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<th>Smallest complex nilpotent orbits with real points (Topics in Combinatorial Representation Theory)</th>
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<td>Okuda, Takayuki</td>
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Kyoto University
Smallest complex nilpotent orbits with real points

Takayuki Okuda

Abstract

In this paper, we show that there uniquely exists a real minimal nilpotent orbit in a non-compact simple Lie algebra $\mathfrak{g}$ if $(\mathfrak{g}, \mathfrak{t})$ is of non-Hermitian type. For the cases where $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$, $\epsilon_6(-26)$ or $\mathfrak{f}_4(-20)$, the complexification $\mathcal{O}_{\text{min},\mathfrak{g}}^{G_{\mathbb{C}}}$ of such the real minimal nilpotent orbit in $\mathfrak{g}$ is not the complex minimal nilpotent orbit in $9c=\mathfrak{g}+\sqrt{-1}\mathfrak{g}$. For such cases, we also determine $\mathcal{O}_{\text{min},\mathfrak{g}}^{G_{\mathbb{C}}}$ by describing the weighted Dynkin diagram of it.

1 Introduction and main results

Let $\mathfrak{g}_C$ be a complex simple Lie algebra. In this paper, an adjoint nilpotent orbit in $\mathfrak{g}_C$ will be simply called a complex nilpotent orbit in $\mathfrak{g}_C$. It is well-known that there exists a unique non-zero complex nilpotent orbit $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_C$, which is called a complex minimal nilpotent orbit, with the following property: The closure of $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_C$ is just $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}} \cup \{0\}$. By the uniqueness of such $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}}$, for any non-zero complex nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}_C$, the closure of $\mathcal{O}$ contains $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}}$. In other words, $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}}$ is minimum in $\mathcal{N}/G_{\mathbb{C}}$ without the zero-orbit, where $\mathcal{N}/G_{\mathbb{C}}$ denotes the set of complex nilpotent orbits in $\mathfrak{g}_C$ with the closure ordering.

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_C$. Namely, $\mathfrak{g}$ is a non-compact real simple Lie algebra without complex structures and $\mathfrak{g}_C$ is the complexification of $\mathfrak{g}$. Our concern in this paper is in real minimal nilpotent orbits in $\mathfrak{g}$. Here, we say that a non-zero real nilpotent orbit $\mathcal{O}^G$ in $\mathfrak{g}$ is minimal if the closure

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of $O^G$ in $\mathfrak{g}$ is just $O^G \sqcup \{0\}$. In general, real minimal nilpotent orbits are not unique for real simple $\mathfrak{g}$.

If the complex minimal nilpotent orbit $O^G_{\text{min}, \mathfrak{g}}$ in $\mathfrak{g}_C$ meets $\mathfrak{g}$, then the intersection $O^G_{\text{min}} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$. It is known that $O^G_{\text{min}}$ meets $\mathfrak{g}$ if and only if $\mathfrak{g}$ is not isomorphic to $\mathfrak{su}^*(2k)$ $(k \geq 2)$, $\mathfrak{so}(n-1,1)$ $(n \geq 5)$, $\mathfrak{sp}(p,q)$ $(p \geq q \geq 1)$, $\mathfrak{f}_4(-20)$ nor $\mathfrak{e}_6(-26)$ (see Brylinski [3, Theorem 4.1]). In particular, if $(\mathfrak{g}, \mathfrak{t})$ is of Hermitian type, then $O^G_{\text{min}}$ meets $\mathfrak{g}$, where $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. Furthermore, for the cases where $O^G_{\text{min}}$ meets $\mathfrak{g}$, the number of real minimal nilpotent orbits (i.e. the number of adjoint orbits in $O^G_{\text{min}} \cap \mathfrak{g}$) is two if $(\mathfrak{g}, \mathfrak{t})$ is of Hermitian type; one if $(\mathfrak{g}, \mathfrak{t})$ is of non-Hermitian type.

In this paper, we study real minimal nilpotent orbits in $\mathfrak{g}$ including the cases where $O^G_{\text{min}}$ does not meets $\mathfrak{g}$. For any real non-compact simple Lie algebra $\mathfrak{g}$ without complex structures, we put

$$\mathcal{N}_{\mathfrak{g}}/G_C := \{\text{Complex nilpotent orbits in } \mathfrak{g}_C \text{ meeting } \mathfrak{g}\}$$

and consider the closure ordering on it. Our first main result is here:

**Theorem 1.1.** There uniquely exists a complex nilpotent orbit $O^G_{\text{min}, \mathfrak{g}}$ in $\mathfrak{g}_C$ which is minimum in $\mathcal{N}_{\mathfrak{g}}/G_C$ without the zero-orbit (i.e. for any non-zero complex nilpotent orbit $O$ in $\mathfrak{g}$, if $O \cap \mathfrak{g} \neq \emptyset$, then the closure of $O$ in $\mathfrak{g}_C$ contains $O^G_{\text{min}, \mathfrak{g}}$). Furthermore, the intersection $O^G_{\text{min}, \mathfrak{g}} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$.

We will construct such $O^G_{\text{min}, \mathfrak{g}}$ as the complex adjoint orbit through a non-zero longest restricted root vector in $\mathfrak{g}$. By the definition of $O^G_{\text{min}, \mathfrak{g}}$, the complex minimal nilpotent orbit $O^G_{\text{min}}$ is not our $O^G_{\text{min}, \mathfrak{g}}$ if and only if $O^G_{\text{min}}$ does not meet $\mathfrak{g}$ (namely, $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$ $(k \geq 2)$, $\mathfrak{so}(n-1,1)$ $(n \geq 5)$, $\mathfrak{sp}(p,q)$ $(p \geq q \geq 1)$, $\mathfrak{f}_4(-20)$ or $\mathfrak{e}_6(-26)$). This means that for such cases, a non-zero longest restricted root vector in $\mathfrak{g}$ is not a longest root vector in $\mathfrak{g}_C$.

Theorem 1.1 claims that $O^G_{\text{min}, \mathfrak{g}} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$. Our second main result is here:

**Theorem 1.2.** For the cases where the complex minimal nilpotent orbit $O^G_{\text{min}}$ does not meet $\mathfrak{g}$, there exists a unique real minimal nilpotent orbit in $\mathfrak{g}$. In particular, the complex nilpotent orbit $O^G_{\text{min}, \mathfrak{g}}$ in Theorem 1.1 (which is not $O^G_{\text{min}}$ in these cases) is the complexification of the unique real minimal nilpotent orbit in $\mathfrak{g}$.
Therefore, we have the following corollary:

**Corollary 1.3.** Let \( \mathfrak{g} \) be a non-compact real simple \( L \)ia algebra without complex structures. If \((\mathfrak{g}, \mathfrak{k})\) is of non-Hermitian type, there uniquely exists a real minimal nilpotent orbit in \( \mathfrak{g} \). If \((\mathfrak{g}, \mathfrak{k})\) is of Hermitian type, there are just two real minimal nilpotent orbits in \( \mathfrak{g} \).

By Theorem 1.2, our \( \mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \) is just the complexification of the unique real minimal nilpotent orbit in \( \mathfrak{g} \) for the cases where \( \mathfrak{g} \) is isomorphic to \( \mathfrak{su}^*(2k) \) \((k \geq 2)\), \( \mathfrak{so}(n - 1, 1) \) \((n \geq 5)\), \( \mathfrak{sp}(p, q) \) \((p \geq q \geq 1)\), \( \mathfrak{f}_{4(-20)} \) or \( \epsilon_{6(-26)} \). We will determine our \( \mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \) by describing the weighted Dynkin diagram of it for such cases (recall that for another cases, \( \mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \) is just \( \mathcal{O}_{\min}^{G_{\mathbb{C}}} \)). The result is here (see also Table 2 in §2 for the weighted Dynkin diagrams of \( \mathcal{O}_{\min}^{G_{\mathbb{C}}} \)):

**Theorem 1.4.** For the cases where \( \mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \neq \mathcal{O}_{\min}^{G_{\mathbb{C}}} \), the weighted Dynkin diagram of \( \mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \) are the following:

<table>
<thead>
<tr>
<th>( \mathfrak{g} )</th>
<th>( \dim_{\mathbb{C}} \mathcal{O}<em>{\min, \mathfrak{g}}^{G</em>{\mathbb{C}}} )</th>
<th>Weighted Dynkin diagram of ( \mathcal{O}<em>{\min, \mathfrak{g}}^{G</em>{\mathbb{C}}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathfrak{su}^*(2k) )</td>
<td>( 8k - 8 )</td>
<td>( 0 1 0 0 \ldots 0 0 1 0 ) ((k \geq 3))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 0 2 0 ) ((k = 2))</td>
</tr>
<tr>
<td>( \mathfrak{so}(n - 1, 1) )</td>
<td>( 2n - 4 )</td>
<td>( 2 0 0 \ldots 0 0 ) ((n \text{ is odd, } n \geq 5))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 2 0 0 \ldots 0 0 ) ((n \text{ is even, } n \geq 6))</td>
</tr>
<tr>
<td>( \mathfrak{sp}(p, q) )</td>
<td>( 4(p + q) - 2 )</td>
<td>( 0 1 0 0 \ldots 0 0 ) ((p + q \geq 3, p \geq q \geq 1))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 0 2 ) ((p = q = 1))</td>
</tr>
<tr>
<td>( \epsilon_{6(-26)} )</td>
<td>( 32 )</td>
<td>( 1 0 0 0 0 1 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \mathfrak{f}_{4(-20)} )</td>
<td>( 22 )</td>
<td>( 0 0 0 0 1 )</td>
</tr>
</tbody>
</table>

*Table 1: List of \( \mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \) for \( \mathfrak{su}^*(2k) \), \( \mathfrak{so}(n - 1, 1) \), \( \mathfrak{sp}(p, q) \), \( \epsilon_{6(-26)} \) and \( \mathfrak{f}_{4(-20)} \).*
This work is motivated by recent works [7], by Joachim Hilgert, Toshiyuki Kobayashi and Jan Möllers, on the construction of an $L^2$-model of irreducible unitary representations of real reductive groups with smallest Gelfand-Kirillov dimension; and [8], by Toshiyuki Kobayashi and Yoshiki Oshima, on the classification of reductive symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ with a $(\mathfrak{g}, K)$-module which is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module.

2 Preliminary results for weighted Dynkin diagrams of complex minimal nilpotent orbits

In this section, we recall weighted Dynkin diagrams of complex minimal nilpotent orbits in complex simple Lie algebras.

Let $\mathfrak{g}_\mathbb{C}$ be a complex semisimple Lie algebra, and denote by $G_\mathbb{C}$ the inner automorphism group of $\mathfrak{g}_\mathbb{C}$. Fix a Cartan subalgebra $\mathfrak{h}_\mathbb{C}$ of $\mathfrak{g}_\mathbb{C}$. We denote by $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ the root system of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Then, the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ can be regarded as a subset of the dual space $\mathfrak{h}^*$ of

$$\mathfrak{h} := \{ H \in \mathfrak{h}_\mathbb{C} | \alpha(H) \in \mathbb{R} (\forall \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})) \}.$$  

We write $W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ for the Weyl group of $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ acting on $\mathfrak{h}$. Take a positive system $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ of the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Then, a closed Weyl chamber

$$\mathfrak{h}_+ := \{ H \in \mathfrak{h} | \alpha(H) \geq 0 (\forall \alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})) \}$$

is a fundamental domain of $\mathfrak{h}$ under the action of $W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$.

Let $\Pi$ be the simple system of $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Then, for any $H \in \mathfrak{h}$, we can define a map

$$\Psi_H : \Pi \to \mathbb{R}, \alpha \mapsto \alpha(H).$$

We call $\Psi_H$ the weighted Dynkin diagram corresponding to $H \in \mathfrak{h}$, and $\alpha(H)$ the weight on a node $\alpha \in \Pi$ of the weighted Dynkin diagram. Since $\Pi$ is a basis of $\mathfrak{h}^*$, the map

$$\Psi : \mathfrak{h} \to \text{Map}(\Pi, \mathbb{R}), \ H \mapsto \Psi_H$$
is a linear isomorphism (between vector spaces). Furthermore,

$$\mathfrak{h}_+ \rightarrow \text{Map}(\Pi, \mathbb{R}_{\geq 0}), \ H \mapsto \Psi_H$$

is also bijective.

A triple \((H, X, Y)\) is said to be an \(\mathfrak{sl}_2\)-triple in \(\mathfrak{g}_C\) if

$$[H, X] = 2X, \ [H, Y] = -2Y, \ [X, Y] = H \ (H, X, Y \in \mathfrak{g}_C).$$

For any \(\mathfrak{sl}_2\)-triple \((H, X, Y)\) in \(\mathfrak{g}_C\), the elements \(X\) and \(Y\) are nilpotent in \(\mathfrak{g}_C\), and \(H\) is hyperbolic in \(\mathfrak{g}_C\) (i.e. \(\text{ad}_{\mathfrak{g}_C} H \in \text{End}(\mathfrak{g}_C)\) is diagonalizable with only real eigenvalues).

Combining the Jacobson–Morozov theorem with Kostant [9], for any complex nilpotent orbit \(\mathcal{O}^{G_C}\), there uniquely exists an element \(H_o\) of \(\mathfrak{h}_+\) with the following property: There exists \(X, Y \in \mathcal{O}^{G_C}\) such that \((H_o, X, Y)\) is an \(\mathfrak{sl}_2\)-triple in \(\mathfrak{g}_C\). Furthermore, by Malcev [10], the following map is injective:

$$\{\text{Complex nilpotent orbits in } \mathfrak{g}_C \} \leftrightarrow \mathfrak{h}_+, \ \mathcal{O}^{G_C} \leftrightarrow H_o.$$

The weighted Dynkin diagram corresponding to \(H_o\) is called the weighted Dynkin diagram of \(\mathcal{O}^{G_C}\). Dynkin [6] proved that for any complex nilpotent orbit \(\mathcal{O}^{G_C}\), any weight of the weighted Dynkin diagram of \(\mathcal{O}^{G_C}\) is given by 0, 1 or 2, and classified weighted Dynkin diagrams of complex nilpotent orbits (More precisely, Dynkin [6] classified \(\mathfrak{sl}_2\)-triples in \(\mathfrak{g}_C\). See Bala–Carter [2] for more details).

In the rest of this subsection, we suppose that \(\mathfrak{g}_C\) is simple. Let \(\phi\) be the highest root of \(\Delta^+(\mathfrak{g}_C, \mathfrak{h}_C)\). Then, the complex minimal nilpotent orbit in \(\mathfrak{g}_C\) can be written by

$$\mathcal{O}_{\text{min}}^{G_C} = G_C \cdot \mathfrak{g}_\phi \setminus \{0\}.$$

We define the element \(H_{\phi^\vee}\) of \(\mathfrak{h}\) by

$$\alpha(H_{\phi^\vee}) = \frac{2 \langle \alpha, \phi \rangle}{\langle \phi, \phi \rangle}$$

for any \(\alpha \in \mathfrak{h}^\ast\) (where \(\langle \ , \ \rangle\) is the inner product on \(\mathfrak{h}^\ast\) induced by the Killing form on \(\mathfrak{g}_C\)). Namley, \(H_{\phi^\vee}\) is the element of \(\mathfrak{h}\) corresponding to the coroot \(\phi^\vee\) of \(\phi\). Since \(\phi\) is dominant, \(H_{\phi^\vee}\) is in \(\mathfrak{h}_+\). Furthermore, \(H_{\phi^\vee}\) is the hyperbolic element corresponding to \(\mathcal{O}_{\text{min}}^{G_C}\) since we can find \(X_\phi \in \mathfrak{g}_\phi, Y_\phi \in \mathfrak{g}_{-\phi}\) such that \((H_{\phi^\vee}, X_\phi, Y_\phi)\) is an \(\mathfrak{sl}_2\)-triple. The list of weighted Dynkin diagrams of \(\mathcal{O}_{\text{min}}^{G_C}\) for all simple \(\mathfrak{g}_C\) can be found in Collingwood–McGovern [4, Ch.5.4 and 8.4].
Recall that our concern in this paper is in real simple Lie algebras $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{e}_6(-26)$ and $\mathfrak{f}_4(-20)$. The complexifications of such algebras are $\mathfrak{sl}(2k,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{sp}(p+q,\mathbb{C})$, $\mathfrak{e}_{6,\mathbb{C}}$ and $\mathfrak{f}_{4,\mathbb{C}}$, respectively. For the convenience of the reader, we give a list of weighted Dynkin diagrams of complex minimal nilpotent orbits in such complex simple Lie algebras.

<table>
<thead>
<tr>
<th>$\mathfrak{g}_{\mathbb{C}}$</th>
<th>$\dim_{\mathbb{C}}\mathcal{O}<em>{\min}^{G</em>{\mathbb{C}}}$</th>
<th>Weighted Dynkin diagram of $\mathcal{O}<em>{\min,\mathfrak{g}}^{G</em>{\mathbb{C}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{sl}(n,\mathbb{C})$</td>
<td>$2n$</td>
<td>1 0 0 0 \ldots 0 0 0 1 ((n \geq 2))</td>
</tr>
<tr>
<td>$\mathfrak{so}(n,\mathbb{C})$</td>
<td>$2n-6$</td>
<td>0 1 0 \ldots 0 0 ((n) is odd, (n \geq 7))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 1 (\quad(n=5))</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 1 0 \ldots 0 0 ((n) is even, (n \geq 6))</td>
</tr>
<tr>
<td>$\mathfrak{sp}(n,\mathbb{C})$</td>
<td>$2n$</td>
<td>1 0 0 0 \ldots 0 0 ((n \geq 2))</td>
</tr>
<tr>
<td>$\mathfrak{e}_{6,\mathbb{C}}$</td>
<td>22</td>
<td>0 0 0 0 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$\mathfrak{f}_{4,\mathbb{C}}$</td>
<td>16</td>
<td>1 0 0 0</td>
</tr>
</tbody>
</table>

Table 2: List of weighted Dynkin diagrams of $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ for $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{sp}(n,\mathbb{C})$, $\mathfrak{e}_{6,\mathbb{C}}$ and $\mathfrak{f}_{4,\mathbb{C}}$.

3 Outline of a proof of Theorem 1.1

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra and $\mathfrak{g}$ a non-compact real form of $\mathfrak{g}$ with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In this section, we describe an idea of the proof of Theorem 1.1.

We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ (such $\mathfrak{a}$ is called a maximally split abelian subspace of $\mathfrak{g}$) and write $\Sigma(\mathfrak{g}, \mathfrak{a})$ for the restricted root system.
for $(g, a)$. For any restricted root $\xi$ of $\Sigma(g, a)$, we define $A_{\xi^\vee} \in a$ by

$$\eta(A_{\xi^\vee}) = \frac{2(\xi, \eta)}{(\xi, \xi)} \quad (\forall \eta \in a^*)$$

(\text{where } (\ , \ ) \text{ is the inner product on } a^* \text{ induced by the Killing form on } g). Namely, $A_{\xi^\vee}$ is the element of $a$ corresponding to the coroot $\xi^\vee$ of $\xi$. Then, the fact below holds:

**Fact 3.1.** For any restricted root $\xi$ of $\Sigma(g, a)$ and any non-zero root vector $X_\xi$ in $g_\xi$, there exists $Y_\xi \in g_{-\xi}$ such that $(A_{\xi^\vee}, X_\xi, Y_\xi)$ is an $sl_2$-triple in $g$.

We fix an ordering on $a$ and write $\Sigma^+(g, a)$ for the positive system of $\Sigma(g, a)$ corresponding to the ordering on $a$. We denote by $\lambda$ the highest root of $\Sigma^+(g, a)$ with respect to the ordering on $a$. Next two lemmas give characterizations of the highest root $\lambda$ of $\Sigma^+(g, a)$ (we omit proofs of the two lemmas in this paper):

**Lemma 3.2.** The highest root $\lambda$ of $\Sigma^+(g, a)$ is a unique dominant longest root of $\Sigma(g, a)$.

**Lemma 3.3.** Let $\xi$ be a root of $\Sigma(g, a)$. If $\xi$ is not the highest root $\lambda$, then for any non-zero root vector $X_\xi$ in $g_\xi$, there exists a positive root $\eta$ in $\Sigma^+(g, a)$ and a root vector $X_\eta \in g_\eta$ such that $[X_\xi, X_\eta] \neq 0$. In particular, $\xi = \lambda$ if and only if $\xi + \eta \in a^*$ is not a root of $\Sigma(g, a)$ for any $\eta \in \Sigma^+(g, a)$.

We write $G_C$ for the inner automorphism group of $g_C$. Then, the following two propositions hold:

**Proposition 3.4.** For any non-zero real nilpotent orbit $O_0'$ in $g$. Then, there exists a non-zero highest root vector $X_\lambda$ in $g_\lambda$ such that $X_\lambda$ is in the closure of $O_0'$ in $g$.

**Proposition 3.5.** For any two highest root vectors $X_\lambda, X'_\lambda$ in $g_\lambda$, there exists $g_C \in G_C$ such that $g_C X_\lambda = X'_\lambda$.

*Proof of Proposition 3.4.* There is no loss of generality in assuming that the ordering on $a$ is lexicographic. Let us put $m = Z_a(a)$. Then, $g$ can be decomposed as

$$g = m \oplus a \oplus \bigoplus_{\xi \in \Sigma(g, a)} g_\xi.$$
For any $X' \in g$, we denote by

$$X' = X'_m + X'_a + \sum_{\xi \in \Sigma(g, a)} X'_\xi \quad (X'_m \in m, \ X'_a \in a, \ X'_\xi \in g_\xi).$$

We put $\overline{O'_0}$ to the closure of $O'_0$ in $g$ and fix an element $X'$ in $\overline{O'_0}$. Let us denote by $\lambda'$ the highest one of

$$\Sigma_{X'} := \{ \xi \in \Sigma(g, a) \mid X'_\xi \neq 0 \}$$

with respect to the ordering on $a$ (if $X' \neq 0$, then $\Sigma_{X'}$ is not empty since $X'$ is nilpotent element in $g$). As a first step of the proof, we shall prove that the root vector $X'_{\lambda'}$ is also in $\overline{O'_0}$. We take $A' \in a$ satisfying

$$\xi(A') < \lambda'(A') \quad (\forall \xi \in \Sigma_{X'} \setminus \{ \lambda' \}).$$

(such $A'$ exists since $\lambda'$ is highest in $\Sigma_{X'}$ with respect to the lexicographic ordering on $a$). Let us put

$$X'_k := \frac{1}{e^{k\lambda'(A')}} \exp(ad_{g} k A') X' \quad (k \in \mathbb{N}).$$

Then, $X'_k$ is in $\overline{O'_0}$ for any $k$ since $\overline{O'_0}$ is stable by positive scalars. Furthermore,

$$\lim_{k \to \infty} X'_k = \lim_{k \to \infty} \sum_{\xi \in \Sigma_{X'}} e^{k(\xi(A') - \lambda'(A'))} X'_\xi = X'_{\lambda'}. \quad \text{(for } k \in \mathbb{N})$$

This means that $X'_{\lambda'}$ is in $\overline{O'_0}$. To complete the proof, we only need to show that there exists $X' \in \overline{O'_0}$ such that $\lambda' = \lambda$ (where $\lambda'$ is the highest one of $\Sigma_{X'}$). Let $\lambda_0$ be the highest one of

$$\Sigma_{\overline{O'_0}} := \{ \xi \in \Sigma(g, a) \mid \exists X' \in \overline{O'_0} \text{ such that } X'_\xi \neq 0 \}$$

(namely, $\Sigma_{\overline{O'_0}} = \bigcup_{X' \in \overline{O'_0}} \Sigma_{X'}$) with respect to the ordering on $a$. Then, we can find a root vector $X'_{\lambda_0}$ in $g_{\lambda_0} \cap \overline{O'_0}$ by the argument above. We assume that $\lambda_0 \neq \lambda$. Then, by Lemma 3.3, there exists $\eta \in \Sigma^+(g, a)$ and $X'_{\eta} \in g_{\eta}$ such that $[X'_{\eta}, X'_{\lambda_0}] \neq 0$. In particular, for the element $X'' := \exp(ad_{g}(X'_{\eta})) X'_{\lambda_0}$ in $\overline{O'_0}$, we obtain that

$$\lambda_0 + \eta \in \Sigma_{X''} \subset \Sigma_{\overline{O'_0}}.$$

This contradicts the definition of $\lambda_0$. Thus, $\lambda_0 = \lambda$. \qed
Proof of Proposition 3.5. Let $A_{\lambda}^{\vee}$ be the element in $\mathfrak{a}$ corresponding to the coroot $\lambda^{\vee}$ of the highest root $\lambda$. We put

$$(\mathfrak{g}_{C})_{2} = \{ X \in \mathfrak{g}_{C} \mid [A_{\lambda}^{\vee}, X] = 2X \}.$$ 

Then, $\mathfrak{g}_{\lambda}$ is included in $(\mathfrak{g}_{C})_{2}$. We note that there exists $X, Y \in \mathfrak{g}_{C}$ such that $(A_{\lambda}^{\vee}, X, Y)$ is an $\mathfrak{s}_{2}$-triple in $\mathfrak{g}_{C}$ (in fact, we can find such $X, Y$ in $\mathfrak{g}_{\lambda}$ by Fact 3.1). Therefore, we can use Malcev’s theorem. Namely, for any two non-zero vectors $X$ and $X'$ in $(\mathfrak{g}_{C})_{2}$, there exists $g_{C} \in G_{C}$ such that $g_{C}X = X'$. Since $\mathfrak{g}_{\lambda} \subset (\mathfrak{g}_{C})_{2}$, the proof is completed. \qed

By using Proposition 3.4 and Proposition 3.5, Theorem 1.1 follows by taking $\mathcal{O}_{\min,\mathfrak{g}}^{G_{C}}$ as

$\mathcal{O}_{\min,\mathfrak{g}}^{G_{C}} := G_{C} \cdot \mathfrak{g}_{\lambda} \setminus \{0\}$.

4 Outline of a proof of Theorem 1.2

Let us consider the same setting in §3. Recall that $\mathcal{O}_{\min,\mathfrak{g}}^{G_{C}}$ is not the complex minimal nilpotent orbit $\mathcal{O}_{\min}^{G_{C}}$ if and only if $\mathcal{O}_{\min}^{G_{C}}$ does not meet $\mathfrak{g}$. The proposition below give a characterization of $\mathfrak{g}$ for which $\mathcal{O}_{\min}^{G_{C}}$ is not $\mathcal{O}_{\min,\mathfrak{g}}^{G_{C}}$ (see Proposition 5.6 for another characterizations of it).

Proposition 4.1. The following conditions on $\mathfrak{g}$ are equivalent:

1. $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$.

2. $\mathcal{O}_{\min}^{G_{C}} \cap \mathfrak{g} = \emptyset$.

We can prove the proposition without any classification, but we omit it in this paper.

Here, we put $m := Z_{\mathfrak{z}}(\mathfrak{a})$ and denote by $M_{0}, A$ to the analytic subgroups of $G$ corresponding to $m, \mathfrak{a}$, respectively. Then, the connected Lie group $M_{0}A$ (which is the analytic subgroup of $G$ corresponding to $m \oplus \mathfrak{a}$) acts on $\mathfrak{a}$. Furthermore, the following proposition holds:

Proposition 4.2. If $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$, then $\mathfrak{g}_{\lambda} \setminus \{0\}$ is a single $M_{0}A$-orbit.

Combining Proposition 3.4, Proposition 4.1 with Proposition 4.2, we obtain Theorem 1.2.

We will use the next lemma to prove Proposition 4.2.
Lemma 4.3. Suppose that \( g \) has real rank one (i.e. \( \dim_{\mathbb{R}} a = 1 \)) and \( \dim_{\mathbb{R}} g_{\lambda} \geq 2 \). Then, \( g_{\lambda} \setminus \{0\} \) is a single \( M_0 A \)-orbit.

Proof of Lemma 4.3. Let \( A_{\lambda} \) be the element of \( a \) corresponding to the coroot \( \lambda' \) of the highest root \( \lambda \) in \( \Sigma^+(g, a) \) (see §3). Since \( g \) has real rank one, we have \( a = \mathbb{R} A_{\lambda} \), and \( g \) can be written by

\[
g = g_{-\lambda} \oplus g_{-\frac{\lambda}{2}} \oplus m \oplus a \oplus g_{\frac{\lambda}{2}} \oplus g_{\lambda}
\]

\( (g_{\pm \frac{\lambda}{2}} \) can be zero). Let us denote by \( g_{\mathbb{C}}, m_{\mathbb{C}}, a_{\mathbb{C}}, (g_{\pm \lambda})_{\mathbb{C}}, (g_{\pm \frac{\lambda}{2}})_{\mathbb{C}} \) the complexification of \( g, m, a, g_{\pm \lambda}, g_{\pm \frac{\lambda}{2}} \), respectively. We set

\[
(g_{\mathbb{C}})_i = \{ X \in g_{\mathbb{C}} \mid [A_{\lambda'}, X] = iX \} \quad \text{(for } i \in \mathbb{Z}).
\]

Then,

\[
(g_{\mathbb{C}})_0 = m_{\mathbb{C}} \oplus a_{\mathbb{C}}, \quad (g_{\mathbb{C}})_{\pm 1} = (g_{\pm \frac{\lambda}{2}})_{\mathbb{C}}, \quad (g_{\mathbb{C}})_{\pm 2} = (g_{\pm \lambda})_{\mathbb{C}}.
\]

By Fact 3.1, for any non-zero highest root vector \( X_{\lambda} \) in \( g_{\lambda} \), there exists \( Y_{\lambda} \in g_{-\lambda} \) such that \( (A_{\lambda'}, X_{\lambda}, Y_{\lambda}) \) is an \( sl_2 \)-triple in \( g_{\mathbb{C}} \). By the theory of representations of \( sl(2, \mathbb{C}) \), we obtain that \( [(g_{\mathbb{C}})_0, X_{\lambda}] = (g_{\mathbb{C}})_2 \). In particular, we have

\[
[m \oplus a, X_{\lambda}] = g_{\lambda}.
\]

Therefore, for the \( M_0 A \)-orbit \( O^{M_0 A}(X_{\lambda}) \) in \( g_{\lambda} \) through \( X_{\lambda} \), we obtain that

\[
\dim_{\mathbb{R}} O^{M_0 A}(X_{\lambda}) = \dim_{\mathbb{R}} g_{\lambda}.
\]

This means that the \( M_0 A \)-orbit \( O^{M_0 A}(X_{\lambda}) \) is open in \( g_{\lambda} \) for any non-zero root vector \( X_{\lambda} \) in \( g_{\lambda} \). Recall that we are assuming that \( \dim_{\mathbb{R}} g_{\lambda} \geq 2 \). Hence, \( g_{\lambda} \setminus \{0\} \) is connected. Therefore, \( g_{\lambda} \setminus \{0\} \) is a single \( M_0 A \)-orbit. \( \square \)

We are ready to prove Proposition 4.2.

Sketch of a proof of Proposition 4.2. Let \( h' := [g_{\lambda}, g_{-\lambda}] \subset m \oplus a \). Then \( g' := g_{-\lambda} \oplus h' \oplus g_{\lambda} \) becomes a subalgebra of \( g \) (since \( \pm 2\lambda \) is not a root). Furthermore, one can prove that \( g' \) is a real rank one simple Lie algebra with a maximally split abelian subspace \( a' := \mathbb{R} A_{\lambda} \), where \( A_{\lambda} \) is the element of \( a \) corresponding to the coroot \( \lambda' \) of the highest root \( \lambda \) in \( \Sigma^+(g, a) \) (see §3). We put \( m' \oplus a' := Z_{g'}(a') \) and denote by \( M'_0 A' \) the analytic subgroup of \( G \) corresponding to \( m' \oplus a' \). Then, by Lemma 4.3, we obtain that \( g_{\lambda} \setminus \{0\} \) is a single \( M'_0 A' \)-orbit. Since \( M'_0 A' \) is a subgroup of \( M_0 A \), the proof is completed. \( \square \)
5 Determination of $\mathcal{O}^{G_C}_{\text{min},g}$

In this section, we determine $\mathcal{O}^{G_C}_{\text{min},g}$ by describing the weighted Dynkin diagram of $\mathcal{O}^{G_C}_{\text{min},g}$. Recall that Proposition 4.1 claims that $\mathcal{O}^{G_C}_{\text{min}} = \mathcal{O}^{G_C}_{\text{min},g}$ if and only if $\dim_{\mathbb{R}} g_\lambda \geq 2$ (i.e. $g$ is isomorphic to $\mathfrak{su}^*(2k), \mathfrak{so}(n-1,1), \mathfrak{sp}(p,q), \mathfrak{e}_6(-26)$ or $\mathfrak{f}_4(-20)$).

5.1 Satake diagrams and weighted Dynkin diagrams

In order to determine the weighted Dynkin diagram of our $\mathcal{O}^{G_C}_{\text{min},g}$, we describe some lemmas of relationship between weighted Dynkin diagrams of $g_C$ and Satake diagrams of $g$ in this subsection.

Let $g_C$ be a semisimple Lie algebra and $g$ a real form of it through this subsection. First, we recall briefly the definition of Satake diagram of a real form $g$ of a complex semisimple Lie algebra $g_C$ (see also [1] for more details). Fix a Cartan decomposition $g = \mathfrak{k} \oplus \mathfrak{p}$ of $g$. We take a maximal abelian subspace $a$ in $\mathfrak{p}$, and extend it to a maximal abelian subspace $\mathfrak{h} = \sqrt{-1}\mathfrak{k} \oplus a$ in $\sqrt{-1}\mathfrak{k} \oplus \mathfrak{p}$. Then, the complexification, denoted by $\mathfrak{h}_C$, of $\mathfrak{h}$ is a Cartan subalgebra of $g_C$, and $\mathfrak{h}$ coincide with the real form

$$\{X \in \mathfrak{h}_C \mid \alpha(X) \in \mathbb{R} (\forall \alpha \in \Delta(\mathfrak{g}_C, \mathfrak{h}_C))\}$$

of $\mathfrak{h}_C$, where $\Delta(\mathfrak{g}_C, \mathfrak{h}_C)$ is the root system of $(\mathfrak{g}_C, \mathfrak{h}_C)$. Let us denote by

$$\Sigma(\mathfrak{g}, a) := \{\alpha|_a \mid \alpha \in \Delta(\mathfrak{g}_C, \mathfrak{h}_C)\} \setminus \{0\} \subset a^*$$

the restricted root system of $(\mathfrak{g}, a)$. We will denote by $W(\mathfrak{g}, a)$, $W(\mathfrak{g}_C, \mathfrak{h}_C)$ the Weyl group of $\Sigma(\mathfrak{g}, a)$, $\Delta(\mathfrak{g}_C, \mathfrak{h}_C)$, respectively. Fix an ordering on $a$ and extend it to an ordering on $\mathfrak{h}$. We write $\Sigma^+(\mathfrak{g}, a)$, $\Delta^+(\mathfrak{g}_C, \mathfrak{h}_C)$ for the positive system of $\Sigma(\mathfrak{g}, a)$, $\Delta(\mathfrak{g}_C, \mathfrak{h}_C)$ corresponding to the ordering on $a$, $\mathfrak{h}$, respectively. Then, $\Sigma^+(\mathfrak{g}, a)$ can be written by

$$\Sigma^+(\mathfrak{g}, a) = \{\alpha|_a \mid \alpha \in \Delta^+(\mathfrak{g}_C, \mathfrak{h}_C)\} \setminus \{0\}.$$ 

We denote by $\Pi$ the fundamental system of $\Delta^+(\mathfrak{g}_C, \mathfrak{h}_C)$. Then,

$$\overline{\Pi} := \{\alpha|_a \mid \alpha \in \Pi\} \setminus \{0\}$$

is the simple system of $\Sigma^+(\mathfrak{g}, a)$. Let $\Pi_0$ be the set of all simple roots in $\Pi$ whose restriction to $a$ is zero. The Satake diagram $S_a$ of $g$ consists of the
following data: The Dynkin diagram of $\mathfrak{g}_C$ with nodes $\Pi$; black nodes $\Pi_0$ in $S_\mathfrak{g}$; and arrows joining $\alpha \in \Pi \setminus \Pi_0$ and $\beta \in \Pi \setminus \Pi_0$ in $S_\mathfrak{g}$ whose restrictions to $\alpha$ are the same.

Second, we define that a weighted Dynkin diagram $\Psi_H \in \text{Map}(\Pi, \mathbb{R})$ "matches" the Satake diagram $S_\mathfrak{g}$ of $\mathfrak{g}$ as follows:

**Definition 5.1.** Let $\Psi_H \in \text{Map}(\Pi, \mathbb{R})$ be a weighted Dynkin diagram (see §2) and $S_\mathfrak{g}$ the Satake diagram of $\mathfrak{g}$ with nodes $\Pi$. We say that $\Psi_H$ matches $S_\mathfrak{g}$ if all the weights on black nodes are zero and any pair of nodes joined by an arrow has the same weights.

**Remark 5.2.** The concept of "match" defined above is same as "weighted Satake diagrams" in Djocovic [5] and the condition described in Sekiguchi [11, Proposition 1.16].

Recall that $\Psi$ is a linear isomorphism from $\mathfrak{h}$ to $\text{Map}(\Pi, \mathbb{R})$ (see §2). Then, the next two lemmas hold (we omit proofs of the two lemmas in this paper):

**Lemma 5.3.** $\Psi : \mathfrak{h} \rightarrow \text{Map}(\Pi, \mathbb{R})$ induces a linear isomorphism below:
$$\alpha \rightarrow \{ \Psi_H \in \text{Map}(\Pi, \mathbb{R}) \mid \Psi_H \text{ matches } S_\mathfrak{g} \}.$$ 

**Lemma 5.4.** For each simple root $\alpha$ of $\Pi$, we denote by $H_\alpha^\vee$ the element in $\mathfrak{h}$ corresponding to the coroot $\alpha^\vee$ of the simple root $\alpha$. Then, the set
$$\{ H_\alpha^\vee \mid \alpha \text{ is black in } S_\mathfrak{g} \} \cup \{ H_\alpha^\vee - H_\beta^\vee \mid \alpha \text{ and } \beta \text{ are joined by an arrow in } S_\mathfrak{g} \}$$
is a basis of $\sqrt{-1}t$.

Lemma 5.3 and Lemma 5.4 will be used to compute the weighted Dynkin diagrams of $\mathcal{O}_{\text{min},\mathfrak{g}}^{G_C}$ for the cases where $\mathcal{O}_{\text{min},\mathfrak{g}}^{G_C}$ is not the complex minimal nilpotent orbit $\mathcal{O}_{\text{min}}^{G_C}$.

Recall that our concern in this paper is in real simple Lie algebras $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n - 1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_6(-26)$ and $\mathfrak{f}_4(-20)$. For the convenience of the reader, we give a list of Satake diagrams of such simple Lie algebras.

<table>
<thead>
<tr>
<th>$\mathfrak{g}$</th>
<th>Satake diagrams of $\mathfrak{g}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{su}^*(2k)$</td>
<td>$\bullet \cdots \bullet \alpha_{2k-1}$</td>
</tr>
</tbody>
</table>
5.2 Computation of weighted Dynkin diagrams of $\mathcal{O}_{\text{min},g}^{G_{\mathbb{C}}}$

We consider the same setting on §5.1 and suppose that $g_{\mathbb{C}}$ is simple and $g$ is non-compact. Let us denote by

$$a_+ := \{ A \in a \mid \xi(A) \geq 0 \forall \xi \in \Sigma^+(g,a) \}.$$ 

Then $a_+$ is a fundamental domain of $a$ under the action of $W(g,a)$. Since

$$\Sigma^+(g,a) = \{ \alpha|_a \mid \alpha \in \Delta(g_{\mathbb{C}}, h_{\mathbb{C}}) \} \setminus \{0\},$$

the domain $a_+ \cap \mathfrak{h}_+ \cap a$. Recall that $\lambda$ is dominant (by Lemma 3.2) and $\mathcal{O}_{\text{min},g}^{G_{\mathbb{C}}}$ contains $g_\lambda \setminus \{0\}$ (by the proof of Theorem 1.1). Thus, $A_{\lambda^\vee}$ is the hyperbolic element in $a_+$ corresponding to $\mathcal{O}_{\text{min},g}^{G_{\mathbb{C}}}$ (see §2) since we can find $X_\lambda \in g_\lambda, Y_\lambda \in g_{-\lambda}$ such that the triple $(A_{\lambda^\vee}, X_\lambda, Y_\lambda)$ is an $s\ell_2$-triple in $g_{\mathbb{C}}$ by Lemma 3.1 (then, $X_\lambda, Y_\lambda \in \mathcal{O}_{\text{min},g}^{G_{\mathbb{C}}}$). Therefore, to determine the weighted Dynkin diagram of $\mathcal{O}_{\text{min},g}^{G_{\mathbb{C}}}$, we shall compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$.

Let $\phi$ be the highest root of $\Delta^+(g_{\mathbb{C}}, h_{\mathbb{C}})$. Recall that the complex minimal nilpotent orbit $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}}$ contains the root space $(g_{\mathbb{C}})_\phi$ without zero, and the weighted Dynkin diagram of $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}}$ is the weighted Dynkin diagram corresponding to $H_{\phi^\vee}$ (see §2). The next lemma gives a formula for $A_{\lambda^\vee}$ by $H_{\phi^\vee}$ (we omit a proof of the lemma):

<table>
<thead>
<tr>
<th>$\mathfrak{so}(n-1,1)$</th>
<th>$\mathfrak{sp}(p,q)$</th>
<th>$\mathfrak{e}_6(-26)$</th>
<th>$\mathfrak{f}_4(-20)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bullet - \bullet - \bullet - \cdots - \bullet - \bullet \Rightarrow \bullet$</td>
<td>$\alpha_{2q} \leftrightarrow \alpha_{p+q}$</td>
<td>$\bullet - \bullet - \bullet - \cdots$</td>
<td>$\bullet - \bullet$</td>
</tr>
</tbody>
</table>

$(n$ is odd, $n \geq 5)$

$(n$ is even, $n \geq 6)$

$(p \geq q \geq 1)$

<table>
<thead>
<tr>
<th>$\mathfrak{e}_6(-26)$</th>
<th>$\mathfrak{f}_4(-20)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\bullet - \bullet - \bullet - \cdot - \bullet - \bullet}{\mathfrak{sp}(p,q) \bullet - \infty - \bullet - \infty \Rightarrow \vec{\alpha}_{2q} - \bullet - \bullet - \cdot - \bullet - 5_p^q}$</td>
<td>$\frac{f_{4(-20)} \bullet - \bullet \Rightarrow -}{Table\ 3: List of Satakediagrams of \epsilon u^{*}(2k), \epsilon o(n-1,1)}$</td>
</tr>
</tbody>
</table>

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Lemma 5.5. We denote by $\tau$ the anti $\mathbb{C}$-linear involution corresponding to $g_{\mathbb{C}} = g \oplus \sqrt{-1}g$ (i.e. $\tau$ is the complex conjugation of $g_{\mathbb{C}}$ with respect to the real form $g$). Then, $H_{\phi^\vee}$ is in $a$ if and only if $\dim_{\mathbb{R}} g_{\lambda} \geq 2$ and

$$A_{\lambda^\vee} = \begin{cases} H_{\phi^\vee} & (\text{if } \dim_{\mathbb{R}} g_{\lambda} = 1), \\ H_{\phi^\vee} + \tau H_{\phi^\vee} & (\text{if } \dim_{\mathbb{R}} g_{\lambda} \geq 2). \end{cases}$$

In particular, we have another characterizations of $g$ for which $O_{\min,g}^{G_{\mathbb{C}}}$ is not $O_{\min}^{G_{\mathbb{C}}}$ from Proposition 4.1.

Proposition 5.6. The following conditions on $g$ are equivalent:

1. $O_{\min,\mathfrak{g}}^{G_{\mathbb{C}}} \neq O_{\min}^{G_{\mathbb{C}}}$. 
2. $O_{\min}^{G_{\mathbb{C}}} \cap g = \emptyset$. 
3. $\dim_{\mathbb{R}} g_{\lambda} \geq 2$. 
4. The highest root $\phi$ in $\Delta^+(g_{\mathbb{C}}, h_{\mathbb{C}})$ is not a real root. 
5. The weighted Dynkin diagram of $O_{\min}^{G_{\mathbb{C}}}$ matches the Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$ (see Definition §5.1). 
6. $g$ is isomorphic to $su^*(2k)$, $so(n-1,1)$, $sp(p,q)$, $e_6(-26)$ or $f_4(-20)$, where $k \geq 2$, $n \geq 5$ and $p \geq q \geq 1$.

We now determine the weighted Dynkin diagram of $O_{\min,\mathfrak{g}}^{G_{\mathbb{C}}}$ for the cases where $g$ is isomorphic to $su^*(2k)$, $so(n-1,1)$, $sp(p,q)$, $e_6(-26)$ or $f_4(-20)$. By Lemma 5.5, our purpose is to compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee} = H_{\phi^\vee} + \tau H_{\phi^\vee}$. We only give the computation for the case $g = e_6(-26)$ below. For the other $g$ with $\dim_{\mathbb{R}} g_{\lambda} \geq 2$, we can compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$ by the same way.

Example 5.7. Let $(g_{\mathbb{C}}, g) = (e_{6,\mathbb{C}}, e_6(-26))$. We denote the Satake diagram of $e_6(-26)$ by

\[ \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \circ \bullet \bullet \bullet - \bullet \alpha_6 \]
By Table 2, the weighted Dynkin diagram corresponding to $H_\phi^\vee$ is

$$
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet \\
& & & & \circ \\
& & & & \circ \\
& & & & \circ \\
\end{array}
$$

We now compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee} = H_\phi^\vee + \tau H_\phi^\vee$. By Lemma 5.3, the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$ matches the Satake diagram of $e_{6(-26)}$. Thus, we can put the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$ as

$$
\begin{array}{c}
\circ & \circ & \circ & \circ & \circ \\
& a & 0 & 0 & b \\
& 0 & & & \\
\end{array}
$$

(a, b \in \mathbb{R}).

To determine $a, b \in \mathbb{R}$, we also put

$$H_\phi^{im} = H_\phi^\vee - \tau H_\phi^\vee \in \sqrt{-1}t.$$

Since $A_{\lambda^\vee} + H_\phi^{im} = 2H_\phi^\vee$, the weighted Dynkin diagram corresponding to $H_\phi^{im}$ can be written by

$$
\begin{array}{c}
\circ & \circ & \circ & \circ & \circ \\
& -a & 0 & 0 & -b \\
& 0 & & & \\
\end{array}
$$

Namely, we have

$$\begin{align*}
\alpha_1(H_\phi^{im}) &= -a, \\
\alpha_2(H_\phi^{im}) &= \alpha_3(H_\phi^{im}) = \alpha_4(H_\phi^{im}) = 0, \\
\alpha_5(H_\phi^{im}) &= -b, \\
\alpha_6(H_\phi^{im}) &= 2.
\end{align*}$$

By Lemma 5.4, the set $\{H_{\alpha_2^\vee}, H_{\alpha_3^\vee}, H_{\alpha_4^\vee}, H_{\alpha_6^\vee}\}$ is a basis of $\sqrt{-1}t$. Thus, $H_\phi^{im} \in \sqrt{-1}t$ can be written by

$$H_\phi^{im} = c_2 H_{\alpha_2^\vee} + c_3 H_{\alpha_3^\vee} + c_4 H_{\alpha_4^\vee} + c_6 H_{\alpha_6^\vee} \quad (c_2, c_3, c_4, c_6 \in \mathbb{R}).$$

By the Dynkin diagram of $e_{6,C}$, we can compute

$$\alpha_i(H_{\alpha_j^\vee}) = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}$$
for each $i, j$. Thus, we also have

\[\begin{align*}
\alpha_1(H_{\phi^\vee}^{im}) &= -c_2, \\
\alpha_2(H_{\phi^\vee}^{im}) &= 2c_2 - c_3, \\
\alpha_3(H_{\phi^\vee}^{im}) &= -c_2 + 2c_3 - c_4 - c_6, \\
\alpha_4(H_{\phi^\vee}^{im}) &= -c_3 + 2c_4, \\
\alpha_5(H_{\phi^\vee}^{im}) &= -c_4, \\
\alpha_6(H_{\phi^\vee}^{im}) &= -c_3 + 2c_6.
\end{align*}\]

Then, we obtain that $a = b = 1$. Therefore, the weighted Dynkin diagram of $\mathcal{O}_{\text{min}, \mathfrak{g}}^{G_{\mathbb{C}}}$ for $\mathfrak{g} = \mathfrak{e}_{6(-26)}$ is

```
1 0 0 0 1
```

The result of our computation for all $\mathfrak{g}$ with $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$ is Table 1 in §1.

**References**


