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Blow-up set for a semilinear heat equation with exponential nonlinearity

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1 Introduction

We consider the blow-up problem for a semilinear heat equation

\[
\begin{cases}
\partial_t u = \epsilon \Delta u + e^u, & x \in \Omega, \ t > 0, \\
u(x, t) = 0, & x \in \partial \Omega, \ t > 0, \\
u(x, 0) = \varphi_\epsilon(x) \geq 0 (\not\equiv 0), & x \in \Omega,
\end{cases}
\]

where $\partial_t = \partial/\partial t$, $\epsilon > 0$, $N \geq 1$, $\Omega$ is a bounded domain in $\mathbb{R}^N$, and the initial function $\varphi_\epsilon$ is a nonnegative bounded continuous function in $\Omega$. Let $T_\epsilon$ be the maximal existence time of the unique classical solution $u_\epsilon$ of problem (1.1). If $T_\epsilon < \infty$, then we define the set $B_\epsilon$ by

\[
B_\epsilon := \{x \in \Omega : \text{there exists a sequence } \{(x_n, t_n)\} \subset \Omega \times (0, T_\epsilon) \text{ such that } \lim_{n \to \infty} (x_n, t_n) = (x, T_\epsilon) \text{ and } \lim_{n \to \infty} |u(x_n, t_n)| = \infty \}.
\]

We call $T_\epsilon$ and $B_\epsilon$ the blow-up time and the blow-up set, respectively. We remark that, if $\epsilon$ is sufficiently small, then $T_\epsilon < \infty$ and the solution $u_\epsilon$ blows up in a finite time.

The blow-up problem for a semilinear heat equation has been studied by many mathematicians. We refer to a survey [6] and references therein. Among others, the author of this paper and Ishige in [3] studied the blow-up problem for

\[
\begin{cases}
\partial_t u = \epsilon \Delta u + u^p, & x \in \Omega, \ t > 0, \\
u(x, t) = 0, & x \in \partial \Omega, \ t > 0, \\
u(x, 0) = \varphi_\epsilon(x) \geq 0 (\not\equiv 0), & x \in \Omega,
\end{cases}
\]

where $p > 1$. Consider a family of initial functions $\{\varphi_\epsilon\}$ satisfying

\[
\liminf_{\epsilon \to 0} \|\varphi_\epsilon\|_{L^\infty(\Omega)} > 0, \quad \limsup_{\epsilon \to 0} \|\varphi_\epsilon\|_{L^\infty(\Omega)} < \infty, \quad \varphi_\epsilon = 0 \text{ on } \partial \Omega,
\]

and

\[
\limsup_{\epsilon \to 0} \left\{|\varphi_\epsilon(x) - \varphi_\epsilon(y)| : x, y \in \Omega, |x - y| \leq \epsilon^{1/2-A}\right\} = 0 \text{ for some } A > 0.
\]

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They studied the location of the blow-up set for problem (1.2), and proved the following: Let $u_\epsilon$ be the solution of (1.2) satisfying
\begin{equation}
\limsup_{\epsilon \to 0} \sup_{0 < t < T_\epsilon} (T_\epsilon - t)^{1/(p-1)} \|u_\epsilon(t)\|_{L^\infty(\Omega)} < \infty.
\end{equation}
Then, for any $\delta > 0$, there holds
\begin{equation}
B_\epsilon \subset \{ x \in \overline{\Omega} : \varphi_\epsilon(x) \geq \|\varphi_\epsilon\|_{L^\infty(\Omega)} - \delta \}
\end{equation}
for all sufficiently small $\epsilon > 0$. Since $\delta > 0$ is arbitrary, we see from (1.4) that the solution $u_\epsilon$ of problem (1.2) blows up only near the maximum points of the initial function $\varphi_\epsilon$ if $\epsilon > 0$ is sufficiently small.

Furthermore, if the initial function is independent of $\epsilon$, then we can obtain more precise information on the location of the blow-up set. In fact, the author of this paper and Ishige in [4] proved the following: Let $\varphi \in C^2(\Omega) \cap C(\overline{\Omega})$ be a nonnegative function such that $\varphi \not\equiv 0$ and $u_\epsilon$ be the solution of (1.2) satisfying (1.3) with the initial function replaced by $\varphi$. Assume that there exist two points $\alpha, \beta \in \Omega$ such that $|\Delta \varphi(\alpha)| < |\Delta \varphi(\beta)|$. Then there exists a positive constant $\delta^*$ such that
\begin{equation}
B_\epsilon \cap \{ y \in \overline{\Omega} : |y - \beta| < \delta^* \} = \emptyset
\end{equation}
for all sufficiently small $\epsilon > 0$. This result implies that the location of the blow-up set for problem (1.2) depends on the mean curvature of the graph of the initial function on its maximum points.

In this paper, we consider a semilinear heat equation having exponential nonlinearity (1.1), and study the location of the blow-up set $B_\epsilon$ of the solution $u_\epsilon$. In particular, we refine the argument of [3], and characterize the location of the blow-up set of $u_\epsilon$ of (1.1) by using the level sets of the initial function $\varphi_\epsilon$.

Before stating our main results, we introduce some notation. For $x \in \mathbb{R}^N$ and $r > 0$, we put $B(x, r) = \{ y \in \mathbb{R}^N : |y - x| < r \}$. Let
\begin{align*}
BC_+ (\overline{\Omega}) &:= \{ f \in L^\infty(\Omega) : f \text{ is a nonnegative continuous function on } \overline{\Omega} \}, \\
BUC_+ (\overline{\Omega}) &:= \{ f \in L^\infty(\Omega) : f \text{ is a nonnegative uniformly continuous function on } \overline{\Omega} \}.
\end{align*}
For any $\epsilon > 0$, $A > 0$, $\eta > 0$, and $\phi \in C(\overline{\Omega})$, put
\begin{align*}
\omega(\epsilon, A, \phi) &:= \sup \left\{ |\phi(x) - \phi(y)| : x, y \in \overline{\Omega}, \ |x - y| \leq A\epsilon^{1/2} \right\}, \\
M(\phi, \eta) &:= \{ x \in \overline{\Omega} : \phi(x) \geq \|\phi\|_{L^\infty(\Omega)} - \eta \}.
\end{align*}
We are ready to state our main result.

**Theorem 1.1** Let $N \geq 1$, $\epsilon_0 > 0$, $\Omega$ be a domain in $\mathbb{R}^N$, and $\{ \varphi_\epsilon \}_{0 < \epsilon < \epsilon_0} \subset BC_+ (\overline{\Omega})$ be a family of initial functions satisfying
\begin{equation}
\inf_{0 < \epsilon < \epsilon_0} \|\varphi_\epsilon\|_{L^\infty(\Omega)} > 0, \quad \sup_{0 < \epsilon < \epsilon_0} \|\varphi_\epsilon\|_{L^\infty(\Omega)} < \infty.
\end{equation}
Assume the followings:
there exists a positive constant \( \eta \) such that \( M(\varphi, \eta) \subset \Omega \) for all \( \epsilon \in (0, \epsilon_0) \);

there exists a family of positive constants \( \{A_\epsilon\}_{0<\epsilon<\epsilon_0} \) such that
\[
\lim_{\epsilon \to 0} A_\epsilon = \infty, \quad \lim_{\epsilon \to 0} \omega(\epsilon, A_\epsilon, \varphi_\epsilon) = 0.
\]

For any \( \epsilon \in (0, \epsilon_0) \), let \( u_\epsilon \) be the solution of (1.1), and assume that
\[
\sup_{0<\epsilon<\epsilon_0} \sup_{0<t<T_\epsilon} \left[ \log(T_\epsilon - t) + \|u_\epsilon(t)\|_{L^\infty(\Omega)} \right] < \infty.
\]

Then, for any \( \delta > 0 \), there exists a positive constant \( \epsilon_\delta \) such that
\[
B_\epsilon \subset M(\varphi, \delta) = \{ x \in \overline{\Omega} : \varphi(x) \geq \|\varphi\|_{L^\infty(\Omega)} - \delta \}
\]
for all \( \epsilon \in (0, \epsilon_\delta) \).

The following corollary immediately follows from Theorem 1.1.

**Corollary 1.1** Let \( N \geq 1 \), \( \epsilon_0 > 0 \), \( \Omega \) be a domain in \( \mathbb{R}^N \), and \( \varphi \in BUC_+ (\overline{\Omega}) \) satisfying \( M(\varphi, \eta) \subset \Omega \) for some \( \eta > 0 \). For any \( \epsilon \in (0, \epsilon_0) \), let \( u_\epsilon \) be the solution of
\[
\left\{ \begin{array}{ll}
\partial_t u = \epsilon \Delta u + e^u, & x \in \Omega, \ t > 0, \\
u(x, t) = 0, & x \in \partial \Omega, \ t > 0 \text{ if } \partial \Omega \neq \emptyset, \\
u(x, 0) = \varphi_\epsilon(x), & x \in \Omega,
\end{array} \right.
\]
and assume that (1.6) holds. Then, for any \( \delta > 0 \), there exists a positive constant \( \epsilon_\delta \) such that
\[
B_\epsilon \subset M(\varphi, \delta) = \{ x \in \overline{\Omega} : \varphi(x) \geq \|\varphi\|_{L^\infty(\Omega)} - \delta \}
\]
for all \( \epsilon \in (0, \epsilon_\delta) \).

By Theorem 1.1 and Corollary 1.1 we see that the location of the blow-up set for problem (1.1) is characterized by using the level sets of the initial function.

For the proof of Theorem 1.1, we refine the argument of [3], which is a modification of [7], and construct a supersolution \( \overline{u}_\epsilon \) with the following properties:

(a) \( \overline{u}_\epsilon \) exists in \( \Omega \times (0, T_\epsilon) \);

(b) For any \( \delta > 0 \), \( \overline{u}_\epsilon \) is bounded in \( M(\varphi, \delta) \times (0, T_\epsilon) \) if \( \epsilon > 0 \) is sufficiently small.

However, the argument of [3] heavily depends on the nonlinearity, and it seems difficult to apply the argument to problem (1.1), which has exponential nonlinearity, directly. In order to generalize the argument of [3], we consider the blow-up problem for a generalized semilinear heat equation
\[
\left\{ \begin{array}{ll}
\partial_t u = \epsilon \Delta u + f(u), & x \in \Omega, \ t > 0, \\
u(x, t) = 0, & x \in \partial \Omega, \ t > 0 \text{ if } \partial \Omega \neq \emptyset, \\
u(x, 0) = \varphi_\epsilon(x) \geq 0, & x \in \Omega.
\end{array} \right.
\]

We find a sufficient condition of \( f \) such that a supersolution satisfying properties (a) and (b) can be constructed for problem (1.7), and construct a supersolution \( \overline{u}_\epsilon \) for problem (1.1). Thus we prove Theorem 1.1.
2 Outline of the proof of Theorem 1.1

In this section we explain the outline of the proof of Theorem 1.1. In order to prove Theorem 1.1, we construct a supersolution $\overline{u}_\epsilon$ for some semilinear heat equation with the following properties:

(a) $\overline{u}_\epsilon$ is a smooth function defined in $\Omega \times (0, T_\epsilon)$;

(b) For any $\delta > 0$, there holds

$$\sup \{\overline{u}_\epsilon(x, t) : (x, t) \in [\Omega \setminus M(\varphi_\epsilon, \delta)] \times (0, T_\epsilon)\} < \infty$$

for all sufficiently small $\epsilon > 0$.

By using above properties (a) and (b) and the comparison principle, for any $x_\epsilon \notin M(\varphi_\epsilon, \delta)$, we can find a constant $r_\epsilon > 0$ such that

$$\limsup \sup_{t \to T_\epsilon} \sup_{x \in B(x_\epsilon, r_\epsilon)} u_\epsilon(x, t) \leq \limsup_{x \to B(x_\epsilon, r_\epsilon)} \overline{u}_\epsilon(x, t) < \infty.$$ 

This together with the definition of the blow-up set $B_\epsilon$ implies that $x_\epsilon \notin B_\epsilon$, and we have

$$B_\epsilon \subset M(\varphi_\epsilon, \delta) = \{x \in \overline{\Omega} : \varphi_\epsilon(x) \geq \|\varphi_\epsilon\|_{L^\infty(\Omega)} - \delta\}$$

for all sufficiently small $\epsilon > 0$.

In order to construct a supersolution $\overline{u}_\epsilon$, we impose the following condition (F) on the nonlinear term $f$:

\[
\begin{cases}
  f \in C^2((0, \infty)) \cap C^1([0, \infty)), \\
  f(s), f'(s), f''(s) > 0 \text{ for all } s > 0, \\
  \int_1^\infty \frac{ds}{f(s)} < \infty, \\
  \limsup_{u \to \infty} f'(u) F(u) < \infty, \\
  \limsup_{u \to \infty} \frac{f(u + M)}{f(u)} < \infty \text{ for some } M > 0,
\end{cases}
\]

where

$$F(u) := \int_u^\infty \frac{ds}{f(s)} < \infty.$$ 

Under condition (F), we can generalize the argument of [3], and construct a supersolution $\overline{u}_\epsilon$ satisfying properties (a) and (b).

Remark 2.1 The following functions satisfy condition (F):

- $f(u) = (u + \lambda)^p$ with $p > 1$ and $\lambda \geq 0$;
- $f(u) = u^p + u^q$ with $p > q > 1$;
\begin{itemize}
\item $f(u) = (u + 1)^p \log(u + 1)^q$ with $p > 1$ and $q > 1$;
\item $f(u) = e^{\alpha u}$ with $\alpha > 0$.
\end{itemize}

In the rest of this section we give some comments on condition (F). The third condition $F(1) < \infty$ is a necessary condition for a finite time blow-up of the solution. If $F(1) = \infty$, then the solution for the following ordinary differential equation

\begin{equation}
\begin{cases}
\partial_t \zeta = f(\zeta), & t > 0, \\
\zeta(0) = \lambda > 0,
\end{cases}
\end{equation}

exists globally in time. Therefore, if $F(1) = \infty$, by the comparison principle we see that the solution of (1.1) does not blow-up in a finite time.

Furthermore, we can give examples of $f$ which do not satisfy the fourth condition and the fifth condition. The function defined by $f(u) = e^{\alpha u}$ satisfies

$$\lim_{u \to \infty} \frac{f(u + M)}{f(u)} = \infty \quad \text{for any } M > 0,$$

and $f$ does not satisfy (F).

On the other hand, the function defined by $f(u) = (u + 1)\log(u + 1)^\alpha$ ($\alpha > 1$) satisfies

$$\lim_{u \to \infty} f'(u)F(u) = \infty,$$

and $f$ does not satisfy (F). However, if $\alpha = 2$, then a regional blow-up may occur even if the initial function has the only one maximum point (see [5]). We suspect that Theorem 1.1 does not hold for problem (1.1) with $f(u) = (u + 1)\log(u + 1)^2$.

\section{Blow-up problem for generalized semilinear heat equation}

In this section we consider the blow-up problem for (1.7), and generalize the result of Section 1. Using the argument of Section 2, we have the following theorem.

\begin{theorem}
Let $N \geq 1$, $\epsilon_0 > 0$, $\Omega$ be a domain in $\mathbb{R}^N$, and $\{\varphi_{\epsilon}\}_{0 < \epsilon < \epsilon_0} \subset BC_+ (\overline{\Omega})$ be a family of initial functions satisfying (1.5). Assume the followings:

\begin{itemize}
\item there exists a positive constant $\eta$ such that $M(\varphi_{\epsilon}, \eta) \subset \Omega$ for all $\epsilon \in (0, \epsilon_0)$;
\item there exists a family of positive constants $\{A_{\epsilon}\}_{0 < \epsilon < \epsilon_0}$ such that
$$\lim_{\epsilon \to 0} A_{\epsilon} = \infty, \quad \lim_{\epsilon \to 0} \omega(\epsilon, A_{\epsilon}, \varphi_{\epsilon}) = 0.$$
\end{itemize}

Let $f$ be a function satisfying (F). For any $\epsilon \in (0, \epsilon_0)$, let $u_{\epsilon}$ be the solution of (1.7). Furthermore assume that there exists a constant $c_0 > 0$ such that

\begin{equation}
\|u_{\epsilon}(t)\|_{L^{\infty}(\Omega)} \leq F^{-1}(T_{\epsilon} - t))
\end{equation}

Let $N \geq 1$, $\epsilon_0 > 0$, $\Omega$ be a domain in $\mathbb{R}^N$, and $\{\varphi_{\epsilon}\}_{0 < \epsilon < \epsilon_0} \subset BC_+ (\overline{\Omega})$ be a family of initial functions satisfying (1.5). Assume the followings:

\begin{itemize}
\item there exists a positive constant $\eta$ such that $M(\varphi_{\epsilon}, \eta) \subset \Omega$ for all $\epsilon \in (0, \epsilon_0)$;
\item there exists a family of positive constants $\{A_{\epsilon}\}_{0 < \epsilon < \epsilon_0}$ such that
$$\lim_{\epsilon \to 0} A_{\epsilon} = \infty, \quad \lim_{\epsilon \to 0} \omega(\epsilon, A_{\epsilon}, \varphi_{\epsilon}) = 0.$$
\end{itemize}

Let $f$ be a function satisfying (F). For any $\epsilon \in (0, \epsilon_0)$, let $u_{\epsilon}$ be the solution of (1.7). Furthermore assume that there exists a constant $c_0 > 0$ such that

\begin{equation}
\|u_{\epsilon}(t)\|_{L^{\infty}(\Omega)} \leq F^{-1}(T_{\epsilon} - t))
\end{equation}
for all $t \in (0, T_\epsilon)$ and $\epsilon \in (0, \epsilon_0)$, where $F^{-1}$ is the inverse function of $F$. Then, for any $\delta > 0$, there exists a positive constant $\epsilon_\delta$ such that

$$B_\epsilon \subset M(\varphi, \delta) = \{ x \in \overline{\Omega} : \varphi(x) \geq \|\varphi\|_{L^\infty(\Omega)} - \delta \}$$

for all $\epsilon \in (0, \epsilon_\delta)$.

Theorem 3.1 is a generalization of Theorem 1.1. In fact, it is easy to show that the functions $f(u) = u^p$ ($p > 1$) and $f(u) = e^u$ satisfy (F). Furthermore, (3.1) is equivalent to (1.3) if $f(u) = u^p$ ($p > 1$) and (1.6) if $f(u) = e^u$.

**Remark 3.1** (i) Let $f$ and $g$ be functions satisfying (F). Then, for any $\alpha > 0$ and $\beta > 0$, the function $\alpha f + \beta g$ also satisfies (F).

(ii) Let $\epsilon_0 > 0$ and $\{ \varphi_\epsilon \}_{0 < \epsilon < \epsilon_0} \subset BC_+(\overline{\Omega}) \cap C^1(\Omega) \setminus \{0\}$ satisfying

$$\lim_{\epsilon \to 0} \epsilon^{1/2} \|\nabla \varphi_\epsilon\|_{L^\infty(\Omega)} = 0.$$

Put $A_\epsilon = [\epsilon^{1/2} \|\nabla \varphi_\epsilon\|_{L^\infty(\Omega)}]^{-1/2}$. Then $\lim_{\epsilon \to 0} A_\epsilon = \infty$ and there holds

$$|\varphi_\epsilon(x) - \varphi_\epsilon(y)| \leq \|\nabla \varphi_\epsilon\|_{L^\infty(\Omega)} |x - y| \leq \epsilon^{1/2} \|\nabla \varphi_\epsilon\|_{L^\infty(\Omega)} \cdot A_\epsilon = A_\epsilon^{-1} \to 0 \quad \text{as} \quad \epsilon \to 0$$

for all $x, y \in \overline{\Omega}$ with $|x - y| \leq A_\epsilon^{1/2}$, that is, $\{ A_\epsilon \}_{0 < \epsilon < \epsilon_0}$ and $\{ \varphi_\epsilon \}_{0 < \epsilon < \epsilon_0}$ satisfy (1.5).

(iii) Let $\epsilon_0 > 0$, $\Omega$ be a $C^2$ smooth domain, $f$ be a function satisfying (F), and $\{ \varphi_\epsilon \}_{0 < \epsilon < \epsilon_0} \subset C^2(\Omega)$ be a family of nonnegative functions satisfying (1.5), and $\varphi_\epsilon = 0$ on $\partial \Omega$. Furthermore assume that there exist a function $F \in C^2([0, \infty))$ and a positive constant $c$ such that

$$\begin{align*}
F(s) > 0, & \quad F'(s) \geq 0, \quad F''(s) \geq 0 \quad \text{in} \quad s \in [0, \infty), \\
f'(s)F(s) - f(s)F'(s) \geq cF'(s)F(s) \quad \text{in} \quad s \in (0, \infty),
\end{align*}$$

$$\int_1^\infty \frac{ds}{F(s)} < \infty,$$

which was introduced by Friedman and McLeod in [1]. If there holds either

(a) $f(0) = 0$ and there exists a constant $\delta \in (0, 1)$ such that $\epsilon \Delta \varphi_\epsilon + f(\varphi_\epsilon) \geq \delta f(\varphi_\epsilon)$ in $\Omega$

for all sufficiently small $\epsilon > 0$

or

(b) there exist a subdomain $\Omega' \subset \subset \Omega$ and positive constants $C$ and $\delta$ such that

$$u_\epsilon(x, t) \leq C \quad \text{in} \quad \Omega \setminus \overline{\Omega'} \times (0, T_\epsilon), \quad \epsilon \Delta \varphi_\epsilon + f(\varphi_\epsilon) \geq \delta \quad \text{in} \quad \Omega'$$

for all sufficiently small $\epsilon > 0$,

then we can prove the uniform blow-up estimate (3.1) by using the same argument as in [1].

Condition (b) is verified if $\Omega$ is a bounded convex domain and there exist positive constants $C$ and $\delta$ such that

$$|\nabla^2 \varphi_\epsilon| \leq C \quad \text{near} \quad \partial \Omega, \quad \delta \varphi_\epsilon / \partial \nu \leq -\delta \quad \text{on} \quad \partial \Omega, \quad \epsilon \Delta \varphi_\epsilon + f(\varphi_\epsilon) \geq \delta \quad \text{in} \quad \Omega,$$

for all sufficiently small $\epsilon > 0$, where $\nu$ is the outer normal unit vector to $\partial \Omega$. 
4 Remarks

In this section we discuss the application of Theorem 1.1, and give an extension of Theorem 1.1 for more general superlinear heat equations.

Consider

\begin{equation}
\begin{array}{l}
\partial_t v = \Delta v + e^v, \quad x \in \mathbb{R}^N, \quad t > 0, \\
v(x, 0) = \phi(x) \geq 0, \quad x \in \mathbb{R}^N,
\end{array}
\end{equation}

where $N \geq 1$ and $\phi \in BC_+(\mathbb{R}^N)$. Let $T$ be the blow-up time of $v$, and assume that there exists a positive constant $C$ such that

\begin{equation}
\log(T - t) + \|v(t)\|_{L^\infty(\Omega)} \leq C
\end{equation}

for all $t \in (0, T)$. Here we remark that the solution of (4.1) blows up in a finite time since the initial function $\phi$ is nonnegative. Let $\epsilon > 0$ be a sufficiently small constant and put

\begin{align*}
&u_\epsilon(x, t) = \log \epsilon + v(x, T - \epsilon + \epsilon t) \quad \text{in} \quad \mathbb{R}^N \times [0, 1), \\
&\varphi_\epsilon(x) = u_\epsilon(x, 0) = \log \epsilon + v(x, T - \epsilon) \quad \text{in} \quad \mathbb{R}^N.
\end{align*}

Then $u_\epsilon$ blows up at $t = 1$ and satisfies

\begin{equation}
\begin{array}{l}
\partial_t u_\epsilon = \epsilon \Delta u_\epsilon + e^{u_\epsilon} \quad \text{in} \quad \mathbb{R}^N \times (0, 1), \\
u_\epsilon(x, 0) = \varphi_\epsilon(x) \quad \text{in} \quad \mathbb{R}^N,
\end{array}
\end{equation}

and by (4.2) we obtain

$$
u_\epsilon(x, t) \leq \log \epsilon + \|v(t)\|_{L^\infty(\mathbb{R}^N)} \leq \log \epsilon + [C - \log(T - (T - \epsilon + \epsilon t))] \leq C - \log(1 - t)$$

for all $(x, t) \in \mathbb{R}^N \times (0, 1)$, which implies the uniform blow-up estimate for $u_\epsilon$. Therefore, under suitable assumptions on $\phi$ and $\epsilon$, we can apply Theorem 1.1 to obtain the location of the blow-up set of $v$. Similar argument was employed in [4] for a semilinear heat equation with power nonlinearity, and the author of this paper and Ishige succeeded in obtaining the location of the blow-up set (see also Remark 1.2 (ii) in [3]).

On the other hand, the above argument can be applied to study the location of the blow-up set even if the equation does not have a self-similarity. For this purpose, we give an extension of Theorem 1.1 for more general superlinear heat equation whose nonlinear term depends on $\epsilon$. Consider

\begin{equation}
\begin{array}{l}
\partial_t u = \epsilon \Delta u + f_\epsilon(u), \quad x \in \Omega, \quad t > 0, \\
u(x, t) = 0, \quad x \in \partial \Omega, \quad t > 0 \quad \text{if} \quad \partial \Omega \neq \emptyset, \\
u(x, 0) = \varphi_\epsilon(x) \geq 0 (\neq 0), \quad x \in \Omega,
\end{array}
\end{equation}
where \( \epsilon_0 > 0 \), \( \{\varphi_{\epsilon}\}_{0<\epsilon<\epsilon_0} \subset BC_+ (\overline{fl}) \) satisfies (1.5), and \( \{f_{\epsilon}\}_{0<\epsilon<\epsilon_0} \subset C^1(0,\infty) \cap C^2((0,\infty)) \) satisfies

\[
(F_{\epsilon}) \begin{cases}
\lim_{\epsilon \to 0} \inf_{s \in (0,\infty)} f_{\epsilon}(s), \lim_{\epsilon \to 0} \inf_{s \in (0,\infty)} f_{\epsilon}'(s), \lim_{\epsilon \to 0} \sup_{s \in (0,\infty)} f_{\epsilon}''(s) > 0, \\
\lim_{\epsilon \to 0} \sup_{s \in (0,\infty)} f_{\epsilon}(s), \lim_{\epsilon \to 0} \sup_{s \in (0,\infty)} f_{\epsilon}'(s), \lim_{\epsilon \to 0} \sup_{s \in (0,\infty)} f_{\epsilon}''(s) < \infty, \\
\lim_{s \to \infty} \sup_{0<\epsilon<\epsilon_0} f_{\epsilon}(u+M) < \infty \text{ for some } M > 0.
\end{cases}
\]

Then we can prove the following theorem with a slight modification of the argument of this paper, which is an extension of Theorem 1.1.

**Theorem 4.1** Let \( N \geq 1, \epsilon_0 > 0, \Omega \) be a domain in \( \mathbb{R}^N \), and \( \{\varphi_{\epsilon}\}_{0<\epsilon<\epsilon_0} \subset BC_+ (\overline{\Omega}) \) satisfying (1.5). Let \( \{f_{\epsilon}\}_{0<\epsilon<\epsilon_0} \) be a family of functions satisfying (F\(_\epsilon\)). For any \( \epsilon \in (0,\epsilon_0) \), let \( u_{\epsilon} \) be the solution of (4.3), and assume that there exists a positive constant \( c_* \) such that

\[
\|u_{\epsilon}(t)\|_{L^\infty(\Omega)} \leq F_{f_{\epsilon}}^{-1}(c_*(T_{\epsilon} - t))
\]

for all \( t \in (0,T_{\epsilon}) \) and all \( \epsilon \in (0,\epsilon_0) \). Then, for any \( \delta > 0 \), there holds

\[
B_{\epsilon} \subset M(\varphi_{\epsilon}, \delta) = \{ x \in \overline{\Omega} : \varphi_{\epsilon}(x) \geq \|\varphi_{\epsilon}\|_{L^\infty(\Omega)} - \delta \}
\]

for all sufficiently small \( \epsilon > 0 \).

We apply Theorem 4.1 to the blow-up problem for a semilinear heat equation

\[
(4.4) \quad \begin{cases}
\partial_t v = \Delta v + v^p + v^q, & x \in \Omega, \quad t > 0, \\
v(x,t) = 0, & x \in \partial \Omega, \quad t > 0 \quad \text{if } \partial \Omega \neq \emptyset, \\
v(x,0) = \phi(x) \geq 0, & x \in \Omega,
\end{cases}
\]

where \( 1 < q < p \) and \( \phi \in BC_+ (\overline{\Omega}) \). Assume that the solution of (4.4) blows up at some time \( t = T < \infty \) and that there exists a positive constant \( C \) such that

\[
\|v(t)\|_{L^\infty(\Omega)} \leq C(T-t)^{-\frac{1}{p-1}}
\]

for all \( t \in (0,T) \). Let \( \epsilon > 0 \) be a sufficiently small constant and put

\[
\begin{align*}
u_{\epsilon}(x,t) &= \epsilon^{\frac{1}{p-1}} v(x, T - \epsilon + \epsilon t) \quad \text{in } \mathbb{R}^N \times [0,1), \\
\varphi_{\epsilon}(x) &= u_{\epsilon}(x, 0) = \epsilon^{-\frac{1}{p-1}} v(x, T - \epsilon) \quad \text{in } \mathbb{R}^N, \\
f_{\epsilon}(s) &= s^p + \epsilon^{\frac{1}{p-1}} s^q \quad \text{in } (0,\infty).
\end{align*}
\]
Then $\{f_{\epsilon}\}_{0<\epsilon<\epsilon_0}$ satisfies $(F_\epsilon)$ for some positive constant $\epsilon_0$, and $u_\epsilon$ satisfies

$$
\begin{cases}
\partial_t u_\epsilon = \epsilon \Delta u_\epsilon + f_\epsilon(u_\epsilon) & \text{in } \Omega \times (0,1), \\
u_\epsilon(x,t) = 0 & \text{on } \partial\Omega \times (0,1), \\
u_\epsilon(x,0) = \varphi_\epsilon(x) & \text{in } \Omega.
\end{cases}
$$

Furthermore, by (4.5) we have

$$
\|u_\epsilon(t)\|_{L^\infty(\Omega)} \leq C(1-t)^{-\frac{1}{p-1}}
$$

for all $t \in (0,1)$. This yields

$$
\|u_\epsilon(t)\|_{L^\infty(\Omega)} \leq F^{-1}_{f_\epsilon}(c_*(1-t))
$$

for some constant $c_* > 0$, provided that $\epsilon > 0$ is sufficiently small. Therefore, under suitable assumptions, we can apply Theorem 4.1 to problem (4.4), and obtain the location of the blow-up set of $v$ by using the maximum points of the solution just before the blow-up time.

**References**


