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<th>Trees and Branching Axioms (Model Theory of Fields and its Applications)</th>
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<tr>
<td>Author(s)</td>
<td>Tsuboi, Akito</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2012), 1794: 50-54</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2012-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/172870">http://hdl.handle.net/2433/172870</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Trees and Branching Axioms

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1 Introduction

First we recall the definition of trees. An ordered set \( O = (O, <) \) is called a tree if, for any \( a \in I \), the initial segment \( O_a = \{b \in O : b < a\} \) is linearly ordered. A mapping \( \sigma : O \to O' \), where \( O \) and \( O' \) are trees, is called a tree embedding if \( \sigma \) preserves \(<\)-structure, i.e. \( \eta < \nu \) if and only if \( \sigma(\eta) <' \sigma(\nu) \). We are mainly interested in trees of the form \( \alpha^{<\beta} \), where \( \alpha \) and \( \beta \) are ordinals and its order is \(<_{ini}: \eta <_{ini} \nu \iff \eta \) is a proper initial segment of \( \nu \). The lexicographic order on \( \alpha^{<\beta} \) is denoted by \(<_{lex} \). The meet operator \( \cap \) is a binary function that gives the greatest common lower bound.

We introduce the following notations:

- \( A \simeq_{l.i} B \) for expressing that \( A \) and \( B \) have the same \( \{<_{lex}, <_{ini}\} \)-atomic type.
- \( A \simeq_{l.i.c} B \) for expressing that \( A \) and \( B \) have the same \( \{<_{lex}, <_{ini}, \cap\} \)-atomic type.

Now let \( M \) be an \( L \)-structure. We consider a set \( A \subset M \) whose elements are indexed by a tree. So \( A \) has the form \( A = (a_{\eta})_{\eta \in O} \), where \( O \) is a tree. Such an indexed set is also called a tree. We introduce the notion of indiscernibility for such a tree \( A \).

- \( A \) is \( l.i \)-indiscernible if whenever \( X \simeq_{l.i} Y \) then \( \text{tp}_L(a_X) = \text{tp}_L(a_Y) \), where \( a_X = (a_{\eta})_{\eta \in X} \).

- \( A \) is \( l.i.c \)-indiscernible if whenever \( X \simeq_{l.i} Y \) then \( \text{tp}_L(a_X) = \text{tp}_L(a_Y) \).

In this short note, we seek to find sufficient conditions for \( \Gamma(x_{\eta})_{\eta \in O} \) to be realized by an indiscernible tree.
2 Indiscernible Trees

Throughout, let $\sigma^* : \omega^{<\omega} \rightarrow \omega^{<\omega}$ be the mapping defined by
\[
\langle m_0, \ldots, m_{n-1} \rangle \mapsto \langle 0, m_0, \ldots, 0, m_{n-1} \rangle.
\]
This $\sigma^*$ preserves $<_{ini}$, hence it is a tree embedding. $<_{lex}$ is also preserved by $\sigma^*$.

**Remark 1** Let $\eta, \nu$ be two $<_{ini}$-incomparable elements. Then $\sigma^*(\eta \cap \nu)$ is a proper initial segment of $\sigma^*(\eta) \cap \sigma^*(\nu)$. So, $A$ and $\sigma^* A$ do not have the same $l.i.c.$-atomic type, unless $A$ is linearly ordered.

**Definition 2** Let $A \subset \omega^{<\omega}$ be a finite set. We say that $A$ is a broom set if there are $\eta_0, \ldots, \eta_{n-1}$ such that
1. $\eta_i \cap \eta_j = \eta_{i'} \cap \eta_{j'}$ for any $i < j < n$ and $i' < j' < n$,
2. $A \subset \bigcup_{i<n} \{ \eta_i | j : j \in \omega \}$.

**Lemma 3** Let $A, B \subset \omega^{<\omega}$.

1. Suppose that $A$ and $B$ be broom sets. Then $A \simeq_{l.i.} B \Rightarrow \sigma^* A \simeq_{l.i.c} \sigma^* B$.

2. Suppose $AC \simeq_{l.i.} BC$, where $A$ and $B$ are broom sets. Suppose that for any incomparable $\eta_1, \eta_2 \in A$ and any $\eta \in C$, $\eta_1 \cap \eta <_{ini} \eta_1 \cap \eta_2$. Then $\sigma^*(AC) \simeq_{l.i.c} \sigma^*(BC)$.

3. $A \simeq_{l.i.c} B \Rightarrow \sigma^* A \simeq_{l.i.c} \sigma^* B$.

**Proof:**

2. We consider the most typical case, where $A = \{ \eta_1, \eta_2, \eta_3, \nu \},$ $C = \{ \eta \}, \nu <_{ini} \eta_i$ $(i = 1, 2, 3), \nu <_{ini} \eta$ and $\eta_1 \cap \eta_2 = \eta_2 \cap \eta_3 = \eta_3 \cap \eta_1$. The $l.i.$-atomic type of $\sigma^*(A)$ is determined by this data. Moreover, we have $\sigma^*(\nu) <_{ini} \sigma^*(\eta_i) \cap \sigma^*(\eta_j)$ for any $i < j$, and $\sigma^*(\nu) <_{ini} \sigma^*(\eta_i) \cap \sigma^*(\eta)$. So the $l.i.c.$-atomic type of $\sigma^*(A)$ is also determined. This argument proves $A \simeq_{l.i.} B \Rightarrow \sigma^* A \simeq_{l.i.c} \sigma^* B$.

3. Easy by the remark above.
Now we prepare the variables $x_\eta$, where $\eta$ is a member of some fixed tree $O$. Usually, we are interested in the case $O = \omega^{<\omega}$. Let $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ be a set of $L$-formulas with free variables from $x_\eta$’s.

**Definition 4** We say that $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ has the subtree property if whenever $I = (a_\eta)_{\eta\in\omega^{<\omega}}$ realizes $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ and $\sigma : \omega^{<\omega} \rightarrow \omega^{<\omega}$ is a tree embedding preserving $l.i.c.$-structure then $I_\sigma = (a_{\sigma(\eta)})_{\eta\in\omega^{<\omega}}$ realizes $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$.

**Lemma 5** Let $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ be a consisten set having the subsequence property. Let $\lambda$ be an infinite cardinal. Then there is a set $J = (a_\eta)_{\eta\in\lambda^{<\omega}}$ such that for any $\{<_{lex}, <_{ini}, <_{len}, P_n\}$-embedding $\sigma : \omega^{<\omega} \rightarrow \lambda^{<\omega}$ the set $J_\sigma = (a_{\sigma(\eta)})_{\eta\in\omega^{<\omega}}$ realizes $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$.

**Proof:** For $A, B \subset \lambda^{<\omega}$, we write $A \simeq^{+} B$ if $A$ and $B$ have the same atomic type in the language $L_{l.i.c.l} \cup \{P_n\}_{n \in \omega}$. We prepare new variables $x_\eta$ ($\eta \in \lambda^{<\omega} \setminus \omega^{<\omega}$). Let $\Gamma^*((x_\eta)_{\eta\in\lambda^{<\omega}})$ be the set obtained from $\Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ by adding all formulas $\varphi(x_A)$ with $A \subset \lambda^{<\omega}$ such that $\varphi(x_B) \in \Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ for some $B \simeq^{+} A$. First we show

**Claim A** $\Gamma^*$ is consistent.

Otherwise, there are $\varphi_i(x_{A_i})$ and $B_i$ ($i < n$) such that

1. $A_i \simeq^{+} B_i$ and $\varphi_i(x_{B_i}) \in \Gamma((x_\eta)_{\eta\in\omega^{<\omega}})$ ($i < n$), and
2. $\Gamma \vdash \forall_{i<n} \neg \varphi_i(x_{A_i})$.

By compactness, there is a finite set $\Gamma_0 \subset \Gamma$ such that $\Gamma_0 \vdash \forall_{i<n} \neg \varphi_i(x_{A_i})$. Hence, we can assume $A_i$’s are subsets of $\omega^{<\omega}$. Let $N = \max\{\eta(n) : \eta \in \bigcup_i B_i, n \in \omega\}$ and let $\sigma_N$ be the shift function mapping $\eta = \langle \eta(0), \ldots, \eta(n-1)\rangle$ to $\langle \eta(0) + N, \ldots, \eta(n-1) + N\rangle$. Then, by the subtree property, we have

$$\Gamma((x_\eta)_{\eta\in\omega^{<\omega}}) \vdash \Gamma((x_{\sigma_N(\eta)})_{\eta\in\omega^{<\omega}}) \vdash \bigvee_{i<n} \neg \varphi_i(x_{\sigma_N(A_i)}).$$

From this, by replacing $A_i$ with $\sigma A_i$, we can assume that $A_i \subset (\omega \setminus N)^{<\omega}$. Hence, for each $i$, there is a tree embedding $\sigma_i$ that maps $B_i$ to $A_i$. Choose a set $(a_\eta)_{\eta\in\omega^{<\omega}}$ realizing $\Gamma$. By the property 2, there is $i < n$ such that $\neg \varphi(a_{A_i})$ holds. On the other hand, we have $\varphi(x_{B_i}) \in \Gamma$ and $\sigma_i(B_i) = A_i$. Therefore, by the subtree property, we must have $\varphi(a_{A_i})$. A contradiction.
Claim B Let \((a_\eta)_\eta\) be a realization of \(\Gamma^*\). Then \((a_\eta)_\eta\) has the desired condition.

Lemma 6 Let \(\Gamma((x_\eta)_{\eta \in \omega < \omega})\) be consistent and suppose that \(\Gamma\) has the subtree property. Then \(\Gamma\) is realized by an l.i.c.-indiscernible tree.

Proof: By Theorem 2.6 of [2, AP], since the width of the tree can be made arbitrarily large, we may assume that the tree \((a_\eta)_{\eta \in \omega < \omega}\) is an indiscernible tree in Shelah’s sense. So, by Ramsey’s theorem, we can choose an indiscernible tree \(I = (a_\eta)_{\eta \in \omega < \omega}\) satisfying \(\Gamma\) such that if \(A\) and \(B\) have the same atomic type in the language \(L_{l.i.c.} = L_{l.i.c.} \cup \{<_{len}\}\) then \(a_A\) and \(a_B\) have the same \(L\)-type, where \(\eta <_{len}\nu\) means that the length of \(\eta\) is less than that of \(\nu\).

By compactness, we can assume that the index set of \(I\) is \(\omega^\kappa\), where \(\kappa\) is very large. By induction on \(n \in \omega\), we show that there is an \(l.i.c.-\)preserving mapping \(\sigma_n\) from \(\omega^{<\kappa}\) to \(I\) such that if \(\eta <_{lex}\nu\) then \(\sigma_n(\eta) <_{len}\sigma_n(\nu)\).

Suppose we have defined \(\sigma_n\). Since \(\kappa\) is sufficiently large, there is \(\kappa_0 < \kappa\) such that the lengths of \(\sigma_n(\eta)\) (\(\eta \in \text{dom}(\sigma_n)\)) are all less than \(\kappa_0\). Now we define \(\sigma_{n+1}\) by the equation

\[
\sigma_{n+1}((i)\sim \eta) = \langle i, i_1, \ldots \rangle \sigma_n(\eta).
\]

This definition implies that \(\kappa_0 \cdot i \leq \text{len}(\sigma_{n+1}((i)\sim \eta)) < \kappa_0 \cdot (i + 1)\). So, in particular, we have \(\text{len}(\sigma_{n+1}((i)\sim \eta)) < \text{len}(\sigma_{n+1}((i')\sim \eta'))\), if \(i < i'\). By induction on the length of \(\eta\), we can prove:

Claim A \(\sigma_{n+1}(\eta\sim \nu) = \sigma_n(\eta\sim \sigma_n(\nu))\), if \(\eta, \nu \in \text{dom}(\sigma_n)\).

So, \(\sigma_{n+1}\) preserves \(l.i.c.-\)structure of the tree. Now we show:

Claim B \(\eta <_{lex}\eta' \Rightarrow \sigma_{n+1}(\eta) <_{len}\sigma_{n+1}(\eta')\).

For proving this claim, let \(\nu = \eta \cap \eta'\). If \(\eta <_{len}\eta'\) (i.e. \(\nu = \eta\)), then clearly we have \(\sigma_{n+1}(\eta) <_{len}\sigma_{n+1}(\eta')\). So we can assume \(\text{len}(\nu) > 0\), \(\eta = \nu\sim \langle i\rangle\sim \eta_0\), \(\eta' = \nu\sim \langle i'\rangle\sim \eta'_0\), and \(i < i'\). By Claim A, using the induction hypothesis, we have

\[
\text{len}(\sigma_{n+1}(\eta)) = \text{len}(\sigma_n(\nu)) + \text{len}(\sigma_n(\langle i\rangle\sim \eta_0)) < \text{len}(\sigma_n(\nu)) + \text{len}(\sigma_n(\langle i'\rangle\sim \eta'_0)) = \text{len}(\sigma_{n+1}(\eta')).
\]

Thus Claim B was shown, and \(\sigma_{n+1}\) has the required property. We have shown the existence of \(\sigma_n\)'s for all \(n\). We fix \(n\) and put \(b_\eta = a_{\sigma_n(\eta)}\). We prove:
Claim C  Let $A, B \subset \text{dom}(\sigma_n)$ satisfy $A \simeq \iota.i.c B$. Then $\text{tp}(b_A) = \text{tp}(b_B)$.

By $A \simeq \iota.i.c B$, we have $\sigma_n(A) \simeq \iota.i.c \sigma_n(B)$. So, by Claim B, we have

$$\sigma_n(A) \simeq \iota.i.c \sigma_n(B).$$

By the $l.i.c.l$-indiscernibility of $I$, we have $\text{tp}(a_{\sigma_n(A)}) = \text{tp}(a_{\sigma_n(B)})$. Hence, from the definition $b_\eta = a_{\sigma_n(\eta)}$, we conclude $\text{tp}(b_A) = \text{tp}(b_B)$.

Now, by compactness and Claim C, we have the existence of $l.i.c.$-indiscernible trees realizing $\Gamma$.

**Theorem 7** Let $I = (a_\eta)_{\eta \in \omega} < \omega$ be an $l.i.c.$-indiscernible tree. Let $\sigma^*$ be the mapping described before. Let $J = (b_\eta)_\eta = \sigma^* I$.

1. $J$ is an $l.i.c.$-indiscernible tree.

2. $J$ is $l.i.$-indiscernible for broom sets: Suppose $AC \simeq l.i. BC$, where $A$ and $B$ are broom sets. Suppose that for any incomparable $\eta_1, \eta_2 \in A$ and any $\nu \in C$, $\eta_1 \cap \nu <_{\text{ini}} \eta_1 \cap \eta_2$. Then $\text{tp}((b_\eta)_{\eta \in AC}) = \text{tp}((b_\eta)_{\eta \in BC})$.

**Proof:** 1. Assume $A \simeq l.i.c. B$. Then, by Lemma 3, $\sigma^* A \simeq l.i.c. \sigma^* B$. By the tree indiscernibility, we have $\text{tp}((a_\eta)_{\eta \in \sigma^* A}) = \text{tp}((a_\eta)_{\eta \in \sigma^* B})$. The last equation is equivalent to

$$\text{tp}((a_{\sigma^*(\eta)})_{\eta \in A}) = \text{tp}((a_{\sigma^*(\eta)})_{\eta \in B}).$$

2. Clear by Lemma 3.

**References**
