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On the existence of indiscernible structures

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1 Introduction

There are many types of indiscernible objects, for example, indiscernible sequences, indiscernible arrays, and indiscernible trees. They are useful to analyze the stability of theories. Therefore, we want to know when they exist. The existence of indiscernible objects was studied in [2], [4], [3], and [5]. In this paper, I show a general method to prove the existence of indiscernible objects. It consists of two steps, checking amalgamation property and proving partition theorem.

2 Notations and Preliminaries

Before starting, I remark some elementary facts.

Definition 1. Let $\mathcal{L}$ be a countable language and M_n $(n \in \omega)$ a countable $\mathcal{L}$-structure. We say the class $\{M_n : n \in \omega\}$ has amalgamation property if: For any $M_{n_i} \ni a_{n_i}$ $(i < 2)$ such that $\mathrm{atp}(a_{n_0}) = \mathrm{atp}(a_{n_1})$, there are embeddings $\sigma_i : M_{n_i} \to M_k$ such that $\sigma_0(a_{n_0}) = \sigma_1(a_{n_1})$ for some $k \in \omega$.

Proposition 2. Suppose $\{M_n : n \in \omega\}$ has amalgamation property. Then there exists the unique countable $\mathcal{L}$-structure M^* such that

- for any $a \in M^*$ and for any $b \in M_k$ with $\mathrm{atp}(a) = \mathrm{atp}(b)$, there is an embedding $\sigma : M_k \to M^*$ with $\sigma(b) = a$,
- for any $a, a', b \in M^*$ such that $\mathrm{atp}(a) = \mathrm{atp}(b)$ there is $b' \in M^*$ such that $\mathrm{atp}(aa') = \mathrm{atp}(bb')$. $\square$

Corollary 3. 1. M^* is homogeneous.

2. Let $a, b \in M^*$. If $\mathrm{atp}(a) = \mathrm{atp}(b)$ then $\mathrm{tp}(a) = \mathrm{tp}(b)$. $\square$

Example 4. Let $M_n = (\omega, <)$ for all $n \in \omega$. Then $\{M_n : n \in \omega\}$ has amalgamation property. The generic model M^* is $(\mathbb{Q}, <)$.

In what follows, we work in a big model $\mathcal{M}$ of complete L-theory T. We often consider a set of L-formulas $\Gamma((x_i)_{i\in\mathcal{I}})$ having free variables subscripted by $i \in \mathcal{I}$. We consider $\mathcal{I}$ as an $\mathcal{L}$-structure with a countable language $\mathcal{L}$. Subsets of $\mathcal{I}$ are denoted by $X, Y, ...$, and subsets of $\mathcal{M}$ are denoted by $A, B, ...$. $I = (a_i)_{i\in\mathcal{I}} \subset \mathcal{M}$ denotes an indiscernible object in some sense, which depends on the structure on $\mathcal{I}$. For example if $\mathcal{I}$ has the empty structure then we call I an indiscernible set. And, if $\mathcal{I}$ has a structure of total order, we call I an indiscernible sequence. For $X \subset \mathcal{I}$ and $I = (a_i)_{i\in\mathcal{I}}$, a_X is the sequence $(a_i)_{i\in X}$. $X \sim_{\mathcal{L}} Y$ means $\mathrm{atp}_{\mathcal{L}}(X) = \mathrm{atp}_{\mathcal{L}}(Y)$.

3 Indiscernibilities and substructure properties

Let $\mathcal{I}$ and $\mathcal{I}'$ be $\mathcal{L}$-structures.

Definition 5. A map $\sigma : \mathcal{I} \to \mathcal{I}'$ is said to be an $\mathcal{L}$-embedding if σ preserves $\mathcal{L}$-atomic types.

Note that $\sigma : \mathcal{I} \to \mathcal{I}'$ is an $\mathcal{L}$-embedding if and only if $\sigma : \mathcal{I} \to \sigma(\mathcal{I})$ is an $\mathcal{L}$-isomorphism. Let $A = (a_i)_{i\in\mathcal{I}'}$ be a sequence in $\mathcal{M}$. $\sigma^{-1}A$ is the sequence $B = (b_i)_{i\in\mathcal{I}}$ such that $b_i = a_{\sigma(i)}$. The following definitions are in [5] and [3].

Definition 6. 1. We say $\Gamma((x_i)_{i\in\mathcal{I}})$ has the $\mathcal{L}$-substructure property if there is a realization $A = (a_i)_{i\in\mathcal{I}}$ of Γ such that for any $\mathcal{L}$-embedding $\sigma : \mathcal{I} \to \mathcal{I}$, $\sigma^{-1}A \models \Gamma$, i.e. $\models \varphi(a_{\sigma(i_1)}, ..., a_{\sigma(i_n)})$ for every $\varphi(x_{i_1}, ..., x_{i_n}) \in \Gamma$.

2. We say $I = (a_i)_{i\in\mathcal{I}}$ is $\mathcal{L}$-indiscernible if for any $X \sim_{\mathcal{L}} Y \subset \mathcal{I}$, $\mathrm{tp}(a_X) = \mathrm{tp}(a_Y)$.

Example 7. Let $\mathcal{I}$ be the structure $(\omega, <)$ and $\Gamma((x_i)_{i\in\mathcal{I}})$ the set

$$\{\varphi(x_{i_1}, ..., x_{i_n}) \leftrightarrow \varphi(x_{j_1}, ..., x_{j_n}) : \varphi \text{ is an } L\text{-formula}, \langle i_1, ..., i_n\rangle \sim_< \langle j_1, ..., j_n\rangle, n \in \omega\}.$$

Any realization I of Γ is an indiscernible sequence, and vice versa. Since I is $<$-indiscernible, so is any infinite subsequence I' of I. Hence Γ has $<$-substructure property.

Let $\mathcal{I}$ be an $\mathcal{L}$-structure and let $\Gamma((x_i)_{i\in\mathcal{I}})$ be a set of L-formulas.

Remark 8. For the following conditions, $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ hold:

1. Γ is realized by an $\mathcal{L}$-indiscernible structure I.
2. $\Gamma^* := \{\varphi(x_Y) : \exists X \sim_{\mathcal{L}} Y \text{ s.t. } \Gamma \models \varphi(x_X)\}$ is consistent.
3. $\Gamma' := \{\varphi(x_Y) : \exists X \sim_{\mathcal{L}} Y \text{ s.t. } \Gamma \ni \varphi(x_X)\}$ is consistent.
4. Γ has $\mathcal{L}$-substructure property.

In this paper, we see that if the $\mathcal{L}$-structure $\mathcal{I}$ has a nice property then the above conditions are equivalent. More precisely, amalgamation properties imply $4 \Rightarrow 2$, and the partition theorems imply $2 \Rightarrow 1$.

Example 9. Let $\Gamma((x_i)_{i\in\omega})$ be the set of formulas expressing the unstability of $\varphi(x,y)$, i.e. $\Gamma((x_i)_{i\in\omega}) = \{\varphi(x_i,x_j) : i<j\} \cup \{\neg\varphi(x_i,x_j) : i \geq j\}$. (If the theory is unstable) Γ has subsequence property. By Ramsey's theorem we have an indiscernible sequence realizing Γ.

4 The amalgamation property and the substructure property

Let $\mathcal{I}$ be an $\mathcal{L}$-structure and $\Gamma((x_i)_{i\in\mathcal{I}})$ a set of L-formulas. Suppose that Γ has $\mathcal{L}$-substructure property, witnessed by $A = (a_i)_{i\in\mathcal{I}}$. Put $\mathrm{End}(\mathcal{I}) = \{\sigma : \mathcal{I} \to \mathcal{I} : \sigma \text{ is an } \mathcal{L}\text{-embedding.}\}$. If $\sigma \in \mathrm{End}(\mathcal{I})$ then $A \models \Gamma((x_i)_{i\in\mathcal{I}})) \cup \Gamma((x_{\sigma(i)})_{i\in\mathcal{I}})$, since $\sigma^{-1}A \models \Gamma$. Therefore, $A \models \bigcup_{\sigma\in\mathrm{End}(\mathcal{I})}\Gamma((x_{\sigma(i)})_{i\in\mathcal{I}})$. However, there may not be $\sigma \in \mathrm{End}(\mathcal{I})$ sending $X \subset \mathcal{I}$ to $Y \subset \mathcal{I}$ even if $X \sim_{\mathcal{L}} Y$. In general $\bigcup_{\sigma\in\mathrm{End}(\mathcal{I})}\Gamma((x_{\sigma(i)})_{i\in\mathcal{I}}) \subsetneq \Gamma^*$. The following is an example of Γ such that Γ has substructure property but Γ^* is inconsistent.

Example 10. Let $L = \{<_{\mathrm{ini}}, <_{\mathrm{lex}}\}$ and $M = \omega^{<\omega}$. Consider an $\mathcal{L}$-structure on M by

- $\eta <_{\mathrm{ini}} \eta' \Leftrightarrow \eta$ is a proper initial segment of η',

- $\eta <_{\text{lex}} \eta' \Leftrightarrow \eta$ is less than η' in the lexicographic order,

for every $\eta, \eta' \in M$. For example, $\langle 0\rangle <_{\text{ini}} \langle 0,0\rangle <_{\text{lex}} \langle 0,1\rangle$. Let $\mathcal{I}$ be also the structure $(\omega^{<\omega}, <_{\text{ini}}, <_{\text{lex}})$ and $\mathcal{L}$ the set $\{<_{\text{ini}}, <_{\text{lex}}\}$. Put $\Gamma((x_\eta)_{\eta\in\mathcal{I}}) = \{x_\eta <_{\text{ini}} x_{\eta'} : \eta <_{\text{ini}} \eta'\} \cup \{x_\eta <_{\text{lex}} x_{\eta'} : \eta <_{\text{lex}} \eta'\}$. Immediately Γ has $\mathcal{L}$-substructure property in $\text{Th}_L(M)$ witnessed by M. Let $\eta \cap \eta'$ be the longest common initial segments of η and η'. ($\cap$ is definable in $\text{Th}_L(M)$.) Then $\Gamma \models x_{\langle 0,1\rangle} \cap x_{\langle 1,1\rangle} <_{\text{ini}} x_{\langle 0,0\rangle} \cap x_{\langle 0,1\rangle} \wedge x_{\langle 1,0\rangle} \cap x_{\langle 1,1\rangle} \geq_{\text{ini}} x_{\langle 0,0\rangle} \cap x_{\langle 1,0\rangle}$. However $\langle 0,0\rangle\langle 0,1\rangle\langle 1,1\rangle \sim_{\mathcal{L}} \langle 0,0\rangle\langle 1,0\rangle\langle 1,1\rangle$. Hence Γ^* is inconsistent in the theory.

Proposition 11. Let $\mathcal{I}$ be an $\mathcal{L}$-structure. Suppose that $\{\mathcal{I}\}$ has amalgamation property, i.e. $\{M_n : n \in \omega\}$ has amalgamation property where $M_n = \mathcal{I}$ $(n \in \omega)$. If Γ has substructure property then Γ^* is consistent.

Proof. Let $\mathcal{I}^*$ be the generic model of $\{\mathcal{I}\}$. Let $\Delta((x_i)_{i\in\mathcal{I}^*})$ be the set of formulas such that $\Delta((x_i)_{i\in\mathcal{I}^*}) \ni \varphi(x_Y)$ if and only if

- $\Gamma \models \varphi(x_X)$,
- there is an $\mathcal{L}$-embedding $\sigma : \mathcal{I} \to \mathcal{I}^*$ sending X to Y.

In the other words, Δ is the set

$$\bigcup_\sigma \Gamma((x_{\sigma_k(i)})_{i\in\mathcal{I}}) \qquad (\sigma \text{ varies over all embeddings from } \mathcal{I} \text{ to } \mathcal{I}^*).$$

Claim A. $\Delta((x_i)_{i\in\mathcal{I}^*})$ is consistent.

It is enough to show that for any embeddings $\sigma_1, ..., \sigma_n$, $\bigcup_{k\leq n} \Gamma((x_{\sigma_k(i)})_{i\in\mathcal{I}})$ is consistent. For simplicity, we assume that $n = 2$ and Γ is closed under taking the conjunction. Take $\varphi_1(x_{X_1}), \varphi_2(x_{X_2}) \in \Gamma((x_i)_{i\in\mathcal{I}})$. Let $\sigma_k(X_k) = Y_k$ $(k = 1,2)$ and $Z_k = \sigma_k^{-1}(Y_1 \cap Y_2)$. (If $Z_k = \emptyset$ then $\bigwedge_k \varphi_k(x_{Y_k})$ is clearly consistent.) Since $Z_1 \sim_{\mathcal{L}} Z_2$, there are embeddings $\tau_k : \mathcal{I} \to \mathcal{I}$ such that $\tau_1(Z_1) = \tau_2(Z_2)$. By the substructure property, $\bigcup_k \Gamma((x_{\tau_k(i)})_{i\in\mathcal{I}})$ is consistent. Therefore, $\bigwedge_k \varphi_k(x_{Y_k})$ is consistent. (End of the proof of the claim.)

By the construction of Δ, if $Y \sim_{\mathcal{L}} Y' \subset \mathcal{I}^*$ and $\Delta \models \varphi(x_Y)$ then $\Delta \models \varphi(x_{Y'})$. Hence Γ^* is consistent. $\square$

5 Partition lemmas and the existence of indiscernible structures

We begin with Erdös–Rado theorem and indiscernible sequence.

Let $\mathcal{I} = (\omega, <)$ and $\mathcal{L} = \{<\}$. Suppose that $\Gamma((x_i)_{i\in\omega})$ has $\mathcal{L}$-subsequence property. Since $\{\mathcal{I}\}$ has amalgamation property, $\Gamma^*((x_i)_{i\in\omega})$ is consistent. By compactness, we may assume $\mathcal{I} = (\kappa, <)$ with sufficiently large κ. Take a realization J of $\Gamma^*((x_i)_{i\in\kappa})$. By Erdös–Rado, we have a subsequence $I = (a_i)_{i\in\omega}$ of J which is indiscernible for n-variable formulas. Clearly $I \models \Gamma$. Using compactness, we have

Fact 12. Let $\mathcal{I} = (\omega, <)$ and $\mathcal{L} = \{<\}$. If $\Gamma((x_i)_{i\in\mathcal{I}})$ has subsequence property then it is realized by $\mathcal{L}$-indiscernible sequence. □

In this section we discuss about three examples of indiscernible structures on an array and a tree. Always, first we show the partition lemma for the structure. Then we get indiscernible objects. Indiscernible arrays are discussed in [3]. Indiscernible trees are discussed in [2],[4],[3], and [5]. The partition theorems are proved in [1] and [4].

Let $\mathcal{L} = \{P_n(x), <_{\mathsf{lex}}\}_{n\in\omega}$. Let $\mathcal{I}$ be the $\mathcal{L}$-structure on $\omega \times \omega$ defined by

- $P_n^{\mathcal{I}} = \{n\} \times \omega$,
- $<_{\mathsf{lex}}^{\mathcal{I}} = \{\langle (n, m), (k, l)\rangle : n < k \text{ or } n = k \wedge m < l\}$

Note that $\omega \times \lambda$ is an $\mathcal{L}$-structure with the natural interpretations, for every λ.

Lemma 13 (the partition lemma). Let $\kappa << \lambda$ and $n \in \omega$. Then for any $f : (\omega \times \lambda)^n \to \kappa$ there is an $\mathcal{L}$-embedding $\sigma : \omega \times \omega \to \omega \times \lambda$ such that if $X \sim_{\mathcal{L}} Y \in (\omega \times \omega)^n$ then $f(\sigma(X)) = f(\sigma(Y))$.

Proof. We will give an $\mathcal{L}$-substructure $\omega \times \chi \subset \omega \times \lambda$ which is the image of σ. We assume $n = 2$, since general cases are similar. For given f, we define a map $g : \lambda^2 \to {}^{\omega^2}\kappa$ by

$$g(\alpha_1, \alpha_2) = f((-, \alpha_1), (-, \alpha_2)) : \omega^2 \to \kappa.$$

By Erdös–Rado, we have a homogeneous set $\chi \subset \lambda$ with respect to g.

Claim A. If $X \sim_{\mathcal{L}} Y \in (\omega \times \chi)^2$ then $f(X) = f(Y)$.

Suppose $X = \{(x_1, \alpha_1), (x_2, \alpha_2)\}$ and $Y = \{(y_1, \beta_1), (y_2, \beta_2)\}$. Because $X \sim_{\mathcal{L}} Y$,

- $x_k = y_k$ $(k = 1, 2)$,

- (α_1, α_2) and (β_1, β_2) have the same order type.

By the homogeneity of χ, $g(\alpha_1, \alpha_2) = g(\beta_1, \beta_2)$. Hence, $f((-, \alpha_1), (-, \alpha_2))$ and $f((-, \beta_1), (-, \beta_2))$ are the same function. Since $x_k = y_k$, $f((x_1, \alpha_1), (x_2, \alpha_2)) = f((y_1, \beta_1), (y_2, \beta_2))$. $\square$

Fact 14. Let $\mathcal{L} = \{P_n, <_{\text{lex}}\}_n$ and $\mathcal{I} = (\omega \times \omega; \mathcal{L})$, and let $\Gamma((x_i)_{i \in \mathcal{I}})$ be a set of L-formulas. Suppose that $\Gamma((x_i)_{i \in \mathcal{I}})$ has $\mathcal{L}$-substructure property. Then it is realized by an $\mathcal{L}$-indiscernible array.

Proof. Notice that $\{\mathcal{I}\}$ has the amalgamation property. Hence Γ^* is consistent. By compactness, we may assume $\mathcal{I} = (\omega \times \lambda; (P_n)_n, <_{\text{lex}})$ with sufficiently large λ. Using the partition lemma and compactness, we have a realization of Γ which is $\mathcal{L}$-indiscernible. $\square$

Note that there are many other structures on $\omega \times \omega$ which give the same indiscernibility. For example, $\mathcal{L} = \{P_n, <_n\}_n$, defined by $<_n^{\mathcal{I}} = \{\langle (n, m), (n, l) \rangle : m < k\}$, is one of such structures.

Next we consider another structure on $\omega \times \omega$. Let $\mathcal{L} = \{E(x, y), <_{\text{lex}}\}$. Let $\mathcal{I}$ be the $\mathcal{L}$-structure on $\omega \times \omega$ defined by

- $E^{\mathcal{I}} = \{\langle (n, m), (k, l) \rangle : n = k\}$,
- $<_{\text{lex}}^{\mathcal{I}} = \{\langle (n, m), (k, l) \rangle : n < k \text{ or } n = k \wedge m < l\}$.

Lemma 15 (the partition lemma). Let $\kappa << \lambda$ and $n \in \omega$. Then for any $f : (\lambda \times \lambda)^n \to \kappa$ there is an $\mathcal{L}$-embedding $\sigma : \omega \times \omega \to \lambda \times \lambda$ such that if $X \sim_{\mathcal{L}} Y \in (\omega \times \omega)^n$ then $f(\sigma(X)) = f(\sigma(Y))$.

Proof. We will give an $\mathcal{L}$-substructure $\chi \times \chi' \subset \lambda \times \lambda$ which is the image of σ. Assume that $\kappa << \lambda' << \lambda$. By the same argument of the proof of the previous partition lemma, we have an infinite subset $\chi' \subset \lambda$ such that if $x = \langle (k_1, l_1), ..., (k_n, l_n) \rangle \sim_{\mathcal{L}} \langle (k_1, m_1), ..., (k_n, m_n) \rangle = y \in \lambda' \times \chi'$ then $f(x) = f(y)$. By Erdös–Rado, there is an infinite set $\chi \subset \lambda'$ such that if $x = \langle (l_1, k_1), ..., (l_n, k_n) \rangle \sim_{\mathcal{L}} \langle (m_1, k_1), ..., (m_n, k_n) \rangle = y \in \chi \times \chi'$ then $f(x) = f(y)$. Suppose $\langle (k_1, l_1), ..., (k_n, l_n) \rangle \sim_{\mathcal{L}} \langle (j_1, m_1), ..., (j_n, m_n) \rangle \in \chi \times \chi'$. Then we have

- $\langle (k_1, l_1), ..., (k_n, l_n) \rangle \sim_{\mathcal{L}} \langle (k_1, m_1), ..., (k_n, m_n) \rangle$,
- $\langle (k_1, m_1), ..., (k_n, m_n) \rangle \sim_{\mathcal{L}} \langle (j_1, m_1), ..., (j_n, m_n) \rangle$.

Hence

$$\begin{aligned} f(\langle (k_1, l_1), \ldots, (k_n, l_n)\rangle) &= f(\langle (k_1, m_1), \ldots, (k_n, m_n)\rangle) \\ &= f(\langle (j_1, m_1), \ldots, (j_n, m_n)\rangle). \end{aligned}$$

□

Fact 16. Let $\mathcal{L} = \{E, <_{\mathrm{lex}}\}_n$ and $\mathcal{I} = (\omega \times \omega, \mathcal{L})$, and let $\Gamma((x_i)_{i\in\mathcal{I}})$ be a set of L-formulas. Suppose that $\Gamma((x_i)_{i\in\mathcal{I}})$ has $\mathcal{L}$-substructure property then it is realized by an $\mathcal{L}$-indiscernible array.

Proof. Because $\{\mathcal{I}\}$ has amalgamation property, the proof is similar to the proof of Fact 14. □

Next we consider a structure on the tree $\omega^{<\omega}$.

Definition 17. Let $\mathcal{L} = \{<_{\mathrm{ini}}, <_{\mathrm{lex}}, \cap, <_{\mathrm{len}}, (P_n)_{n\in\omega}\}$. We consider the following structure on $\omega^{<\omega}$: For $\eta, \nu \in \omega^{<\omega}$,

1. $\eta <_{\mathrm{ini}} \nu \Leftrightarrow \eta$ is a proper initial segment of ν;
2. $\eta <_{\mathrm{lex}} \nu \Leftrightarrow \eta$ is less than ν in the lexicographic order;
3. $\eta \cap \nu =$ the longest common initial segment of η and ν;
4. $\eta <_{\mathrm{len}} \nu \Leftrightarrow \mathrm{len}(\eta) < \mathrm{len}(\nu)$, where $\mathrm{len}(\eta)$ is the length of the sequence η;
5. $P_n(\eta) \Leftrightarrow$ the length of η is n.

Let $\mathcal{I}$ be the structure $(\omega^{<\omega}; <_{\mathrm{ini}}, <_{\mathrm{lex}}, \cap, <_{\mathrm{len}}, (P_n)_{n\in\omega})$. Note that $\{\mathcal{I}\}$ has amalgamation property.

The following lemma is in [1, p.662] and [4]. We omit the proof.

Lemma 18 (Shelah). Let $O = \lambda^{<n}$ be a tree, and $f : O^k \to \mu$ a k-palace function. If λ is sufficiently large (depending only on μ), then there is an $\mathcal{L}$-embedding $\sigma : \omega^{<n} \to \lambda^{<n}$ such that $f(\sigma(X)) = f(\sigma(Y))$ for any k-tuples $X, Y \subset \omega^{<n}$ with $X \sim_{\mathcal{L}} Y$. □

By similar way to the above proofs of the existence of indiscernible array, we also have

Fact 19. Let $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ be a set of L-formulas. If Γ has the $\mathcal{L}$-subtree property, then Γ is realized by an $\mathcal{L}$-indiscernible tree. □

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