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<th>Smallest complex nilpotent orbits with real points (Topics in Combinatorial Representation Theory)</th>
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Smallest complex nilpotent orbits with real points

Takayuki Okuda*

Abstract

In this paper, we show that there uniquely exists a real minimal nilpotent orbit in a non-compact simple Lie algebra $\mathfrak{g}$ if $(\mathfrak{g}, \mathfrak{t})$ is of non-Hermitian type. For the cases where $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n - 1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_6(-26)$ or $\mathfrak{f}_4(-20)$, the complexification $\mathcal{O}_{\text{min}, \mathfrak{g}}^{G_{\mathbb{C}}}$ of such the real minimal nilpotent orbit in $\mathfrak{g}$ is not the complex minimal nilpotent orbit in $\mathfrak{g}_C = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$. For such cases, we also determine $\mathcal{O}_{\text{min}, \mathfrak{g}}^{G_{\mathbb{C}}}$ by describing the weighted Dynkin diagram of it.

1 Introduction and main results

Let $\mathfrak{g}_C$ be a complex simple Lie algebra. In this paper, an adjoint nilpotent orbit in $\mathfrak{g}_C$ will be simply called a complex nilpotent orbit in $\mathfrak{g}_C$. It is well-known that there exists a unique non-zero complex nilpotent orbit $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_C$, which is called a complex minimal nilpotent orbit, with the following property: The closure of $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_C$ is just $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}} \cup \{0\}$. By the uniqueness of such $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}}$, for any non-zero complex nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}_C$, the closure of $\mathcal{O}$ contains $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}}$. In other words, $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}}$ is minimum in $\mathcal{N}/G_C$ without the zero-orbit, where $\mathcal{N}/G_C$ denotes the set of complex nilpotent orbits in $\mathfrak{g}_C$ with the closure ordering.

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_C$. Namely, $\mathfrak{g}$ is a non-compact real simple Lie algebra without complex structures and $\mathfrak{g}_C$ is the complexification of $\mathfrak{g}$. Our concern in this paper is in real minimal nilpotent orbits in $\mathfrak{g}$. Here, we say that a non-zero real nilpotent orbit $\mathcal{O}^G$ in $\mathfrak{g}$ is minimal if the closure

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of $\mathcal{O}^{G}$ in $\mathfrak{g}$ is just $\mathcal{O}^{G} \cup \{0\}$. In general, real minimal nilpotent orbits are not unique for real simple $\mathfrak{g}$.

If the complex minimal nilpotent orbit $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_{\mathbb{C}}$ meets $\mathfrak{g}$, then the intersection $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$. It is known that $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ meets $\mathfrak{g}$ if and only if $\mathfrak{g}$ is not isomorphic to $\text{su}^{\ast}(2k)$ ($k \geq 2$), $\text{so}(n-1,1)$ ($n \geq 5$), $\text{sp}(p,q)$ ($p \geq q \geq 1$), $f_{4(-20)}$ nor $\epsilon_{6(-26)}$ (see Brylinski [3, Theorem 4.1]). In particular, if $(\mathfrak{g}, \mathfrak{t})$ is of Hermitian type, then $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ meets $\mathfrak{g}$, where $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. Furthermore, for the cases where $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ meets $\mathfrak{g}$, the number of real minimal nilpotent orbits (i.e. the number of adjoint orbits in $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{g}$) is two if $(\mathfrak{g}, \mathfrak{t})$ is of Hermitian type; one if $(\mathfrak{g}, \mathfrak{t})$ is of non-Hermitian type.

In this paper, we study real minimal nilpotent orbits in $\mathfrak{g}$ including the cases where $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ does not meets $\mathfrak{g}$. For any real non-compact simple Lie algebra $\mathfrak{g}$ without complex structures, we put

$$\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}} := \{ \text{Complex nilpotent orbits in } \mathfrak{g}_{\mathbb{C}} \text{ meeting } \mathfrak{g} \}$$

and consider the closure ordering on it. Our first main result is here:

**Theorem 1.1.** There uniquely exists a complex nilpotent orbit $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_{\mathbb{C}}$ which is minimum in $\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}$ without the zero-orbit (i.e. for any non-zero complex nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$, if $\mathcal{O} \cap \mathfrak{g} \neq \emptyset$, then the closure of $\mathcal{O}$ in $\mathfrak{g}_{\mathbb{C}}$ contains $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$). Furthermore, the intersection $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$.

We will construct such $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ as the complex adjoint orbit through a non-zero longest restricted root vector in $\mathfrak{g}$. By the definition of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$, the complex minimal nilpotent orbit $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ is not our $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ if and only if $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ does not meet $\mathfrak{g}$ (namely, $\mathfrak{g}$ is isomorphic to $\text{su}^{\ast}(2k)$ ($k \geq 2$), $\text{so}(n-1,1)$ ($n \geq 5$), $\text{sp}(p,q)$ ($p \geq q \geq 1$), $f_{4(-20)}$ or $\epsilon_{6(-26)}$). This means that for such cases, a non-zero longest restricted root vector in $\mathfrak{g}$ is not a longest root vector in $\mathfrak{g}_{\mathbb{C}}$.

Theorem 1.1 claims that $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$. Our second main result is here:

**Theorem 1.2.** For the cases where the complex minimal nilpotent orbit $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ does not meet $\mathfrak{g}$, there exists a unique real minimal nilpotent orbit in $\mathfrak{g}$. In particular, the complex nilpotent orbit $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ in Theorem 1.1 (which is not $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ in these cases) is the complexification of the unique real minimal nilpotent orbit in $\mathfrak{g}$.
Therefore, we have the following corollary:

**Corollary 1.3.** Let $\mathfrak{g}$ be a non-compact real simple Lie algebra without complex structures. If $(\mathfrak{g}, \mathfrak{k})$ is of non-Hermitian type, there uniquely exists a real minimal nilpotent orbit in $\mathfrak{g}$. If $(\mathfrak{g}, \mathfrak{k})$ is of Hermitian type, there are just two real minimal nilpotent orbits in $\mathfrak{g}$.

By Theorem 1.2, our $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ is just the complexification of the unique real minimal nilpotent orbit in $\mathfrak{g}$ for the cases where $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$ ($k \geq 2$), $\mathfrak{so}(n - 1, 1)$ ($n \geq 5$), $\mathfrak{sp}(p, q)$ ($p \geq q \geq 1$), $\mathfrak{f}_4(-20)$ or $\mathfrak{e}_6(-26)$. We will determine our $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ by describing the weighted Dynkin diagram of it for such cases (recall that for another cases, $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ is just $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$). The result is here (see also Table 2 in §2 for the weighted Dynkin diagrams of $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$):

**Theorem 1.4.** For the cases where $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \neq \mathcal{O}_{\min}^{G_{\mathbb{C}}}$, the weighted Dynkin diagram of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ are the following:

<table>
<thead>
<tr>
<th>$\mathfrak{g}$</th>
<th>$\dim_{\mathbb{C}} \mathcal{O}<em>{\min, \mathfrak{g}}^{G</em>{\mathbb{C}}}$</th>
<th>Weighted Dynkin diagram of $\mathcal{O}<em>{\min, \mathfrak{g}}^{G</em>{\mathbb{C}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{su}^*(2k)$</td>
<td>$8k - 8$</td>
<td>$\begin{array}{c} 0 1 0 0 \cdots 0 0 1 0 \ (k \geq 3) \ 0 2 0 \ (k = 2) \end{array}$</td>
</tr>
<tr>
<td>$\mathfrak{so}(n - 1, 1)$</td>
<td>$2n - 4$</td>
<td>$\begin{array}{c} 2 0 0 \cdots 0 0 \ (n \text{ is odd}, \ n \geq 5) \ 2 0 0 \cdots 0 0 \ (n \text{ is even}, \ n \geq 6) \end{array}$</td>
</tr>
<tr>
<td>$\mathfrak{sp}(p, q)$</td>
<td>$4(p + q) - 2$</td>
<td>$\begin{array}{c} 0 1 0 0 \cdots 0 0 \ (p + q \geq 3, \ p \geq q \geq 1) \ 0 2 \ (p = q = 1) \end{array}$</td>
</tr>
<tr>
<td>$\mathfrak{e}_6(-26)$</td>
<td>$32$</td>
<td>$\begin{array}{c} 1 0 0 0 0 1 \ (p = q = 1) \end{array}$</td>
</tr>
<tr>
<td>$\mathfrak{f}_4(-20)$</td>
<td>$22$</td>
<td>$\begin{array}{c} 0 0 0 0 1 \end{array}$</td>
</tr>
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</table>

*Table 1: List of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ for $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n - 1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_6(-26)$ and $\mathfrak{f}_4(-20)$.**
This works motivated by recent works [7], by Joachim Hilgert, Toshiyuki Kobayashi and Jan Möllers, on the construction of an $L^2$-model of irreducible unitary representations of real reductive groups with smallest Gelfand-Kirillov dimension; and [8], by Toshiyuki Kobayashi and Yoshiki Oshima, on the classification of reductive symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ with a $(\mathfrak{g}, K)$-module which is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module.

2 Preliminary results for weighted Dynkin diagrams of complex minimal nilpotent orbits

In this section, we recall weighted Dynkin diagrams of complex minimal nilpotent orbits in complex simple Lie algebras.

Let $\mathfrak{g}_\mathbb{C}$ be a complex semisimple Lie algebra, and denote by $G_\mathbb{C}$ the inner automorphism group of $\mathfrak{g}_\mathbb{C}$. Fix a Cartan subalgebra $\mathfrak{h}_\mathbb{C}$ of $\mathfrak{g}_\mathbb{C}$. We denote by $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ the root system of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Then, the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ can be regarded as a subset of the dual space $\mathfrak{h}^*$ of

$$\mathfrak{h} := \{ H \in \mathfrak{h}_\mathbb{C} \mid \alpha(H) \in \mathbb{R} \ (\forall \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})) \}.$$

We write $W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ for the Weyl group of $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ acting on $\mathfrak{h}$. Take a positive system $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ of the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Then, a closed Weyl chamber

$$\mathfrak{h}_+ := \{ H \in \mathfrak{h} \mid \alpha(H) \geq 0 \ (\forall \alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})) \}$$

is a fundamental domain of $\mathfrak{h}$ under the action of $W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$.

Let $\Pi$ be the simple system of $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Then, for any $H \in \mathfrak{h}$, we can define a map

$$\Psi_H : \Pi \to \mathbb{R}, \ \alpha \mapsto \alpha(H).$$

We call $\Psi_H$ the weighted Dynkin diagram corresponding to $H \in \mathfrak{h}$, and $\alpha(H)$ the weight on a node $\alpha \in \Pi$ of the weighted Dynkin diagram. Since $\Pi$ is a basis of $\mathfrak{h}^*$, the map

$$\Psi : \mathfrak{h} \to \text{Map}(\Pi, \mathbb{R}), \ H \mapsto \Psi_H$$
is a linear isomorphism (between vector spaces). Furthermore,
\[ \mathfrak{h}_+ \to \text{Map}(\Pi, \mathbb{R}_{\geq 0}), \quad H \mapsto \Psi_H \]
is also bijective.

A triple \((H, X, Y)\) is said to be an \(sl_2\)-triple in \(g_C\) if
\[ [H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H \quad (H, X, Y \in g_C). \]

For any \(sl_2\)-triple \((H, X, Y)\) in \(g_C\), the elements \(X\) and \(Y\) are nilpotent in \(g_C\), and \(H\) is hyperbolic in \(g_C\) (i.e. \(\text{ad}_{g_C} H \in \text{End}(g_C)\) is diagonalizable with only real eigenvalues).

Combining the Jacobson–Morozov theorem with Kostant [9], for any complex nilpotent orbit \(O^{G_C}\), there uniquely exists an element \(H_\mathcal{O}\) of \(\mathfrak{h}_+\) with the following property: There exists \(X, Y \in O^{G_C}\) such that \((H_\mathcal{O}, X, Y)\) is an \(sl_2\)-triple in \(g_C\). Furthermore, by Malcev [10], the following map is injective:
\[ \{ \text{Complex nilpotent orbits in } g_C \} \hookrightarrow \mathfrak{h}_+, \quad O^{G_C} \mapsto H_\mathcal{O}. \]

The weighted Dynkin diagram corresponding to \(H_\mathcal{O}\) is called the weighted Dynkin diagram of \(O^{G_C}\). Dynkin [6] proved that for any complex nilpotent orbit \(O^{G_C}\), any weight of the weighted Dynkin diagram of \(O^{G_C}\) is given by 0, 1 or 2, and classified weighted Dynkin diagrams of complex nilpotent orbits (More precisely, Dynkin [6] classified \(sl_2\)-triples in \(g_C\). See Bala–Carter [2] for more details).

In the rest of this subsection, we suppose that \(g_C\) is simple. Let \(\phi\) be the highest root of \(\Delta^+(g_C, h_C)\). Then, the complex minimal nilpotent orbit in \(g_C\) can be written by
\[ O^{G_C}_{\min} = G_C \cdot g_\phi \setminus \{0\}. \]

We define the element \(H_{\phi^\vee}\) of \(h\) by
\[ \alpha(H_{\phi^\vee}) = \frac{2\langle \alpha, \phi \rangle}{\langle \phi, \phi \rangle} \]
for any \(\alpha \in h^*\) (where \(\langle , \rangle\) is the inner product on \(h^*\) induced by the Killing form on \(g_C\)). Namely, \(H_{\phi^\vee}\) is the element of \(h\) corresponding to the coroot \(\phi^\vee\) of \(\phi\). Since \(\phi\) is dominant, \(H_{\phi^\vee}\) is in \(h_+\). Furthermore, \(H_{\phi^\vee}\) is the hyperbolic element corresponding to \(O^{G_C}_{\min}\) since we can find \(X_\phi \in g_\phi, Y_\phi \in g_{-\phi}\) such that \((H_{\phi^\vee}, X_\phi, Y_\phi)\) is an \(sl_2\)-triple. The list of weighted Dynkin diagrams of \(O^{G_C}_{\min}\) for all simple \(g_C\) can be found in Collingwood–McGovern [4, Ch.5.4 and 8.4].
Recall that our concern in this paper is in real simple Lie algebras $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n - 1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_6(-26)$ and $\mathfrak{f}_4(-20)$. The complexifications of such algebras are $\mathfrak{sl}(2k, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{sp}(p + q, \mathbb{C})$, $\mathfrak{e}_{6,\mathbb{C}}$ and $\mathfrak{f}_{4,\mathbb{C}}$, respectively. For the convenience of the reader, we give a list of weighted Dynkin diagrams of complex minimal nilpotent orbits in such complex simple Lie algebras.

<table>
<thead>
<tr>
<th>$\mathfrak{g}_\mathbb{C}$</th>
<th>$\dim_{\mathbb{C}} \mathcal{O}<em>{\min}^{G</em>{\mathbb{C}}}$</th>
<th>Weighted Dynkin diagram of $\mathcal{O}<em>{\min}^{G</em>{\mathbb{C}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{sl}(n, \mathbb{C})$</td>
<td>$2n$</td>
<td>$\begin{array}{ccccccccccccc} 1 &amp; 0 &amp; 0 &amp; 0 &amp; \cdots &amp; 0 &amp; 0 &amp; 0 &amp; 1 \ \end{array}$ ($n \geq 2$)</td>
</tr>
<tr>
<td>$\mathfrak{so}(n, \mathbb{C})$</td>
<td>$2n - 6$</td>
<td>$\begin{array}{ccccccccccccc} 0 &amp; 1 &amp; 0 &amp; \cdots &amp; 0 &amp; 0 \ \end{array}$ ($n$ is odd, $n \geq 7$)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{array}{ccccccccccccc} 0 &amp; 1 &amp; \cdots &amp; 0 &amp; 0 \ \end{array}$ ($n = 5$)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{array}{ccccccccccccc} 0 &amp; 1 &amp; 0 &amp; \cdots &amp; 0 &amp; 0 \ \end{array}$ ($n$ is even, $n \geq 6$)</td>
</tr>
<tr>
<td>$\mathfrak{sp}(n, \mathbb{C})$</td>
<td>$2n$</td>
<td>$\begin{array}{ccccccccccccc} 1 &amp; 0 &amp; 0 &amp; 0 &amp; \cdots &amp; 0 &amp; 0 \ \end{array}$ ($n \geq 2$)</td>
</tr>
<tr>
<td>$\mathfrak{e}_{6,\mathbb{C}}$</td>
<td>$22$</td>
<td>$\begin{array}{ccccccccccccc} 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ \end{array}$</td>
</tr>
<tr>
<td>$\mathfrak{f}_{4,\mathbb{C}}$</td>
<td>$16$</td>
<td>$\begin{array}{ccccccccccccc} 1 &amp; 0 &amp; 0 &amp; 0 \ \end{array}$</td>
</tr>
</tbody>
</table>

Table 2: List of weighted Dynkin diagrams of $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ for $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{sp}(n, \mathbb{C})$, $\mathfrak{e}_{6,\mathbb{C}}$ and $\mathfrak{f}_{4,\mathbb{C}}$.

### 3 Outline of a proof of Theorem 1.1

Let $\mathfrak{g}_\mathbb{C}$ be a complex simple Lie algebra and $\mathfrak{g}$ a non-compact real form of $\mathfrak{g}$ with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In this section, we describe an idea of the proof of Theorem 1.1.

We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ (such $\mathfrak{a}$ is called a maximally split abelian subspace of $\mathfrak{g}$) and write $\Sigma(\mathfrak{g}, \mathfrak{a})$ for the restricted root system
for \((g, a)\). For any restricted root \(\xi\) of \(\Sigma(g, a)\), we define \(A_{\xi^\vee} \in a\) by
\[
\eta(A_{\xi^\vee}) = \frac{2(\xi, \eta)}{\langle \xi, \xi \rangle} \quad (\forall \eta \in a^*)
\]
(where \((\ , \ )\) is the inner product on \(a^*\) induced by the Killing form on \(g\)). Namely, \(A_{\xi^\vee}\) is the element of \(a\) corresponding to the coroot \(\xi^\vee\) of \(\xi\). Then, the fact below holds:

**Fact 3.1.** For any restricted root \(\xi\) of \(\Sigma(g, a)\) and any non-zero root vector \(X_\xi\) in \(g_\xi\), there exists \(Y_\xi \in g_{-\xi}\) such that \((A_{\xi^\vee}, X_\xi, Y_\xi)\) is an \(sL_2\)-triple in \(g\).

We fix an ordering on \(a\) and write \(\Sigma^+(g, a)\) for the positive system of \(\Sigma(g, a)\) corresponding to the ordering on \(a\). We denote by \(\lambda\) the highest root of \(\Sigma^+(g, a)\) with respect to the ordering on \(a\). Next two lemmas give characterizations of the highest root \(\lambda\) of \(\Sigma^+(g, a)\) (we omit proofs of the two lemmas in this paper):

**Lemma 3.2.** The highest root \(\lambda\) of \(\Sigma^+(g, a)\) is a unique dominant longest root of \(\Sigma(g, a)\).

**Lemma 3.3.** Let \(\xi\) be a root of \(\Sigma(g, a)\). If \(\xi\) is not the highest root \(\lambda\), then for any non-zero root vector \(X_\xi\) in \(g_\xi\), there exists a positive root \(\eta\) in \(\Sigma^+(g, a)\) and a root vector \(X_\eta \in g_\eta\) such that \([X_\xi, X_\eta] \neq 0\). In particular, \(\xi = \lambda\) if and only if \(\xi + \eta \in a^*\) is not a root of \(\Sigma(g, a)\) for any \(\eta \in \Sigma^+(g, a)\).

We write \(G_C\) for the inner automorphism group of \(g_C\). Then, the following two propositions hold:

**Proposition 3.4.** For any non-zero real nilpotent orbit \(O'_0\) in \(g\). Then, there exists a non-zero highest root vector \(X_\lambda\) in \(g_\lambda\) such that \(X_\lambda\) is in the closure of \(O'_0\) in \(g\).

**Proposition 3.5.** For any two highest root vectors \(X_\lambda, X'_\lambda\) in \(g_\lambda\), there exists \(g_C \in G_C\) such that \(g_C X_\lambda = X'_\lambda\).

**Proof of Proposition 3.4.** There is no loss of generality in assuming that the ordering on \(a\) is lexicographic. Let us put \(m = Z_\xi(a)\). Then, \(g\) can be decomposed as
\[
g = m \oplus a \oplus \bigoplus_{\xi \in \Sigma(g, a)} g_\xi.
\]
For any $X' \in g$, we denote by

$$X' = X'_m + X'_a + \sum_{\xi \in \Sigma(g, a)} X'_\xi \quad (X'_m \in m, \ X'_a \in a, \ X'_\xi \in g_\xi).$$

We put $\overline{O'_0}$ to the closure of $O'_0$ in $g$ and fix an element $X'$ in $\overline{O'_0}$. Let us denote by $\lambda'$ the highest one of

$$\Sigma_{X'} := \{ \xi \in \Sigma(g, a) \mid X'_\xi \neq 0 \}$$

with respect to the ordering on $a$ (if $X' \neq 0$, then $\Sigma_{X'}$ is not empty since $X'$ is nilpotent element in $g$). As a first step of the proof, we shall prove that the root vector $X'_{\lambda'}$ is also in $\overline{O'_0}$. We take $A' \in a$ satisfying that

$$\xi(A') < \lambda'(A') \quad (\forall \xi \in \Sigma_{X'} \setminus \{ \lambda' \}).$$

(such $A'$ exists since $\lambda'$ is highest in $\Sigma_{X'}$ with respect to the lexicographic ordering on $a$). Let us put

$$X'_k := \frac{1}{e^{k\lambda'(A')}} \exp(\text{ad}_{g} kA') X' \quad (\text{for } k \in \mathbb{N})$$

Then, $X'_k$ is in $\overline{O'_0}$ for any $k$ since $\overline{O'_0}$ is stable by positive scalars. Furthermore,

$$\lim_{k \to \infty} X'_k = \lim_{k \to \infty} \sum_{\xi \in \Sigma_{X'}} e^{k(\xi' - \lambda')} X'_\xi = X'_{\lambda'}.$$

This means that $X'_{\lambda'}$ is in $\overline{O'_0}$. To complete the proof, we only need to show that there exists $X' \in \overline{O'_0}$ such that $\lambda' = \lambda$ (where $\lambda'$ is the highest one of $\Sigma_{X'}$). Let $\lambda_0$ be the highest one of

$$\Sigma_{\overline{O'_0}} := \{ \xi \in \Sigma(g, a) \mid \exists X' \in \overline{O'_0} \text{ such that } X'_\xi \neq 0 \}$$

(namely, $\Sigma_{\overline{O'_0}} = \bigcup_{X' \in \overline{O'_0}} \Sigma_{X'}$) with respect to the ordering on $a$. Then, we can find a root vector $X'_{\lambda_0}$ in $g_{\lambda_0} \cap \overline{O'_0}$ by the argument above. We assume that $\lambda_0 \neq \lambda$. Then, by Lemma 3.3, there exists $\eta \in \Sigma^+(g, a)$ and $X_\eta \in g_\eta$ such that $[X_\eta, X'_{\lambda_0}] \neq 0$. In particular, for the element $X'' := \exp(\text{ad}_{g}(X_\eta)) X'_{\lambda_0}$ in $\overline{O'_0}$, we obtain that

$$\lambda_0 + \eta \in \Sigma_{X''} \subset \Sigma_{\overline{O'_0}}.$$

This contradicts the definition of $\lambda_0$. Thus, $\lambda_0 = \lambda$. \qed
Proof of Proposition 3.5. Let $A_{\lambda^\vee}$ be the element in $\mathfrak{a}$ corresponding to the coroot $\lambda^\vee$ of the highest root $\lambda$. We put

$$(\mathfrak{g}_C)_2 = \{ X \in \mathfrak{g}_C \mid [A_{\lambda^\vee}, X] = 2X \}.$$ 

Then, $\mathfrak{g}_\lambda$ is included in $(\mathfrak{g}_C)_2$. We note that there exists $X, Y \in \mathfrak{g}_C$ such that $(A_{\lambda^\vee}, X, Y)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_C$ (in fact, we can find such $X, Y$ in $\mathfrak{g}_\lambda$ by Fact 3.1). Therefore, we can use Malcev's theorem. Namely, for any two non-zero vectors $X$ and $X'$ in $(\mathfrak{g}_C)_2$, there exists $g_C \in G_C$ such that $g_C X = X'$. Since $\mathfrak{g}_\lambda \subset (\mathfrak{g}_C)_2$, the proof is completed. \qed

By using Proposition 3.4 and Proposition 3.5, Theorem 1.1 follows by taking $\mathcal{O}_{\min,\mathfrak{g}}^{G_C}$ as

$$\mathcal{O}_{\min,\mathfrak{g}}^{G_C} := G_C \cdot \mathfrak{g}_\lambda \setminus \{0\}.$$ 

4 Outline of a proof of Theorem 1.2

Let us consider the same setting in §3. Recall that $\mathcal{O}_{\min,\mathfrak{g}}^{G_C}$ is not the complex minimal nilpotent orbit $\mathcal{O}_{\min}^{G_C}$ if and only if $\mathcal{O}_{\min}^{G_C}$ does not meet $\mathfrak{g}$. The proposition below give a characterization of $\mathfrak{g}$ for which $\mathcal{O}_{\min}^{G_C}$ is not $\mathcal{O}_{\min,\mathfrak{g}}^{G_C}$ (see Proposition 5.6 for another characterizations of it).

Proposition 4.1. The following conditions on $\mathfrak{g}$ are equivalent:

1. $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$.

2. $\mathcal{O}_{\min}^{G_C} \cap \mathfrak{g} = \emptyset$.

We can prove the proposition without any classification, but we omit it in this paper.

Here, we put $\mathfrak{m} := Z_\mathbb{R}(\mathfrak{a})$ and denote by $M_0, A$ to the analytic subgroups of $G$ corresponding to $\mathfrak{m}, \mathfrak{a}$, respectively. Then, the connected Lie group $M_0 A$ (which is the analytic subgroup of $G$ corresponding to $\mathfrak{m} \oplus \mathfrak{a}$) acts on $\mathfrak{a}$. Furthermore, the following proposition holds:

Proposition 4.2. If $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$, then $\mathfrak{g}_\lambda \setminus \{0\}$ is a single $M_0 A$-orbit.

Combining Proposition 3.4, Proposition 4.1 with Proposition 4.2, we obtain Theorem 1.2.

We will use the next lemma to prove Proposition 4.2.
Lemma 4.3. Suppose that $g$ has real rank one (i.e. $\dim_{\mathbb{R}} a = 1$) and $\dim_{\mathbb{R}} g_{\lambda} \geq 2$. Then, $g_{\lambda} \setminus \{0\}$ is a single $M_{0}A$-orbit.

Proof of Lemma 4.3. Let $A_{\lambda}^{\vee}$ be the element of $a$ corresponding to the coroot $\lambda^{\vee}$ of the highest root $\lambda$ in $\Sigma^{+}(g, a)$ (see §3). Since $g$ has real rank one, we have $a = \mathbb{R}A_{\lambda}^{\vee}$, and $g$ can be written by

$$g = g_{-\lambda} \oplus g_{-\frac{\lambda}{2}} \oplus m \oplus a \oplus g_{\frac{\lambda}{2}} \oplus g_{\lambda}$$

($g_{\pm\frac{\lambda}{2}}$ can be zero). Let us denote by $g_{\mathbb{C}}, m_{\mathbb{C}}, a_{\mathbb{C}}, (g_{\pm\lambda})_{\mathbb{C}}, (g_{\pm\frac{\lambda}{2}})_{\mathbb{C}}$ the complexification of $g, m, a, g_{\pm\lambda}, g_{\pm\frac{\lambda}{2}}$, respectively. We set

$$(g_{\mathbb{C}})_{i} = \{ X \in g_{\mathbb{C}} \mid [A_{\lambda}^{\vee}, X] = iX \} \quad \text{(for } i \in \mathbb{Z}).$$

Then,

$$(g_{\mathbb{C}})_{0} = m_{\mathbb{C}} \oplus a_{\mathbb{C}}, \quad (g_{\mathbb{C}})_{\pm 1} = (g_{\pm\frac{\lambda}{2}})_{\mathbb{C}}, \quad (g_{\mathbb{C}})_{\pm 2} = (g_{\pm\lambda})_{\mathbb{C}}.$$  

By Fact 3.1, for any non-zero highest root vector $X_{\lambda}$ in $g_{\lambda}$, there exists $Y_{\lambda} \in g_{-\lambda}$ such that $(A_{\lambda}, X_{\lambda}, Y_{\lambda})$ is an $sl_{2}$-triple in $g_{\mathbb{C}}$. By the theory of representations of $sl(2, \mathbb{C})$, we obtain that $[(g_{\mathbb{C}})_{0}, X_{\lambda}] = (g_{\mathbb{C}})_{2}$. In particular, we have

$$[m \oplus a, X_{\lambda}] = g_{\lambda}.$$  

Therefore, for the $M_{0}A$-orbit $O^{M_{0}A}(X_{\lambda})$ in $g_{\lambda}$ through $X_{\lambda}$, we obtain that

$$\dim_{\mathbb{R}} O^{M_{0}A}(X_{\lambda}) = \dim_{\mathbb{R}} g_{\lambda}.$$  

This means that the $M_{0}A$-orbit $O^{M_{0}A}(X_{\lambda})$ is open in $g_{\lambda}$ for any non-zero root vector $X_{\lambda}$ in $g_{\lambda}$. Recall that we are assuming that $\dim_{\mathbb{R}} g_{\lambda} \geq 2$. Hence, $g_{\lambda} \setminus \{0\}$ is connected. Therefore, $g_{\lambda} \setminus \{0\}$ is a single $M_{0}A$-orbit. \hfill \Box

We are ready to prove Proposition 4.2.

Sketch of a proof of Proposition 4.2. Let $h' := [g_{\lambda}, g_{-\lambda}] \subset m \oplus a$. Then $g' := g_{-\lambda} \oplus h' \oplus g_{\lambda}$ becomes a subalgebra of $g$ (since $\pm 2\lambda$ is not a root). Furthermore, one can prove that $g'$ is a real rank one simple Lie algebra with a maximally split abelian subspace $a' := \mathbb{R}A_{\lambda}^{\vee}$, where $A_{\lambda}^{\vee}$ is the element of $a$ corresponding to the coroot $\lambda^{\vee}$ of the highest root $\lambda$ in $\Sigma^{+}(g, a)$ (see §3). We put $m' \oplus a' := Z_{g'}(a')$ and denote by $M'_{0}A'$ the analytic subgroup of $G$ corresponding to $m' \oplus a'$. Then, by Lemma 4.3, we obtain that $g_{\lambda} \setminus \{0\}$ is a single $M'_{0}A'$-orbit. Since $M'_{0}A'$ is a subgroup of $M_{0}A$, the proof is completed. \hfill \Box
5 Determination of $O_{\text{min},g}^{G_C}$

In this section, we determine $O_{\text{min},g}^{G_C}$ by describing the weighted Dynkin diagram of $O_{\text{min},g}^{G_C}$. Recall that Proposition 4.1 claims that $O_{\text{min}}^{G_C} = O_{\text{min},g}^{G_C}$ if and only if $\dim_R g_\lambda \geq 2$ (i.e. $g$ is isomorphic to $\mathfrak{su}(2k)$, $\mathfrak{so}(n - 1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_{6(-26)}$ or $\mathfrak{f}_{4(-20)}$).

5.1 Satake diagrams and weighted Dynkin diagrams

In order to determine the weighted Dynkin diagram of our $O_{\text{min},g}^{G_C}$, we describe some lemmas of relationship between weighted Dynkin diagrams of $g_C$ and Satake diagrams of $g$ in this subsection.

Let $g_C$ be a semisimple Lie algebra and $g$ a real form of it through this subsection. First, we recall briefly the definition of Satake diagram of a real form $g$ of a complex semisimple Lie algebra $g_C$ (see also [1] for more details). Fix a Cartan decomposition $g = \mathfrak{k} \oplus \mathfrak{p}$ of $g$. We take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$, and extend it to a maximal abelian subspace $\mathfrak{h} = \sqrt{-1} \mathfrak{k} \oplus \mathfrak{a}$ in $\sqrt{-1} \mathfrak{k} \oplus \mathfrak{p}$. Then, the complexification, denoted by $\mathfrak{h}_C$, of $\mathfrak{h}$ is a Cartan subalgebra of $g_C$, and $\mathfrak{h}$ coincide with the real form

$$\{X \in \mathfrak{h}_C \mid \alpha(X) \in \mathbb{R} \text{ if } \alpha \in \Delta(g_C, \mathfrak{h}_C)\}$$

of $\mathfrak{h}_C$, where $\Delta(g_C, \mathfrak{h}_C)$ is the root system of $(g_C, \mathfrak{h}_C)$. Let us denote by

$$\Sigma(g, \mathfrak{a}) := \{\alpha|_a \mid \alpha \in \Delta(g_C, \mathfrak{h}_C)\} \setminus \{0\} \subset \mathfrak{a}^*$$

the restricted root system of $(g, \mathfrak{a})$. We will denote by $W(g, \mathfrak{a})$, $W(g_C, \mathfrak{h}_C)$ the Weyl group of $\Sigma(g, \mathfrak{a})$, $\Delta(g_C, \mathfrak{h}_C)$, respectively. Fix an ordering on $\mathfrak{a}$ and extend it to an ordering on $\mathfrak{h}$. We write $\Sigma^+(g, \mathfrak{a})$, $\Delta^+(g_C, \mathfrak{h}_C)$ for the positive system of $\Sigma(g, \mathfrak{a})$, $\Delta(g_C, \mathfrak{h}_C)$ corresponding to the ordering on $\mathfrak{a}$, $\mathfrak{h}$, respectively. Then, $\Sigma^+(g, \mathfrak{a})$ can be written by

$$\Sigma^+(g, \mathfrak{a}) = \{\alpha|_a \mid \alpha \in \Delta^+(g_C, \mathfrak{h}_C)\} \setminus \{0\}.$$

We denote by $\Pi$ the fundamental system of $\Delta^+(g_C, \mathfrak{h}_C)$. Then,

$$\Pi := \{\alpha|_a \mid \alpha \in \Pi\} \setminus \{0\}$$

is the simple system of $\Sigma^+(g, \mathfrak{a})$. Let $\Pi_0$ be the set of all simple roots in $\Pi$ whose restriction to $\mathfrak{a}$ is zero. The Satake diagram $S_\mathfrak{a}$ of $g$ consists of the
following data: The Dynkin diagram of \( \mathfrak{g}_C \) with nodes \( \Pi \); black nodes \( \Pi_0 \) in \( S_\mathfrak{g} \); and arrows joining \( \alpha \in \Pi \setminus \Pi_0 \) and \( \beta \in \Pi \setminus \Pi_0 \) in \( S_\mathfrak{g} \) whose restrictions to \( \alpha \) are the same.

Second, we define that a weighted Dynkin diagram \( \Psi_H \in \text{Map}(\Pi, \mathbb{R}) \) “matches” the Satake diagram \( S_\mathfrak{g} \) of \( \mathfrak{g} \) as follows:

**Definition 5.1.** Let \( \Psi_H \in \text{Map}(\Pi, \mathbb{R}) \) be a weighted Dynkin diagram (see §2) and \( S_\mathfrak{g} \) the Satake diagram of \( \mathfrak{g} \) with nodes \( \Pi \). We say that \( \Psi_H \) matches \( S_\mathfrak{g} \) if all the weights on black nodes are zero and any pair of nodes joined by an arrow has the same weights.

**Remark 5.2.** The concept of “match” defined above is same as “weighted Satake diagrams” in Djocovic [5] and the condition described in Sekiguchi [11, Proposition 1.16].

Recall that \( \Psi \) is a linear isomorphism from \( \mathfrak{h} \) to \( \text{Map}(\Pi, \mathbb{R}) \) (see §2). Then, the next two lemmas hold (we omit proofs of the two lemmas in this paper):

**Lemma 5.3.** \( \Psi : \mathfrak{h} \rightarrow \text{Map}(\Pi, \mathbb{R}) \) induces a linear isomorphism below:

\[
\alpha \rightarrow \{ \Psi_H \in \text{Map}(\Pi, \mathbb{R}) \mid \Psi_H \text{ matches } S_\mathfrak{g} \}.
\]

**Lemma 5.4.** For each simple root \( \alpha \) of \( \Pi \), we denote by \( H_\alpha^\vee \) the element in \( \mathfrak{h} \) corresponding to the coroot \( \alpha^\vee \) of the simple root \( \alpha \). Then, the set

\[
\{ H_\alpha^\vee \mid \alpha \text{ is black in } S_\mathfrak{g} \}\cup\{ H_\alpha^\vee - H_\beta^\vee \mid \alpha \text{ and } \beta \text{ are joined by an arrow in } S_\mathfrak{g} \}
\]

is a basis of \( \sqrt{-1}t \).

Lemma 5.3 and Lemma 5.4 will be used to compute the weighted Dynkin diagrams of \( \mathcal{O}_{\text{min}, \mathfrak{g}}^{G_C} \) for the cases where \( \mathcal{O}_{\text{min}, \mathfrak{g}}^{G_C} \) is not the complex minimal nilpotent orbit \( \mathcal{O}_{\text{min}}^{G_C} \).

Recall that our concern in this paper is in real simple Lie algebras \( \mathfrak{su}^*(2k) \), \( \mathfrak{so}(n - 1, 1) \), \( \mathfrak{sp}(p, q) \), \( \mathfrak{e}_6(-26) \) and \( \mathfrak{f}_4(-20) \). For the convenience of the reader, we give a list of Satake diagrams of such simple Lie algebras.

<table>
<thead>
<tr>
<th>( \mathfrak{g} )</th>
<th>Satake diagrams of ( \mathfrak{g} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathfrak{su}^*(2k) )</td>
<td><img src="image" alt="Diagram of ( \mathfrak{su}^*(2k) )" /></td>
</tr>
</tbody>
</table>

\( \alpha_{2k-1} \)
\(\text{so}(n-1,1)\) for \((n-1,1)\) (\(n\) is odd, \(n \geq 5\))

\(\text{sp}(p,q)\) for \((p,q)\) (\(p \geq q \geq 1\))

\(\mathfrak{e}_6(-26)\)

\(f_4(-20)\)

| Table 3: List of Satake diagrams of \(\mathfrak{su}^*(2k), \text{so}(n-1,1), \text{sp}(p,q), \mathfrak{e}_6(-26)\) and \(f_4(-20)\). |

| 5.2 Computation of weighted Dynkin diagrams of \(\mathcal{O}^{G_{\mathbb{C}}}_{\text{min},g}\) |

We consider the same setting on §5.1 and suppose that \(g_{\mathbb{C}}\) is simple and \(g\) is non-compact. Let us denote by

\[a_+ := \{ A \in a \mid \xi(A) \geq 0 \ (\forall \xi \in \Sigma^+(g, a)) \}.\]

Then \(a_+\) is a fundamental domain of \(a\) under the action of \(W(g, a)\). Since

\[\Sigma^+(g, a) = \{ \alpha|_a \mid \alpha \in \Delta(g_{\mathbb{C}}, h_{\mathbb{C}}) \} \setminus \{0\},\]

the domain \(a_+\) coincide with \(h_+ \cap a\). Recall that \(\lambda\) is dominant (by Lemma 3.2) and \(\mathcal{O}^{G_{\mathbb{C}}}_{\text{min},g}\) contains \(g_\lambda \setminus \{0\}\) (by the proof of Theorem 1.1). Thus, \(A_{\lambda^\vee}\) is the hyperbolic element in \(a_+\) corresponding to \(\mathcal{O}^{G_{\mathbb{C}}}_{\text{min},g}\) (see §2) since we can find \(X_\lambda \in g_\lambda, Y_\lambda \in g_{-\lambda}\) such that the triple \((A_{\lambda^\vee}, X_\lambda, Y_\lambda)\) is an \(\mathfrak{sl}_2\)-triple in \(g_{\mathbb{C}}\) by Lemma 3.1 (then, \(X_\lambda, Y_\lambda \in \mathcal{O}^{G_{\mathbb{C}}}_{\text{min},g}\)). Therefore, to determine the weighted Dynkin diagram of \(\mathcal{O}^{G_{\mathbb{C}}}_{\text{min},g}\), we shall compute the weighted Dynkin diagram corresponding to \(A_{\lambda^\vee}\).

Let \(\phi\) be the highest root of \(\Delta^+(g_{\mathbb{C}}, h_{\mathbb{C}})\). Recall that the complex minimal nilpotent orbit \(\mathcal{O}^{G_{\mathbb{C}}}_{\text{min}}\) contains the root space \((g_{\mathbb{C}})_\phi\) without zero, and the weighted Dynkin diagram of \(\mathcal{O}^{G_{\mathbb{C}}}_{\text{min}}\) is the weighted Dynkin diagram corresponding to \(H_{\phi^\vee}\) (see §2). The next lemma gives a formula for \(A_{\lambda^\vee}\) by \(H_{\phi^\vee}\) (we omit a proof of the lemma):
Lemma 5.5. We denote by $\tau$ the anti $\mathbb{C}$-linear involution corresponding to $g_{\mathbb{C}} = g \oplus \sqrt{-1}g$ (i.e. $\tau$ is the complex conjugation of $g_{\mathbb{C}}$ with respect to the real form $g$). Then, $H_{\phi^\vee}$ is in $a$ if and only if $\dim_{\mathbb{R}} g_\lambda \geq 2$ and

$$A_{\lambda^\vee} = \begin{cases} H_{\phi^\vee} & \text{(if } \dim_{\mathbb{R}} g_\lambda = 1), \\ H_{\phi^\vee} + \tau H_{\phi^\vee} & \text{(if } \dim_{\mathbb{R}} g_\lambda \geq 2). \end{cases}$$

In particular, we have another characterizations of $g$ for which $O_{\min,a}$ is not $O_{\min}^{G_{\mathbb{C}}}$ from Proposition 4.1.

Proposition 5.6. The following conditions on $g$ are equivalent:

1. $O_{\min,a}^{G_{\mathbb{C}}} \neq O_{\min}^{G_{\mathbb{C}}}$. 
2. $O_{\min}^{G_{\mathbb{C}}} \cap g = \emptyset$. 
3. $\dim_{\mathbb{R}} g_\lambda \geq 2$. 
4. The highest root $\phi$ in $\Delta^+(g_{\mathbb{C}}, h_{\mathbb{C}})$ is not a real root. 
5. The weighted Dynkin diagram of $O_{\min}^{G_{\mathbb{C}}}$ matches the Satake diagram $S_\mathfrak{g}$ of $g$ (see Definition §5.1). 
6. $g$ is isomorphic to $su^*(2k)$, $so(n-1,1)$, $sp(p,q)$, $e_{6(-26)}$ or $f_4(-20)$, where $k \geq 2$, $n \geq 5$ and $p \geq q \geq 1$.

We now determine the weighted Dynkin diagram of $O_{\min,a}^{G_{\mathbb{C}}}$ for the cases where $g$ is isomorphic to $su^*(2k)$, $so(n-1,1)$, $sp(p,q)$, $e_{6(-26)}$ or $f_4(-20)$. By Lemma 5.5, our purpose is to compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee} = H_{\phi^\vee} + \tau H_{\phi^\vee}$. We only give the computation for the case $g = e_{6(-26)}$ below. For the other $g$ with $\dim_{\mathbb{R}} g_\lambda \geq 2$, we can compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$ by the same way.

Example 5.7. Let $(g_{\mathbb{C}}, g) = (e_{6,\mathbb{C}}, e_{6(-26)})$. We denote the Satake diagram of $e_{6(-26)}$ by

```
\begin{array}{cccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\
\circ \bullet \bullet \bullet \circ \\
\bullet \alpha_6
\end{array}
```
By Table 2, the weighted Dynkin diagram corresponding to $H_{\phi^\vee}$ is

$$\begin{align*}
0 & 0 & 0 & 0 \\
\circ & \circ & \circ & \circ & 1
\end{align*}$$

We now compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee} = H_{\phi^\vee} + \tau H_{\phi^\vee}$. By Lemma 5.3, the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$ matches the Satake diagram of $\mathfrak{e}_6(-26)$. Thus, we can put the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$ as

$$a \ 0 \ 0 \ 0 \ b \quad (a, b \in \mathbb{R}).$$

To determine $a, b \in \mathbb{R}$, we also put

$$H_{\phi^\vee}^{im} = H_{\phi^\vee} - \tau H_{\phi^\vee} \in \sqrt{-1}t.$$  

Since $A_{\lambda^\vee} + H_{\phi^\vee}^{im} = 2H_{\phi^\vee}$, the weighted Dynkin diagram corresponding to $H_{\phi^\vee}^{im}$ can be written by

$$-a \ 0 \ 0 \ 0 \ -b \quad 2$$

Namely, we have

$$\begin{align*}
\alpha_1(H_{\phi^\vee}^{im}) & = -a, \\
\alpha_2(H_{\phi^\vee}^{im}) & = \alpha_3(H_{\phi^\vee}^{im}) = \alpha_4(H_{\phi^\vee}^{im}) = 0, \\
\alpha_5(H_{\phi^\vee}^{im}) & = -b, \\
\alpha_6(H_{\phi^\vee}^{im}) & = 2.
\end{align*}$$

By Lemma 5.4, the set \{ $H_{\alpha_2^\vee}, H_{\alpha_3^\vee}, H_{\alpha_4^\vee}, H_{\alpha_6^\vee}$ \} is a basis of $\sqrt{-1}t$. Thus, $H_{\phi^\vee}^{im} \in \sqrt{-1}t$ can be written by

$$H_{\phi^\vee}^{im} = c_2 H_{\alpha_2^\vee} + c_3 H_{\alpha_3^\vee} + c_4 H_{\alpha_4^\vee} + c_6 H_{\alpha_6^\vee} \quad (c_2, c_3, c_4, c_6 \in \mathbb{R}).$$

By the Dynkin diagram of $\mathfrak{e}_{6,C}$, we can compute

$$\alpha_i(H_{\alpha_2^\vee}) = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}$$
for each $i,j$. Thus, we also have

\begin{align*}
\alpha_1(H_{\phi^\vee}^{im}) &= -c_2, \\
\alpha_2(H_{\phi^\vee}^{im}) &= 2c_2 - c_3, \\
\alpha_3(H_{\phi^\vee}^{im}) &= -c_2 + 2c_3 - c_4 - c_6, \\
\alpha_4(H_{\phi^\vee}^{im}) &= -c_3 + 2c_4, \\
\alpha_5(H_{\phi^\vee}^{im}) &= -c_4, \\
\alpha_6(H_{\phi^\vee}^{im}) &= -c_3 + 2c_6.
\end{align*}

Then, we obtain that $a = b = 1$. Therefore, the weighted Dynkin diagram of $O_{\min,\mathfrak{g}}^{G_{\mathbb{C}}}$ for $\mathfrak{g} = \mathfrak{e}_6(-26)$ is

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,0) -- (2,0) -- (3,0) -- (4,0) -- (5,0) -- (6,0);
\node at (0,0) [below] {$0$};
\node at (5,0) [below] {$1$};
\node at (3.5,0) [below] {$0$};
\node at (2.5,0) [below] {$0$};
\node at (1.5,0) [below] {$0$};
\end{tikzpicture}
\end{center}

The result of our computation for all $\mathfrak{g}$ with $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$ is Table 1 in §1.

References


