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Trees and Branching Axioms

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1 Introduction

First we recall the definition of trees. An ordered set $O = (O, <)$ is called a tree if, for any $a \in I$, the initial segment $O_a = \{b \in O : b < a\}$ is linearly ordered. A mapping $\sigma : O \rightarrow O'$, where $O$ and $O'$ are trees, is called a tree embedding if $\sigma$ preserves $<$-structure, i.e. $\eta < \nu$ if and only if $\sigma(\eta) < ' \sigma(\nu)$.

We are mainly interested in trees of the form $\alpha^{<\beta}$, where $\alpha$ and $\beta$ are ordinals and its order is $<_{ini}: \eta <_{ini} \nu \iff \eta$ is a proper initial segment of $\nu$. The lexicographic order on $\alpha^{<\beta}$ is denoted by $<_{lex}$. The meet operator $\cap$ is a binary function that gives the greatest common lower bound.

We introduce the following notations:

- $A \simeq_{l.i} B$ for expressing that $A$ and $B$ have the same $\{<_{lex}, <_{ini}\}$-atomic type.
- $A \simeq_{l.i.c.} B$ for expressing that $A$ and $B$ have the same $\{<_{lex}, <_{ini}, \cap\}$-atomic type.

Now let $M$ be an $L$-structure. We consider a set $A \subset M$ whose elements are indexed by a tree. So $A$ has the form $A = (a_\eta)_{\eta \in O}$, where $O$ is a tree. Such an indexed set is also called a tree. We introduce the notion of indiscernibility for such a tree $A$.

- $A$ is $l.i.$-indiscernible if whenever $X \simeq_{l.i.} Y$ then $tp_L(a_X) = tp_L(a_Y)$, where $a_X = (a_\eta)_{\eta \in X}$.
- $A$ is $l.i.c.$-indiscernible if whenever $X \simeq_{l.i.c.} Y$ then $tp_L(a_X) = tp_L(a_Y)$.

In this short note, we seek to find sufficient conditions for $\Gamma(x_\eta)_{\eta \in O}$ to be realized by an indiscernible tree.
2 Indiscernible Trees

Throughout, let \( \sigma^*: \omega^{<\omega} \to \omega^{<\omega} \) be the mapping defined by
\[
\langle m_0, \ldots, m_{n-1} \rangle \mapsto \langle 0, m_0, \ldots, 0, m_{n-1} \rangle.
\]
This \( \sigma^* \) preserves \(<_{ini} \), hence it is a tree embedding. \(<_{lex} \) is also preserved by \( \sigma^* \).

**Remark 1** Let \( \eta, \nu \) be two \(<_{ini} \)-incomparable elements. Then \( \sigma^*(\eta \cap \nu) \) is a proper initial segment of \( \sigma^*(\eta) \cap \sigma^*(\nu) \). So, \( A \) and \( \sigma^*A \) do not have the same l.i.c.-atomic type, unless \( A \) is linearly ordered.

**Definition 2** Let \( A \subset \omega^{<\omega} \) be a finite set. We say that \( A \) is a broom set if there are \( \eta_0, \ldots, \eta_{n-1} \) such that
1. \( \eta_i \cap \eta_j = \eta_i' \cap \eta_j' \) for any \( i < j < n \) and \( i' < j' < n \),
2. \( A \subset \bigcup_{i<n} \{ \eta_i | j : j \in \omega \} \).

**Lemma 3** Let \( A, B \subset \omega^{<\omega} \).
1. Suppose that \( A \) and \( B \) be broom sets. Then \( A \simeq_{l.i.c} B \Rightarrow \sigma^*A \simeq_{l.i.c} \sigma^*B \).
2. Suppose \( AC \simeq_{l.i.c} BC \), where \( A \) and \( B \) are broom sets. Suppose that for any incomparable \( \eta_1, \eta_2 \in A \) and any \( \eta \in C \), \( \eta_1 \cap \eta <_{ini} \eta_1 \cap \eta_2 \). Then \( \sigma^*(AC) \simeq_{l.i.c} \sigma^*(BC) \).
3. \( A \simeq_{l.i.c} B \Rightarrow \sigma^*A \simeq_{l.i.c} \sigma^*B \).

**Proof:** 2. We consider the most typical case, where \( A = \{ \eta_1, \eta_2, \eta_3, \nu \} \), \( C = \{ \eta \} \), \( \nu <_{ini} \eta_i \ (i = 1, 2, 3) \), \( \nu <_{ini} \eta \) and \( \eta_1 \cap \eta_2 = \eta_2 \cap \eta_3 = \eta_3 \cap \eta_1 \). The l.i.-atomic type of \( \sigma^*(A) \) is determined by this data. Moreover, we have \( \sigma^*(\nu) <_{ini} \sigma^*(\eta_i) \cap \sigma^*(\eta_j) \) for any \( i < j \), and \( \sigma^*(\nu) <_{ini} \sigma^*(\eta_i) \cap \sigma^*(\eta) \). So the l.i.c.-atomic type of \( \sigma^*(A) \) is also determined. This argument proves \( A \simeq_{l.i.c} B \Rightarrow \sigma^*A \simeq_{l.i.c} \sigma^*B \).
3. Easy by the remark above.
Now we prepare the variables \( x_{\eta} \), where \( \eta \) is a member of some fixed tree \( O \). Usually, we are interested in the case \( O = \omega^{<\omega} \). Let \( \Gamma((x_{\eta})_{\eta\in\omega^{<\omega}}) \) be a set of \( L \)-formulas with free variables from \( x_{\eta} \)'s.

**Definition 4** We say that \( \Gamma((x_{\eta})_{\eta\in\omega^{<\omega}}) \) has the subtree property if whenever \( I = (a_{\eta})_{\eta\in\omega^{<\omega}} \) realizes \( \Gamma((x_{\eta})_{\eta\in\omega^{<\omega}}) \) and \( \sigma : \omega^{<\omega} \to \omega^{<\omega} \) is a tree embedding preserving \( l.i.c.-\)structure then \( I_{\sigma} = (a_{\sigma(\eta)})_{\eta\in\omega^{<\omega}} \) realizes \( \Gamma((x_{\eta})_{\eta\in\omega^{<\omega}}) \).

**Lemma 5** Let \( \Gamma((x_{\eta})_{\eta\in\omega^{<\omega}}) \) be a consisten set having the subsequence property. Let \( \lambda \) be an infinite cardinal. Then there is a set \( J = (a_{\eta})_{\eta\in\lambda^{<\omega}} \) such that for any \( \{<\text{lex}, <\text{ini}, <\text{len}, P_{n}\} \)-embedding \( \sigma : \omega^{<\omega} \to \lambda^{<\omega} \) the set \( J_{\sigma} = (a_{\sigma(\eta)})_{\eta\in\omega^{<\omega}} \) realizes \( \Gamma((x_{\eta})_{\eta\in\omega^{<\omega}}) \).

**Proof:** For \( A, B \subset \lambda^{<\omega} \), we write \( A \simeq^{+} B \) if \( A \) and \( B \) have the same atomic type in the language \( L_{l.i.c.l} \cup \{P_{n}\}_{n\in\omega} \). We prepare new variables \( x_{\eta} \) (\( \eta \in \lambda^{<\omega} \setminus \omega^{<\omega} \)). Let \( \Gamma^{*}((x_{\eta})_{\eta\in\lambda^{<\omega}}) \) be the set obtained from \( \Gamma((x_{\eta})_{\eta\in\omega^{<\omega}}) \) by adding all formulas \( \varphi(x_{A}) \) with \( A \subset \lambda^{<\omega} \) such that \( \varphi(x_{B}) \in \Gamma((x_{\eta})_{\eta\in\omega^{<\omega}}) \) for some \( B \simeq^{+} A \). First we show

**Claim A** \( \Gamma^{*} \) is consistent.

Otherwise, there are \( \varphi_{i}(x_{A_{i}}) \) and \( B_{i} \) (\( i < n \)) such that

1. \( A_{i} \simeq^{+} B_{i} \) and \( \varphi_{i}(x_{B_{i}}) \in \Gamma((x_{\eta})_{\eta\in\omega^{<\omega}}) \) (\( i < n \)), and

2. \( \Gamma \vdash \bigvee_{i<n} \neg\varphi_{i}(x_{A_{i}}) \).

By compactness, there is a finite set \( \Gamma_{0} \subset \Gamma \) such that \( \Gamma_{0} \vdash \bigvee_{i<n} \neg\varphi_{i}(x_{A_{i}}) \). Hence, we can assume \( A_{i} \)'s are subsets of \( \omega^{<\omega} \). Let \( N = \max\{\eta(n) : \eta \in \bigcup_{i}B_{i} \setminus \omega \} \) and let \( \sigma_{N} \) be the shift function mapping \( \eta = \langle \eta(0), ..., \eta(n-1) \rangle \) to \( \langle \eta(0) + N, ..., \eta(n-1) + N \rangle \). Then, by the subtree property, we have

\[
\Gamma((x_{\eta})_{\eta\in\omega^{<\omega}}) \vdash \Gamma((x_{\sigma_{N}(\eta)})_{\eta\in\omega^{<\omega}}) \vdash \bigvee_{i<n} \neg\varphi_{i}(x_{\sigma_{N}(A_{i})}).
\]

From this, by replacing \( A_{i} \) with \( \sigma A_{i} \), we can assume that \( A_{i} \subset (\omega \setminus N)^{<\omega} \). Hence, for each \( i \), there is a tree embedding \( \sigma_{i} \) that maps \( B_{i} \) to \( A_{i} \). Choose a set \( (a_{\eta})_{\eta\in\omega^{<\omega}} \) realizing \( \Gamma \). By the property 2, there is \( i < n \) such that \( \neg\varphi(a_{A_{i}}) \) holds. On the other hand, we have \( \varphi(x_{B_{i}}) \in \Gamma \) and \( \sigma_{i}(B_{i}) = A_{i} \). Therefore, by the subtree property, we must have \( \varphi(a_{A_{i}}) \). A contradiction.
Claim B Let \((a_\eta)_\eta\) be a realization of \(\Gamma^*\). Then \((a_\eta)_\eta\) has the desired condition.

Lemma 6 Let \(\Gamma((x_\eta)_{\eta\in\omega<\omega})\) be consistent and suppose that \(\Gamma\) has the subtree property. Then \(\Gamma\) is realized by an l.i.c.-indiscernible tree.

Proof: By Theorem 2.6 of [2, AP], since the width of the tree can be made arbitrarily large, we may assume that the tree \((a_\eta)_{\eta\in\omega<\omega}\) is an indiscernible tree in Shelah’s sense. So, by Ramsey’s theorem, we can choose an indiscernible tree \(I = (a_\eta)_{\eta\in\omega<\omega}\) satisfying \(\Gamma\) such that if \(A\) and \(B\) have the same atomic type in the language \(L_{i.c.l.} = L_{i.c.} \cup \{<_{\text{len}}\}\) then \(a_A\) and \(a_B\) have the same \(L\)-type, where \(\eta <_{\text{len}} \nu\) means that the length of \(\eta\) is less than that of \(\nu\).

By compactness, we can assume that the index set of \(I\) is \(\omega^\kappa\), where \(\kappa\) is very large. By induction on \(n \in \omega\), we show that there is an l.i.-preserving mapping \(\sigma_n\) from \(\omega^{<\kappa}\) to \(I\) such that if \(\eta <_{\text{lex}} \nu\) then \(\sigma_n(\eta) <_{\text{len}} \sigma_n(\nu)\).

Suppose we have defined \(\sigma_n\). Since \(\kappa\) is sufficiently large, there is \(\kappa_0 < \kappa\) such that the lengths of \(\sigma_n(\eta)\) (\(\eta \in \text{dom}(\sigma_n)\)) are all less than \(\kappa_0\). Now we define \(\sigma_{n+1}\) by the equation

\[
\sigma_{n+1}(\langle i \rangle^\kappa_0 \eta) = (i, i_1, \ldots, \sigma_n(\eta)).
\]

This definition implies that \(\kappa_0 \cdot i \leq \text{len}(\sigma_{n+1}(\langle i \rangle^\kappa_0 \eta)) < \kappa_0 \cdot (i + 1)\). So, in particular, we have \(\text{len}(\sigma_{n+1}(\langle i \rangle^\kappa_0 \eta)) < \text{len}(\sigma_{n+1}(\langle i' \rangle^\kappa_0 \eta'))\), if \(i < i'\). By induction on the length of \(\eta\), we can prove:

Claim A \(\sigma_{n+1}(\eta^\kappa \nu) = \sigma_n(\eta)^\kappa \sigma_n(\nu)\), if \(\eta, \nu \in \text{dom}(\sigma_n)\).

So, \(\sigma_{n+1}\) preserves l.i.c.-structure of the tree. Now we show:

Claim B \(\eta <_{\text{lex}} \eta' \Rightarrow \sigma_{n+1}(\eta) <_{\text{len}} \sigma_{n+1}(\eta')\).

For proving this claim, let \(\nu = \eta \cap \eta'\). If \(\eta <_{\text{len}} \eta'\) (i.e. \(\nu = \eta\), then clearly we have \(\sigma_{n+1}(\eta) <_{\text{len}} \sigma_{n+1}(\eta')\). So we can assume \(\text{len}(\nu) > 0\), \(\eta = \nu^\langle i \rangle^\kappa_0 \eta_0\), \(\eta' = \nu^\langle i' \rangle^\kappa_0 \eta'_0\), and \(i < i'\). By Claim A, using the induction hypothesis, we have

\[
\begin{align*}
\text{len}(\sigma_{n+1}(\eta)) &= \text{len}(\sigma_n(\nu)) + \text{len}(\sigma_n(\langle i \rangle^\kappa_0 \eta_0)) \\
&= \text{len}(\sigma_n(\nu)) + \text{len}(\sigma_n(\langle i' \rangle^\kappa_0 \eta'_0)) \\
&= \text{len}(\sigma_{n+1}(\eta')).
\end{align*}
\]

Thus Claim B was shown, and \(\sigma_{n+1}\) has the required property. We have shown the existence of \(\sigma_n\)'s for all \(n\). We fix \(n\) and put \(b_\eta = a_{\sigma_n(\eta)}\). We prove:
Claim C Let $A, B \subset \text{dom}(\sigma_n)$ satisfy $A \simeq_{l.i.c} B$. Then $\text{tp}(b_A) = \text{tp}(b_B)$.

By $A \simeq_{l.i.c} B$, we have $\sigma_n(A) \simeq_{l.i.c} \sigma_n(B)$. So, by Claim B, we have
$$\sigma_n(A) \simeq_{l.i.c} \sigma_n(B).$$

By the $l.i.c.l$-indiscernibility of $I$, we have $\text{tp}(a_{\sigma_n(A)}) = \text{tp}(a_{\sigma_n(B)})$. Hence, from the definition $b_\eta = a_{\sigma_n(\eta)}$, we conclude $\text{tp}(b_A) = \text{tp}(b_B)$.

Now, by compactness and Claim C, we have the existence of $l.i.c.$-indiscernible trees realizing $\Gamma$.

**Theorem 7** Let $I = (a_\eta)_{\eta \in \omega^{<\omega}}$ be an $l.i.c.$-indiscernible tree. Let $\sigma^*$ be the mapping described before. Let $J = (b_\eta)_\eta = \sigma^*I$.

1. $J$ is an $l.i.c.$-indiscernible tree.

2. $J$ is $l.i.$-indiscernible for broom sets: Suppose $AC \simeq_{l.i.} BC$, where $A$ and $B$ are broom sets. Suppose that for any incomparable $\eta_1, \eta_2 \in A$ and any $\nu \in C$, $\eta_1 \cap \nu <_{\text{ini}} \eta_1 \cap \eta_2$. Then $\text{tp}((b_\eta)_{\eta \in AC}) = \text{tp}((b_\eta)_{\eta \in BC})$.

**Proof:**

1. Assume $A \simeq_{l.i.c} B$. Then, by Lemma 3, $\sigma^*A \simeq_{l.i.c} \sigma^*B$. By the tree indiscernibility, we have $\text{tp}((a_\eta)_{\eta \in \sigma^*A}) = \text{tp}((a_\eta)_{\eta \in \sigma^*B})$. The last equation is equivalent to
$$\text{tp}((a_{\sigma^*(\eta)})_{\eta \in A}) = \text{tp}((a_{\sigma^*(\eta)})_{\eta \in B}).$$

2. Clear by Lemma 3.

**References**
