A definition of a field for Euclid's Elements without any set theories (Model Theory of Fields and its Applications)

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A definition of a field for Euclid’s Elements without any set theories

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We introduce a definition of a field for Euclid’s Elements [E] without any set theories. Precisely, we define the non-negative part of a ordered field.

Since we never use any notion of set theory, we never say the language is \{+,-,0,1\}, but we say the symbols of binary operations “+” and “-” and the symbols of constants “0” and “1”. Since we never use any notion of set theory, we never say infinitely many variable symbols \(v_1, v_2, \ldots\), but we say variable symbols \(a, b, c, \ldots\) etc. as many as we need. because the words “infinite” and “finite” are notions of set theories. We introduce a unary predicate \(N(\bullet)\), and we say \(n\) is a natural number if \(N(n)\).

The symbol of equality is “=”, and the logical connections are “\&”, “\lor”, “\implies” and “\neg” and the quantifiers are “\forall” and “\exists”.

By usual way of BNF, we define terms, equations, formulas. They are not sets but they are on a paper, in our brain, in storages of computers, or etc.. We never say a set of formulas against model theory.

By usual way we adopt the axiom of equality — replacing the terms which are connected by “=”.

Finally, we work on the classical predicate logic. any proofs or Any deductions are never set.

Here are the definition. Every free variable is bound by universal quantifier.

\[A0 \quad N(0) \land N(1)\]
A1 \((a + b) + c = a + (b + c)\).
A3 \(a + b = b = 1\)
A4 \[P(0, a, b, c, d, e) \land \forall n[\mathbb{N}(n) \land P(n, a, b, c, d, e) \Rightarrow P(n + 1, a, b, c, d, e)]]
\Rightarrow \forall n[\mathbb{N}(n) \Rightarrow P(n, a, b, c, d, e)],\) where \(P(x, u, v, w, y, z)\) is a formula on a paper, in our brain, in storages of computers, or etc., and \(x, u, v, w, y, z\) are meta-symbols to replace terms.
A5 \(a + c = b + c \Rightarrow a = b\).
A6 \(\neg [a + 1 = 0]\).
A7 \(a \cdot 1 = a\).
A8 \((a \cdot b) \cdot c = a \cdot (b \cdot c)\).
A9 \(a \cdot b = b \cdot a\).
A10 \(a \cdot (b + c) = (a \cdot b) + (a \cdot c)\).
A11 \(\exists c [a = 0 \lor a \cdot c = b]\).
A12 \(\exists b a = b \cdot b\).
A13 \(\exists n \exists b [\mathbb{N}(n) \land a + b = n]\).
We denote \(\exists c a + c = b\) by \(a \leq b\) and denote \(\exists n [\mathbb{N}(n) \land a \cdot n = b]\) by \(a | b\).

Reference
