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Mapping class group, Donaldson-Thomas theory and S-duality

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Abstract

We study Donaldson-Thomas theory associated to a triangulated surface. We show that the generating function of the Donaldson-Thomas invariants is "invariant" under an action of the mapping class group, which is identified with the mapping class group action in the (decorated) Teichmüller theory. This gives an example of constraints of the generating function induced by the derived auto-equivalences. From the view point of string theory, this is nothing but S-duality of the BPS spectrum of the 4d gauge theory given by Gaiotto-type construction.

Introduction

The DT invariant for a Calabi-Yau 3-fold Y is a counting invariant of coherent sheaves on Y, which is introduced in [Tho00] as a holomorphic analogue of the Casson invariant on a real 3-manifold. Although the category of coherent sheaves on Y is an Abelian category, it has been known that we take it as a counting invariant of objects in the derived category.

An ideal application of this formulation might be the following: The derived category sometimes have a non-trivial auto-equivalence group. In such a case, the generating function might have a good transformation formula with respect to this action, which would help us to determine the generating function.

In this notes, we will show a new example[1] of such a phenomenon. We study Donaldson-Thomas theory associated to a triangulated surface. The mapping class group acts on the derived category and the generating function of the Donaldson-Thomas invariants is "invariant" under this action.

[1]As far as the author understand, it is the first example.

Plan

In §1, we briefly review the construction in [FST08. LF09] of quivers with potential associated to triangulated surfaces. In §2, we study the mapping class group actions on the derived category and the associated Poisson torus. The later is identified with the mapping class group action on the decorated Teichmüller space as is shown in §5. The main result of this paper appears In §3. Finally, we explain an interpretation of the main result in terms of S-duality ([Gai]) in §4.

Acknowledgement

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1 QP for a triangulated surface

In this section, we briefly explain how to associate a quiver with a potential for a triangulated surface [FST08, LF09].

1.1 Ideal triangulations of a surface

Let Σ be a compact connected oriented surface with (possibly non-empty) boundary and M be a finite set of points on Σ. called marked points. We assume that M is non-empty and has at least one point on each connected component of the boundary of Σ. The marked points that lie in the interior of Σ will be called **punctures**, and the set of punctures of (Σ, M) will be denoted P. [2]

We decompose Σ into "triangles" (in the topological sense) so that each edge is either

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[2]We will always assume that (Σ, M) is none of the following:

- a sphere with less than five punctures;
- an unpunctured monogon, digon or triangle;
- a once-punctured monogon.

Here, by a monogon (resp. digon, triangle) we mean a disk with exactly one (resp. two, three) marked point(s) on the boundary.

- a curve (which is called an *arc*) whose endpoints are in M or
- a connected component of $\partial\Sigma\backslash M$.

A triangle may contains exactly two arcs (see Figure 1). Such a triangle (and its doubled arc) is said to be *self-folded.*

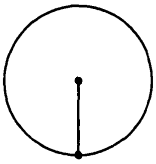

Figure 1: A self-folded triangle

Given a triangulation τ and a (non self-folded) arc i, we can *flip* i to get a new triangulation $f_i(\tau)$ (see Figure 2).

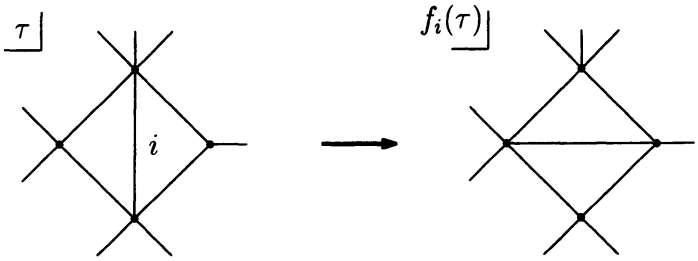

Figure 2: A flip of a triangulation

Theorem 1.1 ([FST08]). *Any two triangulations are related by a sequence of flips.*

1.2 Quiver for a triangulation

Let τ be a triangulation. We will define a quiver $Q(\tau)$ without loops and 2-cycles whose vertex set I is the set of arcs in τ.

Let $\pi\colon \tau_1 \to \tau_1$ be the map which is the identity on the set of non-self-folded arcs and sends the a self-folded arc to the unique loop of τ enclosing it.

For a triangle Δ and arcs i and j, we define a skew-symmetric integer matrix B^Δ by

$$B^\Delta_{i,j} := \begin{cases} 1 & \Delta \text{ has sides } \pi(i) \text{ and } \pi(j), \text{ with } \pi(i) \text{ following } \pi(j) \text{ in the clockwise order,} \\ -1 & \text{the same holds, but in the counter-clockwise order,} \\ 0 & \text{otherwise.} \end{cases}$$

We put

$$B(\tau) := \sum_{\Delta} B^{\Delta}$$

where the sum is taken over all triangles in τ. Let $Q(\tau)$ denote the quiver without loops and 2-cycles associated to the matrix $B(\tau)$.

Theorem 1.2 ([FST08]). *Given a triangulation τ and its (non self-folded) arc i, we have*

$$Q(f_i(r)) = \mu_i(Q(\tau))$$

where μ_i denote the mutation of the quiver at the vertex i.

1.3 Potential for a triangulation

For a triangle Δ in τ, we define a potential ω_Δ as in Figure 3. For a puncture

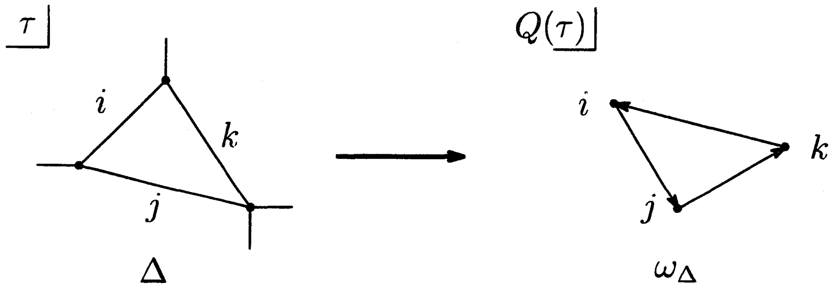

Figure 3: ω_Δ

P in τ, we define a potential ω_P as in Figure 4. We omit the definitions of

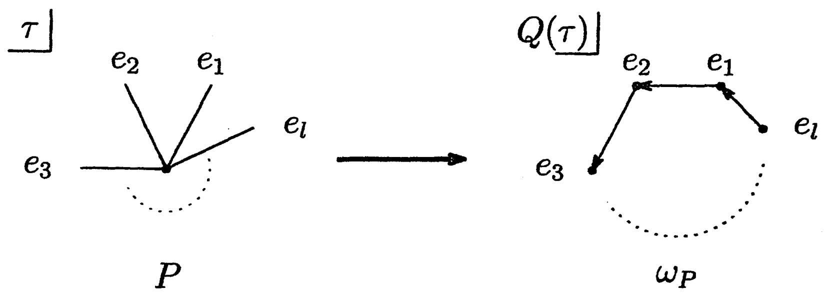

Figure 4: ω_P

ω_Δ and ω_P in the cases when self-folded arcs appear (see [LF09, §3]).

Finally, we put

$$\omega(\tau) := \sum_{\Sigma} \omega_\Sigma + \sum_{P} \omega_P.$$

Theorem 1.3 ([LF09]). *Given a triangulation τ and its (non self-folded) arc i, we have*

$$\omega(f_i(r)) = \mu_i(\omega(\tau))$$

where μ_i denote the mutation of the potential at the vertex i in the sense of [DWZ08].

2 Mapping class group action

2.1 Mapping class group

We define

$$\mathrm{Diffeo}(\Sigma, M) := \{\phi\colon \Sigma \to \Sigma \mid \phi\colon \text{diffeomorphism}, \phi(M) = M\}.$$

Let $\mathrm{Diffeo}(\Sigma, M)_0$ denote the connected component of $\mathrm{Diffeo}(\Sigma, M)$ which contains id_Σ. The quotient

$$\mathrm{MCG}(\Sigma, M) := \mathrm{Diffeo}(\Sigma, M)/\mathrm{Diffeo}(\Sigma, M)_0$$

is called the *mapping class group*.

2.2 Derived category for a triangulation

Let $\Gamma(\tau)$ be Ginzburg's dg algebra associated to the quiver with the potential $(Q(\tau), \omega(\tau))$ and $\mathcal{D}(\tau) = \mathcal{D}\Gamma(\tau)$ be the derived category of right dg-modules over Γ. By the result of Keller ([Kel11]), $\Gamma(\tau)$ and $\Gamma(f_i(\tau))$ are equivalent[3].

For a triangulation τ and an element $\phi \in \mathrm{MCG}(\Sigma, M)$, we get another triangulation $\phi(\tau)$. Note that $(Q(\tau), \omega(\tau))$ and $(Q(\phi(\tau)), \omega(\phi(\tau)))$ (and hence $\mathcal{D}(\tau)$ and $\mathcal{D}(\phi(\tau))$) are canonically identified.

By Theorem 1.1, τ and $\phi(\tau)$ are related by a sequence of flips. Each flips gives a derived equivalence. By composing the derived equivalences, we get a derived equivalence

$$\Psi_\phi\colon \mathcal{D}(\tau) \xrightarrow{\sim} \mathcal{D}(\phi(\tau)) = \mathcal{D}(\tau).$$

Thanks to the result [FST08, Theorem 3.10] and the pentagonal identity for the derived equivalences, Ψ_ϕ is independent of choices of a sequence of flips and well-defined. Finally we get an action of the mapping class group on the derived category:

$$\Psi\colon \mathrm{MCG}(\Sigma, M) \to \mathrm{Aut}(\mathcal{D}(\tau)).$$

[3]Since we have two derived equivalences, we have to choose one of them. Given a sequence of flips, we have a canonical choice. See [Nag, §2.2].

2.3 Cluster transformation

We put $T = T(\tau) := \mathbb{C}[x_i, x_i^{-1}]_{i\in I}$. We define $\mathrm{CT}_k\colon T(f_k(\tau)) \xrightarrow{\sim} T(\tau)$ by

$$\mathrm{CT}_k(x_i') = \begin{cases} (x_k)^{-1}\left(\prod(x_j)^{Q(j,k)} + \prod(x_j)^{Q(k,j)}\right) & i = k, \\ x_i & i \neq k \end{cases}$$

where $Q(i,k)$ is the number of arrows from i to k and x_i' is the generator of $T(f_k(\tau))$.

In the same way as the previous section, we get

$$\mathrm{CT}_\phi\colon T(\phi(\tau)) \xrightarrow{\sim} T(\tau).$$

Under the identification $T(\phi(\tau)) = T(\tau)$ induced by Ψ_ϕ, we get

$$\mathrm{CT}\colon \mathrm{MCG}(\Sigma, M) \to \mathrm{Aut}(T(\tau)).$$

Remark 2.1. *As we will explain in §5, this is compatible with the action of mapping class group on the decorated Teichmüller space.*

3 Donaldson-Thomas theory

Let J_τ be the Jacobi algebra associated to the quiver with the potential $(Q(\tau), W(\tau))$.

Let P_τ^i be the indecomposable projective J_τ-module associated to $i \in I$. For $\mathbf{v} \in \mathbb{Z}_{\geq 0}^I$, we define

$$\mathrm{Hilb}_\tau^i(\mathbf{v}) := \{P_\tau^i \twoheadrightarrow V \mid \underline{\dim} V = \mathbf{v}\}.$$

This is called the *Hilbert scheme*[4].

Definition 3.1. *We define* $\mathrm{DT}_\tau\colon T \xrightarrow{\sim} T$ *by*

$$\mathrm{DT}_\tau(x_i) := (x_i)^{-1} \cdot \sum_{\mathbf{v}} \mathrm{Eu}(\mathrm{Hilb}_\tau^i(\mathbf{v})) \cdot y^{-\mathbf{v}}$$

where

$$y^{-\mathbf{v}} := \prod_i (y_i)^{-v_i}, \quad y_i := \prod_j (x_i)^{Q(i,j)}.$$

As a direct application of the main theorem in [Nag], we get the following:

Theorem 3.2. *For any element* $\phi \in \mathrm{MCG}(\Sigma, M)$*, we have*

$$\mathrm{DT}_\tau \circ \mathrm{CT}_\phi = \mathrm{CT}_\phi \circ \mathrm{DT}_\tau.$$

[4]The name comes from the Hilbert scheme in algebraic geometry which parameterizes quotient sheaves of the structure sheaf.

4 S-duality interpretation

4.1 Gaiotto functor

Let $\mathcal{F}$ is an n-dimensional quantum field theory. Then for any fixed k-dimensional manifold K, the correspondence

$$M \longmapsto \mathcal{F}(K \times M)$$

provides an $(n-k)$-dimensional quantum field theory.

We take a 6d $\mathcal{N} = (2,0)$ quantum field theory S_G, where G is compact Lie group of type ADE. Fixing a Riemann surface C, we get a 4d $\mathcal{N} = 2$ theory by the construction above. Let $S_{G,C}$ denote this theory ([Gai]).

In summary, 6d $\mathcal{N} = (2,0)$ theory provides the following correspondence :

$$\begin{array}{ccc} \{\text{Riemann surfaces}\} & \longrightarrow & \{4\text{d}\ \mathcal{N} = 2\ \text{QFT}\} \\ C & \longmapsto & S_{G,C}. \end{array}$$

Following Y. Tachikawa, we call this "Gaiotto functor"[5] (see [MT]).

4.2 4d BPS spectrum

In this paper, we have studied the DT theory associated to a triangulation of a surface C. It is expected that the generating function provides the BPS spectrum of the 4d QFT $S_{\mathrm{SU}(2),C}$ ([GMN, ACC$^+$a, ACC$^+$b]). [6]

Remark 4.1. *For BPS spectrum of the 4d QFT, there should be a wall-crossing theory which is compatible with those of DT theory under the expectation above [GMN, Moo].*

4.3 S-duality

Fixing a topological type of a 2-dimensional manifold, the Teichmüller space is the space of complex structures on it. The mapping class group acts on the Teichmüller space so that the quotient space gives the moduli space of complex structures.

Under Gaiotto functor, Teichmüller space should give the space of parameters[7] of 4-dimensional quantum field theories. Since two points on a mapping class group orbit in the Teichmüller space give a common complex

[5] This is not a functor of categories in mathematical sense.

[6] For $S_{\mathrm{SU}(N),C}$, we need to take the triangulations which appear in the higher Teichmüller space ([FG06]).

[7] parameter space of exactly marginal gauge couplings.

structure, they provide a common 4d theory. This is the S-duality in the sense of Gaiotto ([Gai]).

Remark 4.2. *The original S-duality is the duality between strongly/weakly coupled regions in the space of parameters. Strongly or weakly coupled regions appear as neighborhoods of cusps in the fundamental region.*

Combining the observations in §4.2, we get an interpretation of Theorem 3.2 as the S-duality on the BPS spectrum.

Remark 4.3. *In this paper, we understand the mapping class group action in DT theory via wall-crossing. We can understand the S-duality as a consequence of wall-crossing of 4d QFT, without passing through DT theory (Figure 5).*

$$\begin{array}{ccc} \boxed{4d \text{ QFT } S_{\mathrm{SU}(2),C}} & = & \boxed{\text{DT for a triangutlation}} \\ \vdots & & \Big| \\ \text{(wall-crossing)} & = & \text{(wall-crossing)} \\ \vdots & & \downarrow \\ \boxed{\text{S-duality}} & = & \boxed{\text{MCG action}} \end{array}$$

Figure 5: Summary

5 Appendix : Teichmüller theory

Let $\mathcal{T}(\Sigma)$ denote the Teichmüller space and $\widetilde{\mathcal{T}}(\Sigma)$ denote the decorated Teichmüller space, which is a $(\mathbb{R}_{>0})^s$-bundle over $\mathcal{T}(\Sigma)$ whose fiber is the set of s-tuples of horocycles around each of the marked points ([Pen87, Pen92]).

We assume that a triangulation τ does not contain self-folded arcs. Let τ_1 be the set of edges of a triangulation τ. Each edge e in τ_1, we take the (unique) geodesic represents e. The coordinate $l_e(P)$ is defined as the hyperbolic length of the segment of the geodesic that lies between the two horocycles surrounding the punctures connected by e, taken with positive sign if the two horocycles are disjoint, with negative sign otherwise.

Theorem 5.1. *[Pen87, Pen92]*

(1) *For an ideal triangulation τ without self-folded arcs, the function*

$$\vec{l}: \ \widetilde{\mathcal{T}}(\Sigma) \to \mathbb{R}^{\tau_1}, \quad P \mapsto (l_e(P))_{e\in\tau_1}$$

is a homeomorphism. (This is called the Penner coordinate of the decorated Teichmüller space.)

(2) *We put*

$$\lambda_e := \sqrt{2}\exp(l_e/2)$$

which is called the Lambda length of e. Let τ' be the triangulation obtained by flipping the edge e. The coordinates associated to τ and τ' agree for each edge which the two triangulations have in common, and

$$\lambda_{e'} = \frac{\lambda_a\lambda_c + \lambda_b\lambda_d}{\lambda_e}.$$

We define the inclusion

$$\begin{array}{ccccccc} \widetilde{\mathcal{T}}(\Sigma) & \simeq & \mathbb{R}^{\tau_1} & \hookrightarrow & (\mathbb{C}^*)^{\tau_1} & \simeq & \mathrm{Spec}(T(\tau)) \\ & & (l_e) & \mapsto & (x_e) = (\lambda_e) & & \end{array} .$$

We call $T(\tau)$ as *the complexified decorated Teichmüller space*. The mapping class group action on $\mathrm{Spec}(T(\tau))$ given in §2.3 preserves $\widetilde{\mathcal{T}}(\Sigma)$. If we can realize all the mapping classes by a sequence of flips without self-folded arcs then restricted action coincides with the geometric one.

Remark 5.2. *In [NTM], we study hyperbolic structures on the mapping torus of a pseudo-Anosov mapping class g of a surface. We show that a fixed point on $T(\tau)$ with respect to the action of g gives a hyperbolic structures on the mapping torus, while the fixed point set on $\widetilde{\mathcal{T}}(\Sigma)$ is empty due to the Nielsen-Thurston classification.*

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