2.2 Numerical examples

We now check the method for the following simple two cases: We start from the initial shapes, $\Omega^0 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$, $\Gamma_D^0 = \{(\cos \theta, \sin \theta) : 0 < \theta < \pi\}$, $\Gamma_N^0 = \{(\cos \theta, \sin \theta) : \pi < \theta < 2\pi\}$.

![Optimization process under the conditions $|\Omega^i| = \pi$ and $\Gamma_D^i = \Gamma_D$](image)
When $\Omega^i$ was already obtained, find a shape $\Omega^{i+1}$ such that

$$\mathcal{E}(u^{i+1}; f, \Omega^{i+1}) < \mathcal{E}(u^i; f, \Omega^i)$$

$$\mathcal{E}(v; f, \Omega^i) = \int_{\Omega^i} \left\{ \frac{1}{2} |\nabla v|^2 - fv \right\} dx$$

$$\mathcal{E}(u^i; f, \Omega^i) = \min_{v \in V(\Omega^i, \Gamma_D^i)} \mathcal{E}(v; f, \Omega^i)$$

under the condition: $|\Omega^i| = \pi, i = 0, 1, \cdots$, where $|\Omega^i|$ denotes the area of $\Omega^i$. We find $\Omega^{i+1}$ by Azegami's method using the vector field $V^i$ calculated by (10) and for some $t_0 > 0$

$$\Omega^{i+1} = \{x + t_0 V^i(x) : x \in \Omega^i\}$$

In the first example, $f = 0.5$, $\Gamma_D$ is fixed and $\Gamma_N$ is changeable, so we use for $\mathcal{V}(\Omega)$ in (10), the following

$$\mathcal{V}(\Omega) = \{A \in W^{1,2}(\Omega, \mathbb{R}^d) : A = 0 \text{ on } \Gamma_D\}$$

We get the shapes in Fig.1 with finite element programming language FreeFem++ [3].

At the initial stage of the optimization, the stress concentrations at points $\gamma_1 = (1, 0), \gamma_2 = (-1, 0)$ are weaken by making a small circular hole near $\gamma_1$ and $\gamma_2$. The optimization is going to be divided to two parts $\Omega_D^o$ and $\Omega_N^o$ in which $\Omega_D^o$ ruled by $\Gamma_D$ and $\Omega_N^o$ ruled by Neumann boundary condition.

![Contour maps](image)

Figure 2: Optimization process under the conditions $|\Omega^2| = \pi$, in the condition to permit a change of $\Gamma_D^i$.

In the second example, find the optimum shape under the conditions $f = 0.5$, and that $\Gamma_D$ is changeable. The results are in Fig. 2, and we see that $|\Gamma_D^{i+1}| < |\Gamma_D^i|$ where
$|\Gamma_D^i|$ is the length of the curve $\Gamma_D^i$. It's natural because
\[
\min_{v \in V(\Omega, \Gamma_D^1)} \mathcal{E}(v; f, \Omega) \leq \min_{w \in V(\Omega, \Gamma_D^2)} \mathcal{E}(w; f, \Omega) \quad \text{if} \quad \Gamma_D^1 \subset \Gamma_D^2
\]

3 Application to shape optimization (general form)

We now consider the cost functional as follows: For the solution $u$ of $P(f, V(\Omega, \Gamma_D))$,
\[
\mathcal{J}^o(u, \Omega) = \int_{\Omega} g(u) \, dx
\]

3.1 Shape derivative of the solution

In this section, we limit $P(f, V(\Omega, \Gamma_D))$ to linear case, because the adjoint problem will be used. The following is the main theorem in the section.

Theorem 8  For any $\varphi \in C_0^\infty(\Omega; \mathbb{R}^m)$, let $u_\varphi$ be the solution of $P(\varphi, V(\Omega, \Gamma_D))$. Then we have
\[
\frac{d}{dt} \int_{\Omega(t)} u(t) \cdot \varphi \, dx \bigg|_{t=0} = \delta R_{\Omega}(u, u_\varphi; \mu_\varphi) + \int_{\partial\Omega} f \cdot u_\varphi(\mu_\varphi \cdot n) \, ds
\]
where $\delta R_{\Omega}(u, u_\varphi; \mu_\varphi) = \lim_{\epsilon \to 0} \epsilon^{-1} \{ R_{\Omega}(u + \epsilon u_\varphi; \mu_\varphi) - R_{\Omega}(u; \mu_\varphi) \}$. By the estimation
\[
\left| \delta R_{\Omega}(u, u_\varphi; \mu_\varphi) + \int_{\partial\Omega} f \cdot u_\varphi(\mu_\varphi \cdot n) \right| \leq C_3 \| f \|_{1,2,\Omega} \| \varphi \|_{0,2,\Omega} \| \mu_\varphi \|_{1,\infty,\Omega}
\]
and the result that $C_0^\infty(\Omega; \mathbb{R}^m)$ is dense in $W^{0,2}(\Omega; \mathbb{R}^m) = L^2(\Omega; \mathbb{R}^m)$, so $t^{-1}(u(t) \circ \varphi_t - u)$ converges weakly in $L^2(\Omega; \mathbb{R}^m)$ and
\[
\frac{d}{dt} \int_{\Omega(t)} u(t) \cdot \varphi \, dx \bigg|_{t=0} = \int_{\Omega} \left( \dot{u} - \mu_\varphi \cdot \nabla u \right) \varphi \, dx, \quad \dot{u} = \lim_{t \to 0} t^{-1}(u(t) \circ \varphi_t - u)
\]
See [8] for the proof.

If $[z \mapsto g(z)] \in W^{1,2}(\mathbb{R}^m; \mathbb{R})$, then $[u(t) \to g(u(t))] \in W^{1,2}(\Omega(t); \mathbb{R})$ and
\[
\int_{\Omega(t)} g(u(t)) \, dx - \int_{\Omega} g(u) \, dx = \int_{\Omega} \{ g(u(t) \circ \varphi_t) \kappa(\varphi_t) - g(u) \} \, dx
\]
\[
= \int_{\Omega} \{ (g(u(t) \circ \varphi_t) - g(u)) \kappa(\varphi_t) + g(u) (\kappa(\varphi_t) - 1) \} \, dx
\]
from which it follows that
\[
\frac{d}{dt} \int_{\Omega(t)} g(u(t)) \, dx \bigg|_{t=0} = \int_{\Omega} \{ \nabla_z g(u) \dot{u} + g(u) \text{div} \mu_\varphi \} \, dx
\]
\[
= \int_{\Omega} \{ \nabla_z g(u) \dot{u} - \nabla_z g(u) (\mu_\varphi \cdot \nabla u) \} \, dx + \int_{\partial\Omega} g(u)(\mu_\varphi \cdot n) \, ds
\]
\[
= \int_{\Omega} \nabla_z g(u) u' \, dx + \int_{\partial\Omega} g(u)(\mu_\varphi \cdot n) \, ds
\]
Proposition 9  Let $u_{g}$ be the solution of $P(\nabla_{z}g(u), V(\Omega, \Gamma_{D}))$, then
\[\int_{\Omega} \nabla_{z}g(u)u'dx = \delta R_{\Omega}(u, u_{g}; \mu_{\varphi}) + \int_{\partial\Omega} f \cdot u_{g}(\mu_{\varphi} \cdot n)ds\]
which implies
\[\frac{d}{dt} \int_{\Omega(t)} g(u(t))dx |_{t=0} = \delta R_{\Omega}(u, u_{g}; \mu_{\varphi}) + \int_{\partial\Omega} (f \cdot u_{g} + g(u))(\mu_{\varphi} \cdot n)ds \quad (11)\]

Acknowledgement
This work was supported by the Grant-in-Aid for Scientific Research (B)#1934007 and Research(C)#23540258, Japan. I would like to thank Prof. M.Kimura who made enormous contribution to prove Theorem 1 and to thank Prof. H.Azegami who provided great idea to solve shape optimization.

References