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Blow-up set for a semilinear heat equation with exponential nonlinearity

東北大学大学院理学研究科数学専攻　藤嶋 陽平 (Yohei Fujishima)[1]
Mathematical Institute, Tohoku University

1 Introduction

We consider the blow-up problem for a semilinear heat equation

$$
(1.1)\qquad \begin{cases} \partial_t u = \epsilon\Delta u + e^u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \varphi_\epsilon(x) \geq 0\,(\not\equiv 0), & x \in \Omega, \end{cases}
$$

where $\partial_t = \partial/\partial t$, $\epsilon > 0$, $N \geq 1$, Ω is a bounded domain in $\mathbf{R}^N$, and the initial function φ_ϵ is a nonnegative bounded continuous function in $\overline{\Omega}$. Let T_ϵ be the maximal existence time of the unique classical solution u_ϵ of problem (1.1). If $T_\epsilon < \infty$, then we define the set B_ϵ by

$$
\begin{aligned} B_\epsilon := \Big\{ x \in \overline{\Omega} : &\text{ there exists a sequence } \{(x_n, t_n)\} \subset \overline{\Omega} \times (0, T_\epsilon) \\ &\text{such that } \lim_{n\to\infty}(x_n, t_n) = (x, T_\epsilon) \text{ and } \lim_{n\to\infty}|u(x_n, t_n)| = \infty \Big\}. \end{aligned}
$$

We call T_ϵ and B_ϵ the blow-up time and the blow-up set, respectively. We remark that, if ϵ is sufficiently small, then $T_\epsilon < \infty$ and the solution u_ϵ blows up in a finite time.

The blow-up problem for a semilinear heat equation has been studied by many mathematicians. We refer to a survey [6] and references therein. Among others, the author of this paper and Ishige in [3] studied the blow-up problem for

$$
(1.2)\qquad \begin{cases} \partial_t u = \epsilon\Delta u + u^p, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \varphi_\epsilon(x) \geq 0\,(\not\equiv 0), & x \in \Omega, \end{cases}
$$

where $p > 1$. Consider a family of initial functions $\{\varphi_\epsilon\}$ satisfying

$$
\liminf_{\epsilon\to 0} \|\varphi_\epsilon\|_{L^\infty(\Omega)} > 0, \quad \limsup_{\epsilon\to 0} \|\varphi_\epsilon\|_{L^\infty(\Omega)} < \infty, \quad \varphi_\epsilon = 0 \text{ on } \partial\Omega,
$$

and

$$
\limsup_{\epsilon\to 0} \Big\{ |\varphi_\epsilon(x) - \varphi_\epsilon(y)| : x, y \in \overline{\Omega}, |x - y| \leq \epsilon^{1/2-A} \Big\} = 0 \text{ for some } A > 0.
$$

[1]e-mail: sa8m31@math.tohoku.ac.jp

They studied the location of the blow-up set for problem (1.2), and proved the following: Let u_ϵ be the solution of (1.2) satisfying

$$\limsup_{\epsilon\to 0}\sup_{0<t<T_\epsilon}(T_\epsilon - t)^{1/(p-1)}\|u_\epsilon(t)\|_{L^\infty(\Omega)} < \infty. \tag{1.3}$$

Then, for any $\delta > 0$, there holds

$$B_\epsilon \subset \{x \in \overline{\Omega} : \varphi_\epsilon(x) \geq \|\varphi_\epsilon\|_{L^\infty(\Omega)} - \delta\} \tag{1.4}$$

for all sufficiently small $\epsilon > 0$. Since $\delta > 0$ is arbitrary, we see from (1.4) that the solution u_ϵ of problem (1.2) blows up only near the maximum points of the initial function φ_ϵ if $\epsilon > 0$ is sufficiently small.

Furthermore, if the initial function is independent of ϵ, then we can obtain more precise information on the location of the blow-up set. In fact, the author of this paper and Ishige in [4] proved the following: Let $\varphi \in C^2(\Omega) \cap C(\overline{\Omega})$ be a nonnegative function such that $\varphi \not\equiv 0$ and u_ϵ be the solution of (1.2) satisfying (1.3) with the initial function replaced by φ. Assume that there exist two points α, $\beta \in \Omega$ such that $|\Delta\varphi(\alpha)| < |\Delta\varphi(\beta)|$. Then there exists a positive constant δ_* such that

$$B_\epsilon \cap \{y \in \overline{\Omega} : |y - \beta| < \delta_*\} = \emptyset$$

for all sufficiently small $\epsilon > 0$. This result implies that the location of the blow-up set for problem (1.2) depends on the mean curvature of the graph of the initial function on its maximum points.

In this paper we consider a semilinear heat equation having exponential nonlinearity (1.1), and study the location of the blow-up set B_ϵ of the solution u_ϵ. In particular, we refine the argument of [3], and characterize the location of the blow-up set of u_ϵ of (1.1) by using the level sets of the initial function φ_ϵ.

Before stating our main results, we introduce some notation. For $x \in \mathbf{R}^N$ and $r > 0$, we put $B(x,r) = \{y \in \mathbf{R}^N : |y - x| < r\}$. Let

$$BC_+(\overline{\Omega}) := \{f \in L^\infty(\Omega) : f \text{ is a nonnegative continuous function on } \overline{\Omega}\},$$
$$BUC_+(\overline{\Omega}) := \{f \in L^\infty(\Omega) : f \text{ is a nonnegative uniformly continuous function on } \overline{\Omega}\}.$$

For any $\epsilon > 0$, $A > 0$, $\eta > 0$, and $\phi \in C(\overline{\Omega})$, put

$$\omega(\epsilon, A, \phi) := \sup\left\{|\phi(x) - \phi(y)| : x, y \in \overline{\Omega},\ |x - y| \leq A\epsilon^{1/2}\right\},$$
$$M(\phi, \eta) := \left\{x \in \overline{\Omega} : \phi(x) \geq \|\phi\|_{L^\infty(\Omega)} - \eta\right\}.$$

We are ready to state our main result.

Theorem 1.1 *Let $N \geq 1$, $\epsilon_0 > 0$, Ω be a domain in $\mathbf{R}^N$, and $\{\varphi_\epsilon\}_{0<\epsilon<\epsilon_0} \subset BC_+(\overline{\Omega})$ be a family of initial functions satisfying*

$$\inf_{0<\epsilon<\epsilon_0}\|\varphi_\epsilon\|_{L^\infty(\Omega)} > 0, \qquad \sup_{0<\epsilon<\epsilon_0}\|\varphi_\epsilon\|_{L^\infty(\Omega)} < \infty. \tag{1.5}$$

Assume the followings:

- *there exists a positive constant η such that $M(\varphi_\epsilon, \eta) \subset \Omega$ for all $\epsilon \in (0, \epsilon_0)$;*
- *there exists a family of positive constants $\{A_\epsilon\}_{0<\epsilon<\epsilon_0}$ such that*

$$\lim_{\epsilon \to 0} A_\epsilon = \infty, \quad \lim_{\epsilon \to 0} \omega(\epsilon, A_\epsilon, \varphi_\epsilon) = 0.$$

For any $\epsilon \in (0, \epsilon_0)$, let u_ϵ be the solution of (1.1), *and assume that*

$$\sup_{0<\epsilon<\epsilon_0} \sup_{0<t<T_\epsilon} \left[\log(T_\epsilon - t) + \|u_\epsilon(t)\|_{L^\infty(\Omega)}\right] < \infty. \tag{1.6}$$

Then, for any $\delta > 0$, there exists a positive constant ϵ_δ such that

$$B_\epsilon \subset M(\varphi_\epsilon, \delta) = \{x \in \overline{\Omega} : \varphi_\epsilon(x) \geq \|\varphi_\epsilon\|_{L^\infty(\Omega)} - \delta\}$$

for all $\epsilon \in (0, \epsilon_\delta)$.

The following corollary immediately follows from Theorem 1.1.

Corollary 1.1 *Let $N \geq 1$, $\epsilon_0 > 0$, Ω be a domain in $\mathbf{R}^N$, and $\varphi \in BUC_+(\overline{\Omega})$ satisfying $M(\varphi, \eta) \subset \Omega$ for some $\eta > 0$. For any $\epsilon \in (0, \epsilon_0)$, let u_ϵ be the solution of*

$$\begin{cases} \partial_t u = \epsilon \Delta u + e^u, & x \in \Omega, \ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \ t > 0 \quad \text{if} \quad \partial\Omega \neq \emptyset, \\ u(x,0) = \varphi(x), & x \in \Omega, \end{cases}$$

and assume that (1.6) *holds. Then, for any $\delta > 0$, there exists a positive constant ϵ_δ such that*

$$B_\epsilon \subset M(\varphi, \delta) = \{x \in \overline{\Omega} : \varphi(x) \geq \|\varphi\|_{L^\infty(\Omega)} - \delta\}$$

for all $\epsilon \in (0, \epsilon_\delta)$.

By Theorem 1.1 and Corollary 1.1 we see that the location of the blow-up set for problem (1.1) is characterized by using the level sets of the initial function.

For the proof of Theorem 1.1, we refine the argument of [3], which is a modification of [7], and construct a supersolution $\overline{u}_\epsilon$ with the following properties:

(a) $\overline{u}_\epsilon$ exists in $\Omega \times (0, T_\epsilon)$;

(b) For any $\delta > 0$, $\overline{u}_\epsilon$ is bounded in $M(\varphi_\epsilon, \delta) \times (0, T_\epsilon)$ if $\epsilon > 0$ is sufficiently small.

However, the argument of [3] heavily depends on the nonlinearity, and it seems difficult to apply the argument to problem (1.1), which has exponential nonlinearity, directly. In order to generalize the argument of [3], we consider the blow-up problem for a generalized semilinear heat equation

$$\begin{cases} \partial_t u = \epsilon \Delta u + f(u), & x \in \Omega, \ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \ t > 0 \quad \text{if} \quad \partial\Omega \neq \emptyset, \\ u(x,0) = \varphi_\epsilon(x) \geq 0, & x \in \Omega. \end{cases} \tag{1.7}$$

We find a sufficient condition of f such that a supersolution satisfying properties (a) and (b) can be constructed for problem (1.7), and construct a supersolution $\overline{u}_\epsilon$ for problem (1.1). Thus we prove Theorem 1.1.

2 Outline of the proof of Theorem 1.1

In this section we explain the outline of the proof of Theorem 1.1. In order to prove Theorem 1.1, we construct a supersolution $\overline{u}_\epsilon$ for some semilinear heat equation with the following properties:

(a) $\overline{u}_\epsilon$ is a smooth function defined in $\Omega \times (0, T_\epsilon)$;

(b) For any $\delta > 0$, there holds

$$\sup \{\overline{u}_\epsilon(x,t) : (x,t) \in [\Omega \setminus M(\varphi_\epsilon, \delta)] \times (0, T_\epsilon)\} < \infty$$

for all sufficiently small $\epsilon > 0$.

By using above properties (a) and (b) and the comparison principle, for any $x_\epsilon \notin M(\varphi_\epsilon, \delta)$, we can find a constant $r_\epsilon > 0$ such that

$$\limsup_{t \nearrow T_\epsilon} \sup_{x \in B(x_\epsilon, r_\epsilon)} u_\epsilon(x,t) \le \limsup_{t \nearrow T_\epsilon} \sup_{x \in B(x_\epsilon, r_\epsilon)} \overline{u}_\epsilon(x,t) < \infty.$$

This together with the definition of the blow-up set B_ϵ implies that $x_\epsilon \notin B_\epsilon$, and we have

$$B_\epsilon \subset M(\varphi_\epsilon, \delta) = \left\{ x \in \overline{\Omega} : \varphi_\epsilon(x) \ge \|\varphi_\epsilon\|_{L^\infty(\Omega)} - \delta \right\}$$

for all sufficiently small $\epsilon > 0$.

In order to construct a supersolution $\overline{u}_\epsilon$, we impose the following condition (F) on the nonlinear term f:

$$\text{(F)} \qquad \begin{cases} f \in C^2((0,\infty)) \cap C^1([0,\infty)), \\ f(s),\ f'(s),\ f''(s) > 0 \text{ for all } s > 0, \\ \displaystyle\int_1^\infty \frac{ds}{f(s)} < \infty, \\ \displaystyle\limsup_{u \to \infty} f'(u) F(u) < \infty, \\ \displaystyle\limsup_{u \to \infty} \frac{f(u+M)}{f(u)} < \infty \text{ for some } M > 0, \end{cases}$$

where

$$F(u) := \int_u^\infty \frac{ds}{f(s)} < \infty.$$

Under condition (F), we can generalize the argument of [3], and construct a supersolution $\overline{u}_\epsilon$ satisfying properties (a) and (b).

Remark 2.1 *The following functions satisfy condition* (F):

- $f(u) = (u + \lambda)^p$ *with* $p > 1$ *and* $\lambda \ge 0$;
- $f(u) = u^p + u^q$ *with* $p > q > 1$;

- $f(u) = (u+1)^p[\log(u+1)]^q$ *with* $p > 1$ *and* $q > 1$;

- $f(u) = e^{\alpha u}$ *with* $\alpha > 0$.

In the rest of this section we give some comments on condition (F). The third condition $F(1) < \infty$ is a necessary condition for a finite time blow-up of the solution. If $F(1) = \infty$, then the solution for the following ordinary differential equation

$$\begin{cases} \partial_t \zeta = f(\zeta), & t > 0, \\ \zeta(0) = \lambda > 0, \end{cases}$$

exists globally in time. Therefore, if $F(1) = \infty$, by the comparison principle we see that the solution of (1.1) does not blow-up in a finite time.

Furthermore, we can give examples of f which do not satisfy the fourth condition and the fifth condition. The function define by $f(u) = e^{u^2}$ satisfies

$$\lim_{u \to \infty} \frac{f(u+M)}{f(u)} = \infty \quad \text{for any} \quad M > 0,$$

and f does not satisfies (F).

On the other hand, the function defined by $f(u) = (u+1)[\log(u+1)]^\alpha$ $(\alpha > 1)$ satisfies

$$\lim_{u \to \infty} f'(u)F(u) = \infty,$$

and f does not satisfy (F). However, if $\alpha = 2$, then a regional blow-up may occur even if the initial function has the only one maximum point (see [5]). We suspect that Theorem 1.1 does not hold for problem (1.1) with $f(u) = (u+1)[\log(u+1)]^2$.

3 Blow-up problem for generalized semilinear heat equation

In this section we consider the blow-up problem for (1.7), and generalize the result of Section 1. Using the argument of Section 2, we have the following theorem.

Theorem 3.1 *Let* $N \geq 1$, $\epsilon_0 > 0$, Ω *be a domain in* $\mathbf{R}^N$, *and* $\{\varphi_\epsilon\}_{0<\epsilon<\epsilon_0} \subset BC_+(\overline{\Omega})$ *be a family of initial functions satisfying* (1.5). *Assume the followings*:

- *there exists a positive constant* η *such that* $M(\varphi_\epsilon, \eta) \subset \Omega$ *for all* $\epsilon \in (0, \epsilon_0)$;

- *there exists a family of positive constants* $\{A_\epsilon\}_{0<\epsilon<\epsilon_0}$ *such that*

$$\lim_{\epsilon \to 0} A_\epsilon = \infty, \quad \lim_{\epsilon \to 0} \omega(\epsilon, A_\epsilon, \varphi_\epsilon) = 0.$$

Let f *be a function satisfying* (F). *For any* $\epsilon \in (0, \epsilon_0)$, *let* u_ϵ *be the solution of* (1.7). *Furthermore assume that there exists a constant* $c_* > 0$ *such that*

$$\|u_\epsilon(t)\|_{L^\infty(\Omega)} \leq F^{-1}(c_*(T_\epsilon - t)) \tag{3.1}$$

for all $t \in (0, T_\epsilon)$ *and* $\epsilon \in (0, \epsilon_0)$, *where* F^{-1} *is the inverse function of* F. *Then, for any* $\delta > 0$, *there exists a positive constant* ϵ_δ *such that*

$$B_\epsilon \subset M(\varphi, \delta) = \{x \in \overline{\Omega} : \varphi(x) \geq \|\varphi\|_{L^\infty(\Omega)} - \delta\}$$

for all $\epsilon \in (0, \epsilon_\delta)$.

Theorem 3.1 is a generalization of Theorem 1.1. In fact, it is easy to show that the functions $f(u) = u^p$ $(p > 1)$ and $f(u) = e^u$ satisfy (F). Furthermore, (3.1) is equivalent to (1.3) if $f(u) = u^p$ $(p > 1)$ and (1.6) if $f(u) = e^u$.

Remark 3.1 (i) *Let* f *and* g *be functions satisfying* (F). *Then, for any* $\alpha > 0$ *and* $\beta > 0$, *the function* $\alpha f + \beta g$ *also satisfies* (F).
(ii) *Let* $\epsilon_0 > 0$ *and* $\{\varphi_\epsilon\}_{0<\epsilon<\epsilon_0} \subset BC_+(\overline{\Omega}) \cap C^1(\Omega) \setminus \{0\}$ *satisfying*

$$\lim_{\epsilon \to 0} \epsilon^{1/2} \|\nabla \varphi_\epsilon\|_{L^\infty(\Omega)} = 0.$$

Put $A_\epsilon = [\epsilon^{1/2} \|\nabla \varphi_\epsilon\|_{L^\infty(\Omega)}]^{-1/2}$. *Then* $\lim_{\epsilon \to 0} A_\epsilon = \infty$ *and there holds*

$$|\varphi_\epsilon(x) - \varphi_\epsilon(y)| \leq \|\nabla \varphi_\epsilon\|_{L^\infty(\Omega)} |x - y| \leq \epsilon^{1/2} \|\nabla \varphi_\epsilon\|_{L^\infty(\Omega)} \cdot A_\epsilon = A_\epsilon^{-1} \to 0 \quad \textit{as} \quad \epsilon \to 0$$

for all $x, y \in \overline{\Omega}$ *with* $|x - y| \leq A_\epsilon \epsilon^{1/2}$, *that is,* $\{A_\epsilon\}_{0<\epsilon<\epsilon_0}$ *and* $\{\varphi_\epsilon\}_{0<\epsilon<\epsilon_0}$ *satisfy* (1.5).
(iii) *Let* $\epsilon_0 > 0$, Ω *be a* C^2 *smooth domain,* f *be a function satisfying* (F), *and* $\{\varphi_\epsilon\}_{0<\epsilon<\epsilon_0} \subset C^2(\Omega)$ *be a family of nonnegative functions satisfying* (1.5), *and* $\varphi_\epsilon = 0$ *on* $\partial\Omega$. *Furthermore assume that there exist a function* $F \in C^2([0, \infty])$ *and a positive constant* c *such that*

$$\begin{cases} F(s) > 0, \quad F'(s) \geq 0, \quad F''(s) \geq 0 \quad \text{in} \quad s \in [0, \infty), \\ f'(s)F(s) - f(s)F'(s) \geq cF'(s)F(s) \quad \text{in} \quad s \in (0, \infty), \\ \displaystyle\int_1^\infty \frac{ds}{F(s)} < \infty, \end{cases}$$

which was introduced by Friedman and McLeod in [1]. *If there holds either*

(a) $f(0) = 0$ *and there exists a constant* $\delta \in (0, 1)$ *such that* $\epsilon \Delta \varphi_\epsilon + f(\varphi_\epsilon) \geq \delta f(\varphi_\epsilon)$ *in* Ω *for all sufficiently small* $\epsilon > 0$

or

(b) *there exist a subdomain* $\Omega' \subset\subset \Omega$ *and positive constants* C *and* δ *such that*

$$u_\epsilon(x, t) \leq C \ \textit{ in } \ [\Omega \setminus \overline{\Omega'}] \times (0, T_\epsilon), \quad \epsilon \Delta \varphi_\epsilon + f(\varphi_\epsilon) \geq \delta \ \textit{ in } \ \Omega,$$

for all sufficiently small $\epsilon > 0$,

then we can prove the uniform blow-up estimate (3.1) *by using the same argument as in* [1]. *Condition* (b) *is verified if* Ω *is a bounded convex domain and there exist positive constants* C *and* δ *such that*

$$|\nabla^2 \varphi_\epsilon| \leq C \ \textit{ near } \ \partial\Omega, \quad \partial \varphi_\epsilon / \partial \nu \leq -\delta \ \textit{ on } \ \partial\Omega, \quad \epsilon \Delta \varphi_\epsilon + f(\varphi_\epsilon) \geq \delta \ \textit{ in } \ \Omega,$$

for all sufficiently small $\epsilon > 0$, *where* ν *is the outer normal unit vector to* $\partial\Omega$.

4 Remarks

In this section we discuss the application of Theorem 1.1, and give an extension of Theorem 1.1 for more general superlinear heat equations.

Consider

$$(4.1)\qquad \begin{cases} \partial_t v = \Delta v + e^v, & x \in \mathbf{R}^N, \ t > 0, \\ v(x,0) = \phi(x) \geq 0, & x \in \mathbf{R}^N, \end{cases}$$

where $N \geq 1$ and $\phi \in BC_+(\mathbf{R}^N)$. Let T be the blow-up time of v, and assume that there exists a positive constant C such that

$$(4.2)\qquad \log(T-t) + \|v(t)\|_{L^\infty(\Omega)} \leq C$$

for all $t \in (0,T)$. Here we remark that the solution of (4.1) blows up in a finite time since the initial function ϕ is nonnegative. Let $\epsilon > 0$ be a sufficiently small constant and put

$$\begin{aligned} u_\epsilon(x,t) &= \log\epsilon + v(x, T-\epsilon+\epsilon t) \quad \text{in} \quad \mathbf{R}^N \times [0,1), \\ \varphi_\epsilon(x) &= u_\epsilon(x,0) = \log\epsilon + v(x, T-\epsilon) \quad \text{in} \quad \mathbf{R}^N. \end{aligned}$$

Then u_ϵ blows up at $t=1$ and satisfies

$$\begin{cases} \partial_t u_\epsilon = \epsilon\Delta u_\epsilon + e^{u_\epsilon} & \text{in} \quad \mathbf{R}^N \times (0,1), \\ u_\epsilon(x,0) = \varphi_\epsilon(x) & \text{in} \quad \mathbf{R}^N, \end{cases}$$

and by (4.2) we obtain

$$u_\epsilon(x,t) \leq \log\epsilon + \|v(t)\|_{L^\infty(\mathbf{R}^N)} \leq \log\epsilon + [C - \log(T-(T-\epsilon+\epsilon t))] \leq C - \log(1-t)$$

for all $(x,t) \in \mathbf{R}^N \times (0,1)$, which implies the uniform blow-up estimate for u_ϵ. Therefore, under suitable assumptions on ϕ and ϵ, we can apply Theorem 1.1 to obtain the location of the blow-up set of v. Similar argument was employed in [4] for a semilinear heat equation with power nonlinearity, and the author of this paper and Ishige succeeded in obtaining the location of the blow-up set (see also Remark 1.2 (ii) in [3]).

On the other hand, the above argument can be applied to study the location of the blow-up set even if the equation does not have a self-similarity. For this purpose, we give an extension of Theorem 1.1 for more general superlinear heat equation whose nonlinear term depends on ϵ. Consider

$$(4.3)\qquad \begin{cases} \partial_t u = \epsilon\Delta u + f_\epsilon(u), & x \in \Omega, \ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \ t > 0 \quad \text{if} \quad \partial\Omega \neq \emptyset, \\ u(x,0) = \varphi_\epsilon(x) \geq 0 \, (\not\equiv 0), & x \in \Omega, \end{cases}$$

where $\epsilon_0 > 0$, $\{\varphi_\epsilon\}_{0<\epsilon<\epsilon_0} \subset BC_+(\overline{\Omega})$ satisfies (1.5), and $\{f_\epsilon\}_{0<\epsilon<\epsilon_0} \subset C^1([0,\infty)) \cap C^2((0,\infty))$ satisfies

$$(\mathrm{F}_\epsilon) \qquad \begin{cases} \liminf_{\epsilon\to 0} f_\epsilon(s),\ \liminf_{\epsilon\to 0} f_\epsilon'(s),\ \liminf_{\epsilon\to 0} f_\epsilon''(s) > 0 \quad \text{for all} \quad s \in (0,\infty), \\ \limsup_{\epsilon\to 0} f_\epsilon(s),\ \limsup_{\epsilon\to 0} f_\epsilon'(s),\ \limsup_{\epsilon\to 0} f_\epsilon''(s) < \infty \quad \text{for all} \quad s \in (0,\infty), \\ \limsup_{\epsilon\to 0} F_{f_\epsilon}(s) < \infty \quad \text{for all} \quad s > 0, \\ \limsup_{u\to\infty} \sup_{0<\epsilon<\epsilon_0} f_\epsilon(u) F_{f_\epsilon}(u) < \infty, \\ \limsup_{u\to\infty} \sup_{0<\epsilon<\epsilon_0} \dfrac{f_\epsilon(u+M)}{f_\epsilon(u)} < \infty \quad \text{for some} \quad M > 0. \end{cases}$$

Then we can prove the following theorem with a slight modification of the argument of this paper, which is an extension of Theorem 1.1.

Theorem 4.1 *Let $N \geq 1$, $\epsilon_0 > 0$, Ω be a domain in $\mathbf{R}^N$, and $\{\varphi_\epsilon\}_{0<\epsilon<\epsilon_0} \subset BC_+(\overline{\Omega})$ satisfying* (1.5). *Let $\{f_\epsilon\}_{0<\epsilon<\epsilon_0}$ be a family of functions satisfying* (F_ϵ). *For any $\epsilon \in (0,\epsilon_0)$, let u_ϵ be the solution of* (4.3), *and assume that there exists a positive constant c_* such that*

$$\|u_\epsilon(t)\|_{L^\infty(\Omega)} \leq F_{f_\epsilon}^{-1}(c_*(T_\epsilon - t))$$

for all $t \in (0,T_\epsilon)$ and all $\epsilon \in (0,\epsilon_0)$. Then, for any $\delta > 0$, there holds

$$B_\epsilon \subset M(\varphi_\epsilon,\delta) = \left\{x \in \overline{\Omega} : \varphi_\epsilon(x) \geq \|\varphi_\epsilon\|_{L^\infty(\Omega)} - \delta\right\}$$

for all sufficiently small $\epsilon > 0$.

We apply Theorem 4.1 to the blow-up problem for a semilinear heat equation

$$(4.4) \qquad \begin{cases} \partial_t v = \Delta v + v^p + v^q, & x \in \Omega,\ t > 0, \\ v(x,t) = 0, & x \in \partial\Omega,\ t > 0 \quad \text{if} \quad \partial\Omega \neq \emptyset, \\ v(x,0) = \phi(x) \geq 0, & x \in \Omega, \end{cases}$$

where $1 < q < p$ and $\phi \in BC_+(\overline{\Omega})$. Assume that the solution of (4.4) blows up at some time $t = T < \infty$ and that there exists a positive constant C such that

$$(4.5) \qquad \|v(t)\|_{L^\infty(\Omega)} \leq C(T-t)^{-\frac{1}{p-1}}$$

for all $t \in (0,T)$. Let $\epsilon > 0$ be a sufficiently small constant and put

$$u_\epsilon(x,t) = \epsilon^{\frac{1}{p-1}} v(x, T-\epsilon+\epsilon t) \quad \text{in} \quad \mathbf{R}^N \times [0,1),$$
$$\varphi_\epsilon(x) = u_\epsilon(x,0) = \epsilon^{\frac{1}{p-1}} v(x,T-\epsilon) \quad \text{in} \quad \mathbf{R}^N,$$
$$f_\epsilon(s) = s^p + \epsilon^{\frac{p-q}{p-1}} s^q \quad \text{in} \quad (0,\infty).$$

Then $\{f_\epsilon\}_{0<\epsilon<\epsilon_0}$ satisfies (F_ϵ) for some positive constant ϵ_0, and u_ϵ satisfies

$$\begin{cases} \partial_t u_\epsilon = \epsilon\Delta u_\epsilon + f_\epsilon(u_\epsilon) & \text{in} \quad \Omega\times(0,1), \\ u_\epsilon(x,t) = 0 & \text{on} \quad \partial\Omega\times(0,1), \\ u_\epsilon(x,0) = \varphi_\epsilon(x) & \text{in} \quad \Omega. \end{cases}$$

Furthermore, by (4.5) we have

$$\|u_\epsilon(t)\|_{L^\infty(\Omega)} \le C(1-t)^{-\frac{1}{p-1}}$$

for all $t \in (0,1)$. This yields

$$\|u_\epsilon(t)\|_{L^\infty(\Omega)} \le F_{f_\epsilon}^{-1}(c_*(1-t))$$

for some constant $c_* > 0$, provided that $\epsilon > 0$ is sufficiently small. Therefore, under suitable assumptions, we can apply Theorem 4.1 to problem (4.4), and obtain the location of the blow-up set of v by using the maximum points of the solution just before the blow-up time.

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