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Trees and Branching Axioms

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1 Introduction

First we recall the definition of trees. An ordered set $O = (O, <)$ is called a tree if, for any $a \in I$, the initial segment $O_a = \{b \in O : b < a\}$ is linearly ordered. A mapping $\sigma : O \to O'$, where O and O' are trees, is called a tree embedding if σ preserves $<$-structure, i.e. $\eta < \nu$ if and only if $\sigma(\eta) <' \sigma(\nu)$. We are mainly interested in trees of the form $\alpha^{<\beta}$, where α and β are ordinals and its order is $<_{ini}$: $\eta <_{ini} \nu \iff \eta$ is a proper initial segment of ν. The lexicographic order on $\alpha^{<\beta}$ is denoted by $<_{lex}$. The meet operator $\cap$ is a binary function that gives the greatest common lower bound.

We introduce the following notations:

- $A \simeq_{l.i.} B$ for expressing that A and B have the seme $\{<_{lex}, <_{ini}\}$-atomic type.
- $A \simeq_{l.i.c.} B$ for expressing that A and B have the seme $\{<_{lex}, <_{ini}, \cap\}$-atomic type.

Now let M be an L-structure. We consider a set $A \subset M$ whose elements are indexed by a tree. So A has the form $A = (a_\eta)_{\eta \in O}$, where O is a tree. Such an indexed set is also called a tree. We introduce the notion of indiscernibility for such a tree A.

- A is $l.i$-indiscernible if whenever $X \simeq_{l.i.} Y$ then $\mathrm{tp}_L(a_X) = \mathrm{tp}_L(a_Y)$, where $a_X = (a_\eta)_{\eta \in X}$.
- A is $l.i.c$-indiscernible if whenever $X \simeq_{l.i.} Y$ then $\mathrm{tp}_L(a_X) = \mathrm{tp}_L(a_Y)$.

In this short note, we seek to find sufficient conditions for $\Gamma(x_\eta)_{\eta \in O}$ to be realized by an indiscernible tree.

2 Indiscernible Trees

Throughout, let $\sigma^* : \omega^{<\omega} \to \omega^{<\omega}$ be the mapping defined by

$$\langle m_0, \ldots, m_{n-1}\rangle \mapsto \langle 0, m_0, \ldots, 0, m_{n-1}\rangle.$$

This σ^* preserves $<_{ini}$, hence it is a tree embedding. $<_{lex}$ is also preserved by σ^*.

Remark 1 Let η, ν be two $<_{ini}$-incomparable elements. Then $\sigma^*(\eta \cap \nu)$ is a proper initial segment of $\sigma^*(\eta) \cap \sigma^*(\nu)$. So, A and $\sigma^* A$ do not have the same *l.i.c.*-atomic type, unless A is linearly ordered.

Definition 2 Let $A \subset \omega^{<\omega}$ be a finite set. We say that A is a broom set if there are $\eta_0, \ldots, \eta_{n-1}$ such that

1. $\eta_i \cap \eta_j = \eta_{i'} \cap \eta_{j'}$ for any $i < j < n$ and $i' < j' < n$,
2. $A \subset \bigcup_{i<n}\{\eta_i | j : j \in \omega\}$.

Lemma 3 *Let $A, B \subset \omega^{<\omega}$.*

1. *Suppose that A and B be broom sets. Then $A \simeq_{l.i.} B \Rightarrow \sigma^* A \simeq_{l.i.c} \sigma^* B$.*
2. *Suppose $AC \simeq_{l.i.} BC$, where A and B are broom sets. Suppose that for any incomparable $\eta_1, \eta_2 \in A$ and any $\eta \in C$, $\eta_1 \cap \eta <_{ini} \eta_1 \cap \eta_2$. Then $\sigma^*(AC) \simeq_{l.i.c} \sigma^*(BC)$.*
3. *$A \simeq_{l.i.c.} B \Rightarrow \sigma^* A \simeq_{l.i.c} \sigma^* B$.*

Proof: 2. We consider the most typical case, where $A = \{\eta_1, \eta_2, \eta_3, \nu\}$, $C = \{\eta\}$, $\nu <_{ini} \eta_i$ $(i = 1, 2, 3)$, $\nu <_{ini} \eta$ and $\eta_1 \cap \eta_2 = \eta_2 \cap \eta_3 = \eta_3 \cap \eta_1$. The *l.i.*-atomic type of $\sigma^*(A)$ is determined by this data. Moreover, we have $\sigma^*(\nu) <_{ini} \sigma^*(\eta_i) \cap \sigma^*(\eta_j)$ for any $i < j$, and $\sigma^*(\nu) <_{ini} \sigma^*(\eta_i) \cap \sigma^*(\eta)$. So the *l.i.c.*-atomic type of $\sigma^*(A)$ is also determined. This argument proves $A \simeq_{l.i.} B \Rightarrow \sigma^* A \simeq_{l.i.c} \sigma^* B$.

3. Easy by the remark above.

Now we prepare the variables x_η, where η is a member of some fixed tree O. Usually, we are interested in the case $O = \omega^{<\omega}$. Let $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ be a set of L-formulas with free variables from x_η's.

Definition 4 We say that $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ has the subtree property if whenever $I = (a_\eta)_{\eta\in\omega^{<\omega}}$ realizes $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ and $\sigma : \omega^{<\omega} \to \omega^{<\omega}$ is a tree embedding preserving $l.i.c.$-structure then $I_\sigma = (a_{\sigma(\eta)})_{\eta\in\omega^{<\omega}}$ realizes $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$.

Lemma 5 *Let $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ be a consitent set having the subsequence property. Let λ be an infinite cardinal. Then there is a set $J = (a_\eta)_{\eta\in\lambda^{<\omega}}$ such that for any $\{<_{lex}, <_{ini}, <_{len}, P_n\}$-embedding $\sigma : \omega^{<\omega} \to \lambda^{<\omega}$ the set $J_\sigma = (a_{\sigma(\eta)})_{\eta\in\omega^{<\omega}}$ realizes $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$.*

Proof: For $A, B \subset \lambda^{<\omega}$, we write $A \simeq^+ B$ if A and B have the same atomic type in the language $L_{l.i.c.l.} \cup \{P_n\}_{n\in\omega}$. We prepare new variables x_η ($\eta \in \lambda^{<\omega} \setminus \omega^{<\omega}$). Let $\Gamma^*((x_\eta)_{\eta\in\lambda^{<\omega}})$ be the set obtained from $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ by adding all formulas $\varphi(x_A)$ with $A \subset \lambda^{<\omega}$ such that $\varphi(x_B) \in \Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ for some $B \simeq^+ A$. First we show

Claim A *Γ^* is consistent.*

Otherwise, there are $\varphi_i(x_{A_i})$ and B_i ($i < n$) such that

1. $A_i \simeq^+ B_i$ and $\varphi_i(x_{B_i}) \in \Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ ($i < n$), and
2. $\Gamma \vdash \bigvee_{i<n} \neg\varphi_i(x_{A_i})$.

By compactness, there is a finite set $\Gamma_0 \subset \Gamma$ such that $\Gamma_0 \vdash \bigvee_{i<n} \neg\varphi_i(x_{A_i})$. Hence, we can assume A_i's are subsets of $\omega^{<\omega}$. Let $N = \max\{\eta(n) : \eta \in \bigcup_i B_i, n \in \omega\}$ and let σ_N be the shift function mapping $\eta = \langle\eta(0), .., \eta(n-1)\rangle$ to $\langle\eta(0) + N, ..., \eta(n-1) + N\rangle$. Then, by the subtree property, we have

$$\Gamma((x_\eta)_{\eta\in\omega^{<\omega}}) \vdash \Gamma((x_{\sigma_N(\eta)})_{\eta\in\omega^{<\omega}}) \vdash \bigvee_{i<n} \neg\varphi_i(x_{\sigma_N(A_i)}).$$

From this, by replacing A_i with σA_i, we can assume that $A_i \subset (\omega \setminus N)^{<\omega}$. Hence, for each i, there is a tree embedding σ_i that maps B_i to A_i. Choose a set $(a_\eta)_{\eta\in\omega^{<\omega}}$ realizing Γ. By the property 2, there is $i < n$ such that $\neg\varphi(a_{A_i})$ holds. On the other hand, we have $\varphi(x_{B_i}) \in \Gamma$ and $\sigma_i(B_i) = A_i$. Therefore, by the subtree property, we must have $\varphi(a_{A_i})$. A contradiction.

Claim B *Let $(a_\eta)_\eta$ be a realization of Γ^*. Then $(a_\eta)_\eta$ has the desired condition.*

Lemma 6 *Let $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ be consistent and suppose that Γ has the subtree property. Then Γ is realized by an l.i.c.-indiscernible tree.*

Proof: By Theorem 2.6 of [2, AP], since the width of the tree can be made arbitrarily large, we may assume that the tree $(a_\eta)_{\eta\in\omega^{<\omega}}$ is an indiscernible tree in Shelah's sense. So, by Ramsey's theorem, we can choose an indiscernible tree $I = (a_\eta)_{\eta\in\omega^{<\omega}}$ satisfying Γ such that if A and B have the same atomic type in the language $L_{l.i.c.l.} = L_{l.i.c.} \cup \{<_{len}\}$ then a_A and a_B have the same L-type, where $\eta <_{len} \nu$ means that the length of η is less than that of ν.

By compactness, we can assume that the index set of I is $\omega^{<\kappa}$, where κ is very large. By induction on $n \in \omega$, we show that there is an $l.i.$-preserving mapping σ_n from $\omega^{<n}$ to I such that if $\eta <_{lex} \nu$ then $\sigma_n(\eta) <_{len} \sigma_n(\nu)$.

Suppose we have defined σ_n. Since κ is sufficiently large, there is $\kappa_0 < \kappa$ such that the lengths of $\sigma_n(\eta)$ $(\eta \in \mathrm{dom}(\sigma_n))$ are all less than κ_0. Now we define σ_{n+1} by the equation

$$\sigma_{n+1}(\langle i\rangle^\frown\eta) = \underbrace{\langle i, i, \ldots\rangle}_{\kappa_0\cdot i}{}^\frown\sigma_n(\eta).$$

This definition implies that $\kappa_0 \cdot i \leq len(\sigma_{n+1}(\langle i\rangle^\frown\eta)) < \kappa_0 \cdot (i+1)$. So, in particular, we have $len(\sigma_{n+1}(\langle i\rangle^\frown\eta) < len(\sigma_{n+1}(\langle i'\rangle^\frown\eta')$, if $i < i'$. By induction on the length of η, we can prove:

Claim A $\sigma_{n+1}(\eta^\frown\nu) = \sigma_n(\eta)^\frown\sigma_n(\nu)$, *if* $\eta, \nu \in \mathrm{dom}(\sigma_n)$.

So, σ_{n+1} preserves $l.i.c.$-structure of the tree. Now we show:

Claim B $\eta <_{lex} \eta' \Rightarrow \sigma_{n+1}(\eta) <_{len} \sigma_{n+1}(\eta')$.

For proving this claim, let $\nu = \eta \cap \eta'$. If $\eta <_{len} \eta'$ (i.e. $\nu = \eta$), then clearly we have $\sigma_{n+1}(\eta) <_{len} \sigma_{n+1}(\eta')$. So we can assume $len(\nu) > 0$, $\eta = \nu^\frown\langle i\rangle^\frown\eta_0$, $\eta' = \nu^\frown\langle i'\rangle^\frown\eta_0'$, and $i < i'$. By Claim A, using the induction hypothesis, we have

$$\begin{aligned} len(\sigma_{n+1}(\eta)) &= len(\sigma_n(\nu)) + len(\sigma_n(\langle i\rangle^\frown\eta_0)) \\ &< len(\sigma_n(\nu)) + len(\sigma_n(\langle i'\rangle^\frown\eta_0')) \\ &= len(\sigma_{n+1}(\eta')). \end{aligned}$$

Thus Claim B was shown, and σ_{n+1} has the required property. We have shown the existence of σ_n's for all n. We fix n and put $b_\eta = a_{\sigma_n(\eta)}$. We prove:

Claim C *Let $A, B \subset \mathrm{dom}(\sigma_n)$ satisfy $A \simeq_{l.i.c} B$. Then* $\mathrm{tp}(b_A) = \mathrm{tp}(b_B)$.

By $A \simeq_{l.i.c.} B$, we have $\sigma_n(A) \simeq_{l.i.c.} \sigma_n(B)$. So, by Claim B, we have

$$\sigma_n(A) \underset{l.i.c.l.}{\simeq} \sigma_n(B).$$

By the $l.i.c.l$-indiscernibility of I, we have $\mathrm{tp}(a_{\sigma_n(A)}) = \mathrm{tp}(a_{\sigma_n(B)})$. Hence, from the definition $b_\eta = a_{\sigma_n(\eta)}$, we conclude $\mathrm{tp}(b_A) = \mathrm{tp}(b_B)$.

Now, by compactness and Claim C, we have the existence of $l.i.c.$-indiscernible trees realizing Γ.

Theorem 7 *Let $I = (a_\eta)_{\eta \in \omega^{<\omega}}$ be an $l.i.c.$-indiscernible tree. Let σ^* be the mapping described before. Let $J = (b_\eta)_\eta = \sigma^* I$.*

1. *J is an $l.i.c.$-indiscernible tree.*
2. *J is $l.i.$-indiscernible for broom sets: Suppose $AC \simeq_{l.i.} BC$, where A and B are broom sets. Suppose that for any incomparable $\eta_1, \eta_2 \in A$ and any $\nu \in C$, $\eta_1 \cap \nu <_{ini} \eta_1 \cap \eta_2$. Then* $\mathrm{tp}((b_\eta)_{\eta \in AC})) = \mathrm{tp}((b_\eta)_{\eta \in BC})$.

Proof: 1. Assume $A \simeq_{l.i.c.} B$. Then, by Lemma 3, $\sigma^* A \simeq_{l.i.c.} \sigma^* B$. By the tree indiscernibility, we have $\mathrm{tp}((a_\eta)_{\eta \in \sigma^* A}) = \mathrm{tp}((a_\eta)_{\eta \in \sigma^* B})$. The last equation is equivalent to

$$\mathrm{tp}((a_{\sigma^*(\eta)})_{\eta \in A}) = \mathrm{tp}((a_{\sigma^*(\eta)})_{\eta \in B}).$$

2. Clear by Lemma 3.

References

[1] Kota Takeuchi and Akito Tsuboi, On the Existence of Indiscernible Trees, submitted.

[2] Saharon Shelah, *Classification Theory and the Number of Non-Isomorphic Models*, North-Holland, 1990