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THE NUMBER OF ARROWS IN THE QUIVER OF TILTING MODULES OVER A PATH ALGEBRA OF TYPE A AND D

RYOICHI KASE

Department of Pure and Applied Mathematics Graduate School of Information Science and Technology ,Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail:r-kase@cr.math.sci.osaka-u.ac.jp

INTRODUCTION

In this report we use the following notations. Let A be a finite dimensional algebra over an algebraically closed field k, and let mod-A be the category of finite dimensional right A-modules. For $M \in$mod-A we denote by $\mathrm{pd}_A M$ the projective dimension of M, and by $\mathrm{add}\, M$ the full subcategory of direct sums of direct summands of M. Let $Q=(Q_0,Q_1)$ be a finite connected quiver without loops and cycles, and Q_0 (resp.Q_1) be the set of vertices (resp.arrows) of Q (we use this notation for an arbitrary quiver). We denote by kQ the path algebra of Q over k, and by $\mathrm{rep}\, Q$ the category of finite dimensional representations of the quiver Q which is category equivalent to mod-kQ. For $M \in \mathrm{rep}\, Q$, denote by M_a the vector space of M associated to a vertex a, and denote by $M_{a\to b}$ the linear map $M_a \to M_b$ of M. For a vertex a of Q, let $\sigma_a Q$ be the quiver obtained from Q by reversing all arrows starting at a or ending at a. A module $T \in$ mod-A is called a tilting module provided the following three conditions are satisfied:
(a) $\mathrm{pd}\, T < \infty$,
(b) $\mathrm{Ext}^i(T,T)=0$ for all $i>0$,
(c) there exists an exact an sequence

$$0 \longrightarrow A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_r \longrightarrow 0 \ (T_i \in \mathrm{add}\, T)$$

in mod-A. In the hereditary case the tilting condition above is equivalent to the following:
(a) $\mathrm{Ext}^1(T,T)=0$,
(b) the number of indecomposable direct summands of T (up to isomorphism) is equal to the number of simple modules.

Then we determine the number of arrows of the tilting quiver $\vec{\mathcal{K}}(kQ)$ (c.f. section 1) for any Dynkin quiver Q of type A or D. Note that the underlying graph of $\vec{\mathcal{K}}(kQ)$ may be embeded into the exchange graph, or the cluster complex, of the corresponding cluster algebra of finite type:the tilting modules of kQ correspond to positive clusters (cf.[3] and [12]). The number of positive clusters when the orientation is alternating is given in [6, prop. 3.9]. However, the number of edges of this subdiagram of positive clusters is not known in the cluster tilting theory. Note also that if we consider the similar problem for the exchange graph, it is not interesting, because the number of edges is $\frac{n}{2}\times$(the number of vertices), and the number of vertices is given in [6, prop. 3.8]. The following is known.[6, prop. 3.9].

$$\#\vec{\mathcal{K}}(kQ)_0 = \begin{cases} \frac{1}{n+1}\binom{2n}{n} & \text{if } Q \text{ is a Dynkin quiver of type } A_n, \\ \frac{3n-4}{2n}\binom{2(n-1)}{n-1} & \text{if } Q \text{ is a Dynkin quiver of type } D_n. \end{cases}$$

The main result is as follows.

Theorem 0.1. (1) :*Let Q be a Dynkin quiver. Then $\#\vec{\mathcal{K}}(kQ)_1$ is independent of the orientation.*
(2) :

$$\#\vec{\mathcal{K}}(kQ)_1 = \begin{cases} \binom{2n-1}{n+1} & \text{if } Q \text{ is a Dynkin quiver of type } A_n, \\ (3n-4)\binom{2(n-2)}{n-3} & \text{if } Q \text{ is a Dynkin quiver of type } D_n. \end{cases}$$

1. Preliminaries

In this section we define a *tilting quiver* $\vec{\mathcal{K}}(A)$, and recall its some properties. First, for a tilting module T, we define the right perpendicular category

$$T^{\perp} = \{X \in \text{mod-}A \mid \text{Ext}_A^{>0}(T,X) = 0\}.$$

Lemma 1.1. (*cf.* [9, lemma 2.1 (a)]) *For two tilting modules T, T' the following conditions are equivalent:*
(1) : $T^{\perp} \subset T'^{\perp}$,
(2) : $T \in T'^{\perp}$.

Denote by *Tilt*(A) the set of isomorphic classes of basic tilting modules of mod-A.

Definition 1.2. We define a partial order on *Tilt*(A) by

$$T \leq T' \overset{def}{\Longleftrightarrow} T^{\perp} \subset T'^{\perp} \Longleftrightarrow T \in T'^{\perp},$$

for $T, T' \in Tilt(A)$.

1.1. A tilting quiver.

Definition 1.3. The *tilting quiver* $\vec{\mathcal{K}}(A) = (\vec{\mathcal{K}}(A)_0, \vec{\mathcal{K}}(A)_1)$ is defined as follows.
(1)$\vec{\mathcal{K}}(A)_0 = Tilt(A)$,
(2)$T' \to T$ in $\vec{\mathcal{K}}(A)$, for $T, T' \in Tilt(A)$, if $T' = M \oplus X$, $T = M \oplus Y$ with $X, Y \in \text{ind}\,A$ and there is a non-split short exact sequence

$$0 \longrightarrow X \longrightarrow \widetilde{M} \longrightarrow Y \longrightarrow 0$$

with $\widetilde{M} \in \text{add}\,M$.

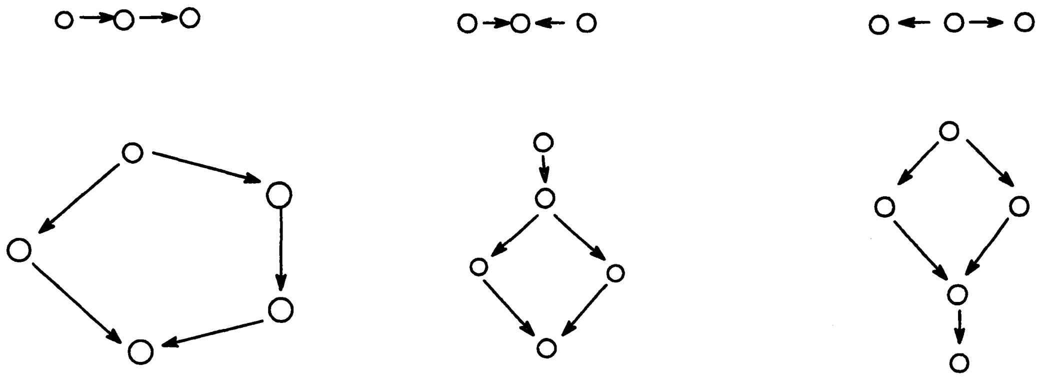
Theorem 1.4. (*cf.* [8, thm 2.1]) *$\vec{\mathcal{K}}(A)$ is the Hasse-diagram of $(Tilt(A), \leq)$ (i.e. if $T \to T' \in \vec{\mathcal{K}}(A)_1$ and $T \geq T'' \geq T'$ then $T'' = T$ or $T'' = T'$).*

Proposition 1.5. (*cf.*[8, cor 2.2]) *If $\vec{\mathcal{K}}(A)$ has a finite component $\mathcal{C}$, then $\vec{\mathcal{K}}(A) = \mathcal{C}$.*

Example 1.6. (A_3)

Q

$\vec{\mathcal{K}}(Q)$



1.2. **A local structure of a tilting quiver.** Let $Q = (Q_0, Q_1)$ be a quiver without loops and cycles and $A = kQ$. For $T \in Tilt(A)$, let

$$\begin{aligned} s(T) &= \#\{T' \in Tilt(A) \mid T \to T' \text{ in } \vec{\mathcal{K}}(A)\} \\ e(T) &= \#\{T' \in Tilt(A) \mid T' \to T \text{ in } \vec{\mathcal{K}}(A)\} \end{aligned}$$

and define $\delta(T) = s(T) + e(T)$. Then the following is key proposition for proof of main result.

Proposition 1.7. (*cf.*[10, prop 3.2]) *$\delta(T) = n - \#\{a \in Q_0 \mid (\underline{\dim} T)_a = 1\}$, where $n = \#Q_0$.*

2. A theorem of Ladkani

In this section, we review [11]. Let Q be a quiver without loops and cycles, x be a source of Q and $Q' = \sigma_x Q$. Let $Tilt(Q) := Tilt(kQ)$ and define

$$Tilt(Q)^x := \{T \in Tilt(Q) \mid S(x) \mid T\},$$

where $S(x)$ is the simple module associated to x.

Definition 2.1. Let $(X, \leq_X)$,$(Y, \leq_Y)$ be two posets and $f : X \to Y$ an order-preserving function. Then we define two partial orders $\leq^f_+$, $\leq^f_-$ of $X \sqcup Y$ as follows.

$$a \leq^f_+ b \Longleftrightarrow \begin{cases} a \leq_X b & \text{if } a, b \in X, \\ a \leq_Y b & \text{if } a, b \in Y, \\ f(a) \leq_Y b & \text{if } a \in X \text{ and } b \in Y. \end{cases}$$

$$a \leq^f_- b \Longleftrightarrow \begin{cases} a \leq_X b & \text{if } a, b \in X, \\ a \leq_Y b & \text{if } a, b \in Y, \\ a \leq_Y f(b) & \text{if } a \in Y \text{ and } b \in X. \end{cases}$$

Define $j_* : \mathrm{rep}(Q \setminus \{x\}) \to \mathrm{rep}Q$ and $j'_* : \mathrm{rep}(Q \setminus \{x\}) \to \mathrm{rep}Q'$ as follows

$$(j_*N)_a = \begin{cases} N_a & (a \neq x) \\ \oplus_{x \to y} N(y) & (a = x) \end{cases}, \quad (j_*N)_{a \to b} = \begin{cases} N_{a \to b} & (a \neq x) \\ (j_*N)_x \overset{projection}{\longrightarrow} N_b & (a = x) \end{cases}.$$

$$(j'_*N)_a = \begin{cases} N_a & (a \neq x) \\ \oplus_{y \to x} N(y) & (a = x) \end{cases}, \quad (j'_*N)_{a \to b} = \begin{cases} N_{a \to b} & (b \neq x) \\ (j'_*N)_a \overset{injection}{\longrightarrow} N_x & (b = x) \end{cases}.$$

Theorem 2.2.

$$\iota_x : T \mapsto S(x) \oplus j_*T$$

and

$$\iota'_x : T' \mapsto S'(x) \oplus j'_*T'$$

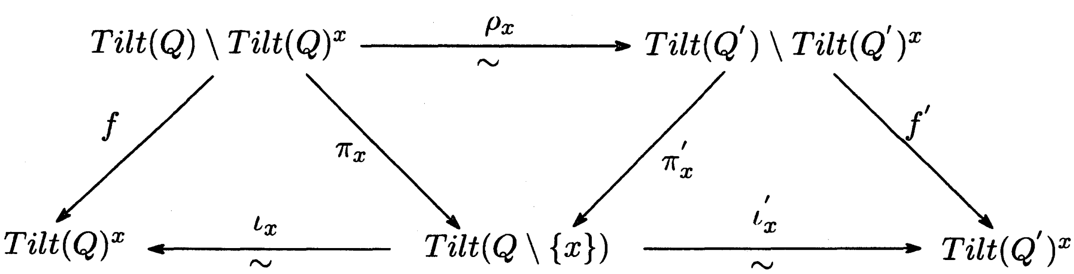
induce the order-preserving functions

$$(Tilt(Q \setminus \{x\}), \leq) \to (Tilt(Q), \leq),$$

and

$$(Tilt(Q \setminus \{x\}), \leq) \to (Tilt(Q'), \leq).$$

More over there is a commutative diagram of the posets



with

$$(Tilt(Q), \leq) \simeq (Tilt(Q) \setminus Tilt(Q)^x \sqcup Tilt(Q)^x, \leq^f_-),$$

and

$$(Tilt(Q'), \leq) \simeq (Tilt(Q') \setminus Tilt(Q')^x \sqcup Tilt(Q')^x, \leq^{f'}_+).$$

In particular if Q is a Dynkin quiver, then $\#Tilt(Q)$ is independent of an orientation.

Remark 2.3. In [11] the partial order on $Tilt(A)$ is defined by

$$T \geq T' \iff T^\perp \subset T'^\perp \text{ (opposite to our definition).}$$

3. Main results

In this section we determine the number of arrows of $\vec{\mathcal{K}}(Q)$ in the case Q is a Dynkin quiver of type A or D. Then, from the facts of section 2, we get following lemma.

Lemma 3.1. *If x is a sink then*

$$\{\alpha \in \vec{\mathcal{K}}(Q)_1 \mid s(\alpha) \in Tilt(Q)^x, t(\alpha) \in Tilt(Q) \setminus Tilt(Q)^x\} \overset{1:1}{\longleftrightarrow} Tilt(Q)^x.$$

If x is a source then

$$\{\alpha \in \vec{\mathcal{K}}(Q)_1 \mid t(\alpha) \in Tilt(Q)^x, s(\alpha) \in Tilt(Q) \setminus Tilt(Q)^x\} \overset{1:1}{\longleftrightarrow} Tilt(Q)^x.$$

Where, for $T \overset{\alpha}{\to} T'$, $s(\alpha) = T$ and $t(\alpha) = T'$.

Corollary 3.2.

$$\#\vec{\mathcal{K}}(Q)_1 = \#\vec{\mathcal{K}}(Q')_1.$$

In particular, if Q is a Dynkin quiver then $\#\vec{\mathcal{K}}(Q)_1$ depends only on the underlying graph of Q.

Proof. By Theorem2.2 and lemma3.1,

$$\begin{aligned}\#\vec{\mathcal{K}}(Q)_1 &= \#\vec{\mathcal{K}}(Q \setminus \{x\})_1 + \#\vec{\mathcal{K}}(Tilt(Q) \setminus Tilt(Q)^x)_1 + \#Tilt(Q)^x \\ &= \#\vec{\mathcal{K}}(Q')_1.\end{aligned}$$

□

3.1. **case** A. In this subsection we consider the quiver,

$$(\vec{A}_n =)Q = \overset{1}{\circ}\to\overset{2}{\circ}\to \cdots \to\overset{n}{\circ}.$$

By Gabriel's theorem, $\operatorname{ind} kQ = \{L(i,j) \mid 0 \le i < j \le n\}$ where

$$L(i,j) = \begin{cases} k & (i < a \le j), \\ 0 & \text{otherwise}, \end{cases} \quad L(i,j)_{a \to b} = \begin{cases} 1 & (i < a, b \le j), \\ 0 & \text{otherwise}. \end{cases}$$

And

$$\tau L(i,j) = \begin{cases} L(i+1, j+1) & (j < n), \\ 0 & (j = n), \end{cases}$$

where τ is a Auslander-Reiten translation.

Definition 3.3. A pair of intervals $([i,j],[i',j'])$ is *compatible* if

$$[i,j] \cap [i',j'] = \emptyset \text{ or } [i,j] \subset [i',j'] \text{ or } [i',j'] \subset [i,j].$$

Applying Auslander-Reiten duality,

$$\mathrm{DExt}(M,N) \cong \mathrm{Hom}(N, \tau M) \ (\mathrm{D} = \mathrm{Hom}_k(-,k)),$$

we get the following lemma.

Lemma 3.4. *We have*

$$\mathrm{Ext}(L(i,j), L(i',j')) = 0 = \mathrm{Ext}(L(i',j'), L(i,j))$$

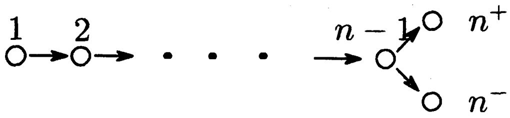
if and only if $([i,j],[i',j'])$ is compatible.

Lemma 3.5. *For any $T \in Tilt(Q)$, we get $\delta(T) = n - 1$.*

Now it is easy to check the number of arrows in $\vec{\mathcal{K}}(Q)$, because it is equal to $\frac{1}{2}\sum_{T \in \mathrm{Tilt(Q)}} \delta(T)$

Corollary 3.6. $\#\vec{\mathcal{K}}(Q)_1 = \frac{n-1}{2(n+1)}\binom{2n}{n} = \binom{2n-1}{n-2}$.

3.2. case D. Through this subsection, we consider the quiver

$Q = Q_n =$ 

Then $\operatorname{ind} kQ = \{L(a,b) \mid 0 \le a < b \le n-1\}$
$\cup \{L^{\pm}(a,n) \mid 0 \le a \le n-1\} \cup \{M(a,b) \mid 0 \le a < b \le n-1\}$

where

$$L(a,b)_i = \begin{cases} k & \text{if } a < i \le b, \\ 0 & \text{otherwise,} \end{cases}$$

$$L(a,b)_{i \to j} = \begin{cases} 1 & \text{if } a < i < b, \\ 0 & \text{otherwise,} \end{cases}$$

$$L(a,n)^{\pm}_i = \begin{cases} k & \text{if } a < i \le n-1 \textit{ or } i = n^{\pm}, \\ 0 & \text{otherwise,} \end{cases}$$

$$L(a,n)^{\pm}_{i \to j} = \begin{cases} 1 & \text{if } a < i < n-1 \textit{ or } i = n-1, j = n^{\pm}, \\ 0 & \text{otherwise,} \end{cases}$$

$$M(a,b)_i = \begin{cases} k & \text{if } a < i \le b \text{ or } i = n^{\pm}, \\ k^2 & \text{if } b < i \le n-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$M(a,b)_{i \to j} = \begin{cases} 1 & \text{if } a < i < b, \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } i = b, \\ (1,0) & \text{if } i = n-1, j = n^+, \\ (0,1) & \text{if } i = n-1, j = n^-, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } b < i < n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\tau L(a,b) = \begin{cases} L(a+1,b+1) & \text{if } b < n-1, \\ M(0,a+1) & \text{if } b = n-1, \end{cases}$$

$\tau L^+(a,n) = L^-(a+1,n),$
$\tau L^-(a,n) = L^+(a+1,n),$

$$\tau M(a,b) = \begin{cases} M(a+1,b+1) & \text{if } b < n-1, \\ 0 & \text{if } b = n-1. \end{cases}$$

Then the Ext $= 0$ conditions are as follows.

Lemma 3.7.

(1) $\mathrm{Ext}(L(a,b), L(a',b')) = 0 = \mathrm{Ext}(L(a',b'), L(a,b))$

$\iff ([a,b],[a',b'])$: compatible.

(2) $\mathrm{Ext}(L(a,b), L^{\pm}(a',n)) = 0 = \mathrm{Ext}(L^{\pm}(a',n), L(a,b))$

$\iff ([a,b],[a',n])$: compatible.

(3) $\mathrm{Ext}(L(a,b), M(a',b')) = 0 = \mathrm{Ext}(M(a',b'), L(a,b))$

$\iff ([a,b],[a',n]), ([a,b],[b',n])$: compatible.

(4) $\mathrm{Ext}(M(a,b), L^{\pm}(a',n)) = 0 = \mathrm{Ext}(L^{\pm}(a',n), M(a,b))$

$\iff a \le a' \le b$.

(5) $\mathrm{Ext}(L^{\pm}(a,n), L^{\pm}(a',n)) = 0 = \mathrm{Ext}(L^{\pm}(a',n), L^{\pm}(a,n))$ for all a, a'.

(6) $\mathrm{Ext}(L^{+}(a,n), L^{-}(a',n)) = 0 = \mathrm{Ext}(L^{-}(a',n), L^{+}(a,n))$

$\iff a = a'$.

(7) $\mathrm{Ext}(M(a,b), M(a',b')) = 0 = \mathrm{Ext}(M(a',b'), M(a,b))$

$\iff [a,b] \subset [a',b']$ *or* $[a',b'] \subset [a,b]$.

Lemma 3.8. *Let $T \in Tilt(Q)$ then $\delta(T) \ge n-1$, and $\delta(T) = n-1$ if and only if $L^{\pm}(0,n) \mid T$ and other indecomposable direct summands of T have the form $L(a,b) (0 \le a < b \le n-1)$. In particular,*

$$\#\{T \in Tilt(kQ) \mid \delta(T) = n-1\} = \frac{1}{n}\binom{2(n-1)}{n-1} = \frac{1}{n-1}\binom{2(n-1)}{n-2}.$$

Now we define subsets of $Tilt(Q)$ by

$$\begin{aligned} \mathcal{T}_0 &:= \{T \in Tilt(Q) \mid \delta(T) = n+1\}, \\ \mathcal{T}_1 &:= \{T \in Tilt(Q) \mid \delta(T) = n\}, \\ \mathcal{T}_2 &:= \{T \in Tilt(Q) \mid \delta(T) = n-1\}. \end{aligned}$$

The above lemma shows that

$$\#\mathcal{T}_2 = \frac{1}{n}\binom{2(n-1)}{n-1} = \frac{1}{n-1}\binom{2(n-1)}{n-2}.$$

Let us define the following subsets of $\mathcal{T}_1$:

$$\mathcal{A}_i := \{T \in \mathcal{T}_1 \mid (\underline{\dim} T)_i = 1\}.$$

Then we get the following correspondences.

Lemma 3.9.

$$(1): \ \mathcal{A}_{n^\pm} \ \overset{1:1}{\longleftrightarrow} \ Tilt(\vec{A}_n) \setminus Tilt(\vec{A}_{n-1}).$$
$$(2): \ \mathcal{A}_i \ \overset{1:1}{\longleftrightarrow} \ Tilt(\vec{A}_{i-1}) \times Tilt(Q_{n-1}) \ (i \neq n^\pm).$$

Remark 3.10. In (1) we identify $\{T' \in Tilt(\vec{A}_n) \mid L(0, n-1) \mid T'\}$ with $Tilt(\vec{A}_{n-1})$.

By lemma 3.9, we can calculate $\#\mathcal{T}_1$ and $\#\mathcal{T}_0$.

Corollary 3.11. *we have*

$$\#\mathcal{T}_1 = 3\binom{2(n-1)}{n-2}, \ \#\mathcal{T}_0 = \frac{3(n-1)}{n+1}\binom{2(n-1)}{n-2}.$$

Theorem 3.12.

$$\#\vec{\mathcal{K}}(Q)_1 = (3n-1)\binom{2(n-1)}{n-2}.$$

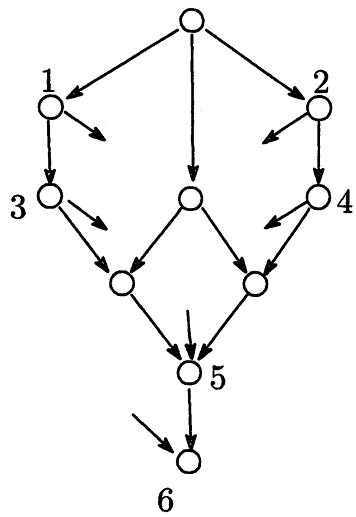
Proof. In fact, $\#\vec{\mathcal{K}}(Q)_1$ is equal to

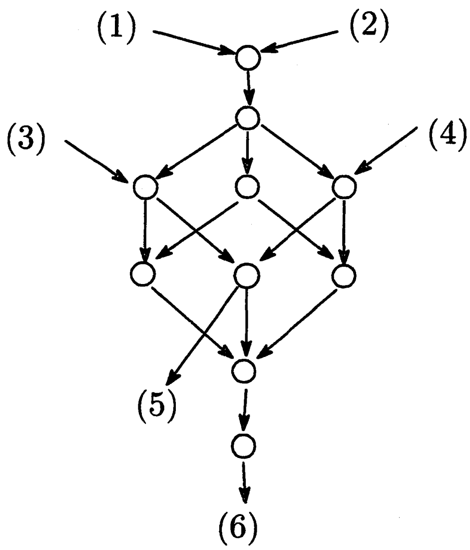
$$\frac{1}{2}\left\{\frac{n-1}{n-1}\binom{2(n-1)}{n-2} + 3n\binom{2(n-1)}{n-2} + 3(n-1)\binom{2(n-1)}{n-2}\right\}$$

$$=(3n-1)\binom{2(n-1)}{n-2}.$$

□

Example 3.13. (D_4)





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