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<td>Author(s)</td>
<td>Tsuboi, Akito</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1794: 50-54</td>
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<tr>
<td>Issue Date</td>
<td>2012-05</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/172870">http://hdl.handle.net/2433/172870</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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<td>Textversion</td>
<td>publisher</td>
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Trees and Branching Axioms

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1 Introduction

First we recall the definition of trees. An ordered set $O = (O, <)$ is called a tree if, for any $a \in I$, the initial segment $O_a = \{ b \in O : b < a \}$ is linearly ordered. A mapping $\sigma : O \rightarrow O'$, where $O$ and $O'$ are trees, is called a tree embedding if $\sigma$ preserves $<$-structure, i.e. $\eta < \nu$ if and only if $\sigma(\eta) <' \sigma(\nu)$. We are mainly interested in trees of the form $\alpha^{<\beta}$, where $\alpha$ and $\beta$ are ordinals and its order is $<_{ini}$: $\eta <_{ini} \nu \iff \eta$ is a proper initial segment of $\nu$. The lexicographic order on $\alpha^{<\beta}$ is denoted by $<_{lex}$. The meet operator $\cap$ is a binary function that gives the greatest common lower bound.

We introduce the following notations:

- $A \simeq_{l.i} B$ for expressing that $A$ and $B$ have the same $\{<_{lex}, <_{ini}\}$-atomic type.
- $A \simeq_{l.i.c} B$ for expressing that $A$ and $B$ have the same $\{<_{lex}, <_{ini}, \cap\}$-atomic type.

Now let $M$ be an $L$-structure. We consider a set $A \subset M$ whose elements are indexed by a tree. So $A$ has the form $A = (a_\eta)_{\eta \in O}$, where $O$ is a tree. Such an indexed set is also called a tree. We introduce the notion of indiscernibility for such a tree $A$.

- $A$ is $l.i$-indiscernible if whenever $X \simeq_{l.i} Y$ then $\text{tp}_L(a_X) = \text{tp}_L(a_Y)$, where $a_X = (a_\eta)_{\eta \in X}$.
- $A$ is $l.i.c$-indiscernible if whenever $X \simeq_{l.i} Y$ then $\text{tp}_L(a_X) = \text{tp}_L(a_Y)$.

In this short note, we seek to find sufficient conditions for $\Gamma(x_\eta)_{\eta \in O}$ to be realized by an indiscernible tree.
2 Indiscernible Trees

Throughout, let $\sigma^* : \omega^{<\omega} \rightarrow \omega^{<\omega}$ be the mapping defined by

$$\langle m_0, \ldots, m_{n-1} \rangle \mapsto \langle 0, m_0, \ldots, 0, m_{n-1} \rangle.$$  

This $\sigma^*$ preserves $<_{ini}$, hence it is a tree embedding. $<_{lex}$ is also preserved by $\sigma^*$.

Remark 1 Let $\eta, \nu$ be two $<_{ini}$-incomparable elements. Then $\sigma^*(\eta \cap \nu)$ is a proper initial segment of $\sigma^*(\eta) \cap \sigma^*(\nu)$. So, $A$ and $\sigma^*A$ do not have the same $l.i.c.$-atomic type, unless $A$ is linearly ordered.

Definition 2 Let $A \subset \omega^{<\omega}$ be a finite set. We say that $A$ is a broom set if there are $\eta_0, \ldots, \eta_{n-1}$ such that

1. $\eta_i \cap \eta_j = \eta_{i'} \cap \eta_{j'}$ for any $i < j < n$ and $i' < j' < n$,
2. $A \subset \bigcup_{i<n}\{\eta_i | j : j \in \omega\}$.

Lemma 3 Let $A, B \subset \omega^{<\omega}$.

1. Suppose that $A$ and $B$ be broom sets. Then $A \preceq_{l.i.c} B \Rightarrow \sigma^*A \simeq_{l.i.c} \sigma^*B$.
2. Suppose $AC \preceq_{l.i.c} BC$, where $A$ and $B$ are broom sets. Suppose that for any incomparable $\eta_1, \eta_2 \in A$ and any $\eta \in C$, $\eta_1 \cap \eta <_{ini} \eta_1 \cap \eta_2$. Then $\sigma^*(AC) \preceq_{l.i.c} \sigma^*(BC)$.
3. $A \preceq_{l.i.c} B \Rightarrow \sigma^*A \simeq_{l.i.e} \sigma^*B$.

Proof: 2. We consider the most typical case, where $A = \{\eta_1, \eta_2, \eta_3, \nu\}$, $C = \{\eta\}$, $\nu <_{ini} \eta_i$ ($i = 1, 2, 3$), $\nu <_{ini} \eta$ and $\eta_1 \cap \eta_2 = \eta_2 \cap \eta_3 = \eta_3 \cap \eta_1$. The $l.i.$-atomic type of $\sigma^*(A)$ is determined by this data. Moreover, we have $\sigma^*(\nu) <_{ini} \sigma^*(\eta_i) \cap \sigma^*(\eta_j)$ for any $i < j$, and $\sigma^*(\nu) <_{ini} \sigma^*(\eta_i) \cap \sigma^*(\eta)$. So the $l.i.c.$-atomic type of $\sigma^*(A)$ is also determined. This argument proves $A \preceq_{l.i.c} B \Rightarrow \sigma^*A \simeq_{l.i.c} \sigma^*B$.
3. Easy by the remark above.
Now we prepare the variables $x_{\eta}$, where $\eta$ is a member of some fixed tree $O$. Usually, we are interested in the case $O = \omega^{<\omega}$. Let $\Gamma((x_{\eta})_{\eta \in \omega^{<\omega}})$ be a set of $L$-formulas with free variables from $x_{\eta}$'s.

**Definition 4** We say that $\Gamma((x_{\eta})_{\eta \in \omega^{<\omega}})$ has the subtree property if whenever $I = (a_{\eta})_{\eta \in \omega^{<\omega}}$ realizes $\Gamma((x_{\eta})_{\eta \in \omega^{<\omega}})$ and $\sigma : \omega^{<\omega} \to \omega^{<\omega}$ is a tree embedding preserving $l.i.c.$-structure then $I_{\sigma} = (a_{\sigma(\eta)})_{\eta \in \omega^{<\omega}}$ realizes $\Gamma((x_{\eta})_{\eta \in \omega^{<\omega}})$.

**Lemma 5** Let $\Gamma((x_{\eta})_{\eta \in \omega^{<\omega}})$ be a consistent set having the subsequence property. Let $\lambda$ be an infinite cardinal. Then there is a set $J = (a_{\eta})_{\eta \in \lambda^{<\omega}}$ such that for any $\{<_{\text{lex}}, <_{\text{ini}}, <_{\text{len}}, P_{n}\}$-embedding $\sigma : \omega^{<\omega} \to \lambda^{<\omega}$ the set $J_{\sigma} = (a_{\sigma(\eta)})_{\eta \in \omega^{<\omega}}$ realizes $\Gamma((x_{\eta})_{\eta \in \omega^{<\omega}})$.

**Proof:** For $A, B \subseteq \lambda^{<\omega}$, we write $A \simeq^{+} B$ if $A$ and $B$ have the same atomic type in the language $L_{l.i.c.l} \cup \{P_{n}\}_{n \in \omega}$. We prepare new variables $x_{\eta}$ ($\eta \in \lambda^{<\omega} \setminus \omega^{<\omega}$). Let $\Gamma^{*}((x_{\eta})_{\eta \in \lambda^{<\omega}})$ be the set obtained from $\Gamma((x_{\eta})_{\eta \in \omega^{<\omega}})$ by adding all formulas $\varphi(x_{A})$ with $A \subseteq \lambda^{<\omega}$ such that $\varphi(x_{B}) \in \Gamma((x_{\eta})_{\eta \in \omega^{<\omega}})$ for some $B \simeq^{+} A$. First we show

**Claim A** $\Gamma^{*}$ is consistent.

Otherwise, there are $\varphi_{i}(x_{A_{i}})$ and $B_{i}$ ($i < n$) such that

1. $A_{i} \simeq^{+} B_{i}$ and $\varphi_{i}(x_{B_{i}}) \in \Gamma((x_{\eta})_{\eta \in \omega^{<\omega}})$ ($i < n$), and

2. $\Gamma \vdash \bigvee_{i<n} \neg \varphi_{i}(x_{A_{i}})$.

By compactness, there is a finite set $\Gamma_{0} \subseteq \Gamma$ such that $\Gamma_{0} \vdash \bigvee_{i<n} \neg \varphi_{i}(x_{A_{i}})$. Hence, we can assume $A_{i}$'s are subsets of $\omega^{<\omega}$. Let $N = \max\{\eta(\eta) : \eta \in \bigcup_{i} B_{i}, n \in \omega\}$ and let $\sigma_{N}$ be the shift function mapping $\eta = \langle \eta(0), ..., \eta(n-1) \rangle$ to $\langle \eta(0) + N, ..., \eta(n-1) + N \rangle$. Then, by the subtree property, we have

$$\Gamma((x_{\eta})_{\eta \in \omega^{<\omega}}) \vdash \Gamma((x_{\sigma_{N}(\eta)})_{\eta \in \omega^{<\omega}}) \vdash \bigvee_{i<n} \neg \varphi_{i}(x_{\sigma_{N}(A_{i})}).$$

From this, by replacing $A_{i}$ with $\sigma A_{i}$, we can assume that $A_{i} \subseteq (\omega \setminus N)^{<\omega}$. Hence, for each $i$, there is a tree embedding $\sigma_{i}$ that maps $B_{i}$ to $A_{i}$. Choose a set $(a_{\eta})_{\eta \in \omega^{<\omega}}$ realizing $\Gamma$. By the property 2, there is $i < n$ such that $\neg \varphi(a_{A_{i}})$ holds. On the other hand, we have $\varphi(x_{B_{i}}) \in \Gamma$ and $\sigma_{i}(B_{i}) = A_{i}$. Therefore, by the subtree property, we must have $\varphi(a_{A_{i}})$. A contradiction.
Claim B  Let \((a_\eta)_\eta\) be a realization of \(\Gamma^*\). Then \((a_\eta)_\eta\) has the desired condition.

Lemma 6  Let \(\Gamma((x_\eta)_{\eta \in \omega^{<\omega}})\) be consistent and suppose that \(\Gamma\) has the subtree property. Then \(\Gamma\) is realized by an l.i.c.-indiscernible tree.

Proof:  By Theorem 2.6 of [2, AP], since the width of the tree can be made arbitrarily large, we may assume that the tree \((a_\eta)_{\eta \in \omega^{<\omega}}\) is an indiscernible tree in Shelah’s sense. So, by Ramsey’s theorem, we can choose an indiscernible tree \(I = (a_\eta)_{\eta \in \omega^{<\omega}}\) satisfying \(\Gamma\) such that if \(A\) and \(B\) have the same atomic type in the language \(L_{l.i.c.l.} = L_{l.i.c.} \cup \{<_\text{len}\}\) then \(a_A\) and \(a_B\) have the same \(L\)-type, where \(\eta < \text{len} \nu\) means that the length of \(\eta\) is less than that of \(\nu\).

By compactness, we can assume that the index set of \(I\) is \(\omega^{<\kappa}\), where \(\kappa\) is very large. By induction on \(n \in \omega\), we show that there is an \(l.i.c.-\)preserving mapping \(\sigma_n\) from \(\omega^{<\kappa}\) to \(I\) such that if \(\eta < \text{lex} \nu\) then \(\sigma_n(\eta) < \text{len} \sigma_n(\nu)\).

Suppose we have defined \(\sigma_n\). Since \(\kappa\) is sufficiently large, there is \(\kappa_0 < \kappa\) such that the lengths of \(\sigma_n(\eta)(\eta \in \text{dom}(\sigma_n))\) are all less than \(\kappa_0\). Now we define \(\sigma_{n+1}\) by the equation

\[
\sigma_{n+1}((i)^\sim\eta) = \langle i, i_1, \ldots \rangle^\sim_{\kappa_0^{i+1}} \sigma_n(\eta).
\]

This definition implies that \(\kappa_0 \cdot i \leq \text{len}(\sigma_{n+1}((i)^\sim\eta)) < \kappa_0 \cdot (i + 1)\). So, in particular, we have \(\text{len}(\sigma_{n+1}((i)^\sim\eta)) < \text{len}(\sigma_{n+1}((i')^\sim\eta'))\), if \(i < i'\). By induction on the length of \(\eta\), we can prove:

Claim A  \(\sigma_{n+1}(\eta^\sim\nu) = \sigma_n(\eta)^\sim\sigma_n(\nu)\), if \(\eta, \nu \in \text{dom}(\sigma_n)\).

So, \(\sigma_{n+1}\) preserves \(l.i.c.-\)structure of the tree. Now we show:

Claim B  \(\eta < \text{lex} \eta' \Rightarrow \sigma_{n+1}(\eta) < \text{len} \sigma_{n+1}(\eta')\).

For proving this claim, let \(\nu = \eta \cap \eta'\). If \(\eta < \text{len} \eta'\) (i.e. \(\nu = \eta\), then clearly we have \(\sigma_{n+1}(\eta) < \text{len} \sigma_{n+1}(\eta')\). So we can assume \(\text{len}(\nu) > 0\), \(\eta = \nu^\sim(i)^\sim\eta_0\), \(\eta' = \nu^\sim(i')^\sim\eta_0'\), and \(i < i'\). By Claim A, using the induction hypothesis, we have

\[
\text{len}(\sigma_{n+1}(\eta)) = \text{len}(\sigma_n(\nu)) + \text{len}(\sigma_n((i)^\sim\eta_0)) < \text{len}(\sigma_n(\nu)) + \text{len}(\sigma_n((i')^\sim\eta_0')) = \text{len}(\sigma_{n+1}(\eta')).
\]

Thus Claim B was shown, and \(\sigma_{n+1}\) has the required property. We have shown the existence of \(\sigma_n\)'s for all \(n\). We fix \(n\) and put \(b_\eta = a_{\sigma_n(\eta)}\). We prove:
Claim C Let $A, B \subset \text{dom}(\sigma_n)$ satisfy $A \simeq_{l.i.c} B$. Then $\text{tp}(b_A) = \text{tp}(b_B)$.

By $A \simeq_{l.i.c} B$, we have $\sigma_n(A) \simeq_{l.i.c} \sigma_n(B)$. So, by Claim B, we have

$$\sigma_n(A) \simeq_{l.i.c.} \sigma_n(B).$$

By the $l.i.c.l$-indiscernibility of $I$, we have $\text{tp}(a_{\sigma_n(A)}) = \text{tp}(a_{\sigma_n(B)})$. Hence, from the definition $b_\eta = a_{\sigma_n(\eta)}$, we conclude $\text{tp}(b_A) = \text{tp}(b_B)$.

Now, by compactness and Claim C, we have the existence of $l.i.c.$-indiscernible trees realizing $\Gamma$.

Theorem 7 Let $I = (a_\eta)_{\eta \in \omega} < \omega$ be an $l.i.c.$-indiscernible tree. Let $\sigma^*$ be the mapping described before. Let $J = (b_\eta)_{\eta} = \sigma^*I$.

1. $J$ is an $l.i.c.$-indiscernible tree.

2. $J$ is $l.i.$-indiscernible for broom sets: Suppose $AC \simeq_{l.i.} BC$, where $A$ and $B$ are broom sets. Suppose that for any incomparable $\eta_1, \eta_2 \in A$ and any $\nu \in C$, $\eta_1 \cap \nu <_{\text{ini}} \eta_1 \cap \eta_2$. Then $\text{tp}((b_\eta)_{\eta \in AC}) = \text{tp}((b_\eta)_{\eta \in BC})$.

Proof: 1. Assume $A \simeq_{l.i.c} B$. Then, by Lemma 3, $\sigma^*A \simeq_{l.i.c.} \sigma^*B$. By the tree indiscernibility, we have $\text{tp}((a_\eta)_{\eta \in \sigma^*A}) = \text{tp}((a_\eta)_{\eta \in \sigma^*B})$. The last equation is equivalent to

$$\text{tp}((a_{\sigma^*(\eta)})_{\eta \in A}) = \text{tp}((a_{\sigma^*(\eta)})_{\eta \in B}).$$

2. Clear by Lemma 3.

References
