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<th>Smallest complex nilpotent orbits with real points (Topics in Combinatorial Representation Theory)</th>
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<td>Okuda, Takayuki</td>
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Kyoto University
Smallest complex nilpotent orbits with real points

Takayuki Okuda*

Abstract

In this paper, we show that there uniquely exists a real minimal nilpotent orbit in a non-compact simple Lie algebra $\mathfrak{g}$ if $(\mathfrak{g}, \mathfrak{t})$ is of non-Hermitian type. For the cases where $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n - 1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_6(-26)$ or $\mathfrak{f}_4(-20)$, the complexification $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ of such the real minimal nilpotent orbit in $\mathfrak{g}$ is not the complex minimal nilpotent orbit in $\mathfrak{g}_C = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$. For such cases, we also determine $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ by describing the weighted Dynkin diagram of it.

1 Introduction and main results

Let $\mathfrak{g}_C$ be a complex simple Lie algebra. In this paper, an adjoint nilpotent orbit in $\mathfrak{g}_C$ will be simply called a complex nilpotent orbit in $\mathfrak{g}_C$. It is well-known that there exists a unique non-zero complex nilpotent orbit $\mathcal{O}_{\min}^{\mathfrak{g}_C}$ in $\mathfrak{g}_C$, which is called a complex minimal nilpotent orbit, with the following property: The closure of $\mathcal{O}_{\min}^{\mathfrak{g}_C}$ in $\mathfrak{g}_C$ is just $\mathcal{O}_{\min}^{\mathfrak{g}_C} \cup \{0\}$. By the uniqueness of such $\mathcal{O}_{\min}^{\mathfrak{g}_C}$, for any non-zero complex nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}_C$, the closure of $\mathcal{O}$ contains $\mathcal{O}_{\min}^{\mathfrak{g}_C}$. In other words, $\mathcal{O}_{\min}^{\mathfrak{g}_C}$ is minimum in $\mathcal{N}/G_C$ without the zero-orbit, where $\mathcal{N}/G_C$ denotes the set of complex nilpotent orbits in $\mathfrak{g}_C$ with the closure ordering.

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_C$. Namely, $\mathfrak{g}$ is a non-compact real simple Lie algebra without complex structures and $\mathfrak{g}_C$ is the complexification of $\mathfrak{g}$. Our concern in this paper is in real minimal nilpotent orbits in $\mathfrak{g}$. Here, we say that a non-zero real nilpotent orbit $\mathcal{O}^G$ in $\mathfrak{g}$ is minimal if the closure

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of $O^G$ in $\mathfrak{g}$ is just $O^G \cup \{0\}$. In general, real minimal nilpotent orbits are not unique for real simple $\mathfrak{g}$.

If the complex minimal nilpotent orbit $O^G_{\min, \mathfrak{g}}$ in $\mathfrak{g}_C$ meets $\mathfrak{g}$, then the intersection $O^G_{\min} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$. It is known that $O^G_{\min}$ meets $\mathfrak{g}$ if and only if $\mathfrak{g}$ is not isomorphic to $\text{su}^*(2k)$ ($k \geq 2$), $\text{so}(n - 1, 1)$ ($n \geq 5$), $\text{sp}(p, q)$ ($p \geq q \geq 1$), $f_{4(-20)}$ nor $e_{6(-26)}$ (see Brylinski [3, Theorem 4.1]). In particular, if $(\mathfrak{g}, \mathfrak{t})$ is of Hermitian type, then $O^G_{\min}$ meets $\mathfrak{g}$, where $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. Furthermore, for the cases where $O^G_{\min}$ meets $\mathfrak{g}$, the number of real minimal nilpotent orbits (i.e. the number of adjoint orbits in $O^G_{\min} \cap \mathfrak{g}$) is two if $(\mathfrak{g}, \mathfrak{t})$ is of Hermitian type; one if $(\mathfrak{g}, \mathfrak{t})$ is of non-Hermitian type.

In this paper, we study real minimal nilpotent orbits in $\mathfrak{g}$ including the cases where $O^G_{\min}$ does not meets $\mathfrak{g}$. For any real non-compact simple Lie algebra $\mathfrak{g}$ without complex structures, we put

$$\mathcal{N}_{\mathfrak{g}}/G_C := \{ \text{Complex nilpotent orbits in } \mathfrak{g}_C \text{ meeting } \mathfrak{g} \}$$

and consider the closure ordering on it. Our first main result is here:

**Theorem 1.1.** There uniquely exists a complex nilpotent orbit $O^G_{\min, \mathfrak{g}}$ in $\mathfrak{g}_C$ which is minimum in $\mathcal{N}_{\mathfrak{g}}/G_C$ without the zero-orbit (i.e. for any non-zero complex nilpotent orbit $O$ in $\mathfrak{g}$, if $O \cap \mathfrak{g} \neq \emptyset$, then the closure of $O$ in $\mathfrak{g}_C$ contains $O^G_{\min, \mathfrak{g}}$). Furthermore, the intersection $O^G_{\min, \mathfrak{g}} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$.

We will construct such $O^G_{\min, \mathfrak{g}}$ as the complex adjoint orbit through a non-zero longest restricted root vector in $\mathfrak{g}$. By the definition of $O^G_{\min, \mathfrak{g}}$, the complex minimal nilpotent orbit $O^G_{\min}$ is not our $O^G_{\min, \mathfrak{g}}$ if and only if $O^G_{\min}$ does not meet $\mathfrak{g}$ (namely, $\mathfrak{g}$ is isomorphic to $\text{su}^*(2k)$ ($k \geq 2$), $\text{so}(n - 1, 1)$ ($n \geq 5$), $\text{sp}(p, q)$ ($p \geq q \geq 1$), $f_{4(-20)}$ or $e_{6(-26)}$). This means that for such cases, a non-zero longest restricted root vector in $\mathfrak{g}$ is not a longest root vector in $\mathfrak{g}_C$.

Theorem 1.1 claims that $O^G_{\min, \mathfrak{g}} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$. Our second main result is here:

**Theorem 1.2.** For the cases where the complex minimal nilpotent orbit $O^G_{\min}$ does not meet $\mathfrak{g}$, there exists a unique real minimal nilpotent orbit in $\mathfrak{g}$. In particular, the complex nilpotent orbit $O^G_{\min, \mathfrak{g}}$ in Theorem 1.1 (which is not $O^G_{\min}$ in these cases) is the complexification of the unique real minimal nilpotent orbit in $\mathfrak{g}$. 
Therefore, we have the following corollary:

**Corollary 1.3.** Let $\mathfrak{g}$ be a non-compact real simple Lie algebra without complex structures. If $(\mathfrak{g}, \mathfrak{k})$ is of non-Hermitian type, there uniquely exists a real minimal nilpotent orbit in $\mathfrak{g}$. If $(\mathfrak{g}, \mathfrak{k})$ is of Hermitian type, there are just two real minimal nilpotent orbits in $\mathfrak{g}$.

By Theorem 1.2, our $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ is just the complexification of the unique real minimal nilpotent orbit in $\mathfrak{g}$ for the cases where $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$ ($k \geq 2$), $\mathfrak{so}(n - 1, 1)$ ($n \geq 5$), $\mathfrak{sp}(p, q)$ ($p \geq q \geq 1$), $\mathfrak{f}_4(-20)$ or $\mathfrak{e}_6(-26)$. We will determine our $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ by describing the weighted Dynkin diagram of it for such cases (recall that for another cases, $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ is just $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$). The result is here (see also Table 2 in §2 for the weighted Dynkin diagrams of $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$):

**Theorem 1.4.** For the cases where $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \neq \mathcal{O}_{\min}^{G_{\mathbb{C}}}$, the weighted Dynkin diagram of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ are the following:

<table>
<thead>
<tr>
<th>$\mathfrak{g}$</th>
<th>$\dim_{\mathbb{C}} \mathcal{O}<em>{\min, \mathfrak{g}}^{G</em>{\mathbb{C}}}$</th>
<th>Weighted Dynkin diagram of $\mathcal{O}<em>{\min, \mathfrak{g}}^{G</em>{\mathbb{C}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{su}^*(2k)$</td>
<td>$8k - 8$</td>
<td>$0 \ 1 \ 0 \ 0 \ \cdots \ 0 \ 0 \ 1 \ 0$ ($k \geq 3$) $0 \ 2 \ 0$ ($k = 2$)</td>
</tr>
<tr>
<td>$\mathfrak{so}(n - 1, 1)$</td>
<td>$2n - 4$</td>
<td>$2 \ 0 \ 0 \ \cdots \ 0 \ 0$ ($n$ is odd, $n \geq 5$) $2 \ 0 \ 0 \ \cdots \ 0 \ 0$ ($n$ is even, $n \geq 6$)</td>
</tr>
<tr>
<td>$\mathfrak{sp}(p, q)$</td>
<td>$4(p + q) - 2$</td>
<td>$0 \ 1 \ 0 \ 0 \ \cdots \ 0 \ 0$ ($p + q \geq 3, p \geq q \geq 1$) $0 \ 2$ ($p = q = 1$)</td>
</tr>
<tr>
<td>$\mathfrak{e}_6(-26)$</td>
<td>$32$</td>
<td>$1 \ 0 \ 0 \ 0 \ 1$ $0$</td>
</tr>
<tr>
<td>$\mathfrak{f}_4(-20)$</td>
<td>$22$</td>
<td>$0 \ 0 \ 0 \ 1$</td>
</tr>
</tbody>
</table>

*Table 1: List of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ for $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n - 1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_6(-26)$ and $\mathfrak{f}_4(-20)$.**
This work is motivated by recent works [7], by Joachim Hilgert, Toshiyuki Kobayashi and Jan Möllers, on the construction of an $L^2$-model of irreducible unitary representations of real reductive groups with smallest Gelfand-Kirillov dimension; and [8], by Toshiyuki Kobayashi and Yoshiki Oshima, on the classification of reductive symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ with a $(\mathfrak{g}, K)$-module which is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module.

2 Preliminary results for weighted Dynkin diagrams of complex minimal nilpotent orbits

In this section, we recall weighted Dynkin diagrams of complex minimal nilpotent orbits in complex simple Lie algebras.

Let $\mathfrak{g}_\mathbb{C}$ be a complex semisimple Lie algebra, and denote by $G_\mathbb{C}$ the inner automorphism group of $\mathfrak{g}_\mathbb{C}$. Fix a Cartan subalgebra $\mathfrak{h}_\mathbb{C}$ of $\mathfrak{g}_\mathbb{C}$. We denote by $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ the root system of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Then, the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ can be regarded as a subset of the dual space $\mathfrak{h}^*$ of

$$\mathfrak{h} := \{ H \in \mathfrak{h}_\mathbb{C} | \alpha(H) \in \mathbb{R} \}.$$ 

We write $W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ for the Weyl group of $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ acting on $\mathfrak{h}$. Take a positive system $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ of the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Then, a closed Weyl chamber

$$\mathfrak{h}_+ := \{ H \in \mathfrak{h} | \alpha(H) \geq 0 \}$$

is a fundamental domain of $\mathfrak{h}$ under the action of $W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$.

Let $\Pi$ be the simple system of $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Then, for any $H \in \mathfrak{h}$, we can define a map

$$\Psi_H : \Pi \to \mathbb{R}, \ \alpha \mapsto \alpha(H).$$

We call $\Psi_H$ the weighted Dynkin diagram corresponding to $H \in \mathfrak{h}$, and $\alpha(H)$ the weight on a node $\alpha \in \Pi$ of the weighted Dynkin diagram. Since $\Pi$ is a basis of $\mathfrak{h}^*$, the map

$$\Psi : \mathfrak{h} \to \text{Map}(\Pi, \mathbb{R}), \ \ H \mapsto \Psi_H$$
is a linear isomorphism (between vector spaces). Furthermore,

\[ \mathfrak{h}_+ \to \text{Map}(\Pi, \mathbb{R}_{\geq 0}), \quad H \mapsto \Psi_H \]

is also bijective.

A triple \((H, X, Y)\) is said to be an \(\mathfrak{sl}_2\)-triple in \(\mathfrak{g}_C\) if

\[ [H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H \quad (H, X, Y \in \mathfrak{g}_C). \]

For any \(\mathfrak{sl}_2\)-triple \((H, X, Y)\) in \(\mathfrak{g}_C\), the elements \(X\) and \(Y\) are nilpotent in \(\mathfrak{g}_C\), and \(H\) is hyperbolic in \(\mathfrak{g}_C\) (i.e. \(\text{ad}_{\mathfrak{g}_C} H \in \text{End}(\mathfrak{g}_C)\) is diagonalizable with only real eigenvalues).

Combining the Jacobson–Morozov theorem with Kostant [9], for any complex nilpotent orbit \(\mathcal{O}^{G_C}\), there uniquely exists an element \(H_{\mathcal{O}}\) of \(\mathfrak{h}_+\) with the following property: There exists \(X, Y \in \mathcal{O}^{G_C}\) such that \((H_{\mathcal{O}}, X, Y)\) is an \(\mathfrak{sl}_2\)-triple in \(\mathfrak{g}_C\). Furthermore, by Malcev [10], the following map is injective:

\[
\{ \text{Complex nilpotent orbits in } \mathfrak{g}_C \} \leftrightarrow \mathfrak{h}_+, \quad \mathcal{O}^{G_C} \mapsto H_{\mathcal{O}}.
\]

The weighted Dynkin diagram corresponding to \(H_{\mathcal{O}}\) is called the weighted Dynkin diagram of \(\mathcal{O}^{G_C}\). Dynkin [6] proved that for any complex nilpotent orbit \(\mathcal{O}^{G_C}\), any weight of the weighted Dynkin diagram of \(\mathcal{O}^{G_C}\) is given by 0, 1 or 2, and classified weighted Dynkin diagrams of complex nilpotent orbits (More precisely, Dynkin [6] classified \(\mathfrak{sl}_2\)-triples in \(\mathfrak{g}_C\). See Bala–Carter [2] for more details).

In the rest of this subsection, we suppose that \(\mathfrak{g}_C\) is simple. Let \(\phi\) be the highest root of \(\Delta^{+}(\mathfrak{g}_C, \mathfrak{h}_C)\). Then, the complex minimal nilpotent orbit in \(\mathfrak{g}_C\) can be written by

\[
\mathcal{O}^{G_C}_{\text{min}} = G_C \cdot \mathfrak{g}_\phi \setminus \{0\}.
\]

We define the element \(H_{\phi^\vee}\) of \(\mathfrak{h}\) by

\[
\alpha(H_{\phi^\vee}) = \frac{2\langle \alpha, \phi \rangle}{\langle \phi, \phi \rangle}
\]

for any \(\alpha \in \mathfrak{h}^*\) (where \(\langle , \rangle\) is the inner product on \(\mathfrak{h}^*\) induced by the Killing form on \(\mathfrak{g}_C\)). Namely, \(H_{\phi^\vee}\) is the element of \(\mathfrak{h}\) corresponding to the coroot \(\phi^\vee\) of \(\phi\). Since \(\phi\) is dominant, \(H_{\phi^\vee}\) is in \(\mathfrak{h}_+\). Furthermore, \(H_{\phi^\vee}\) is the hyperbolic element corresponding to \(\mathcal{O}^{G_C}_{\text{min}}\) since we can find \(X_\phi \in \mathfrak{g}_\phi\), \(Y_\phi \in \mathfrak{g}_{-\phi}\) such that \((H_{\phi^\vee}, X_\phi, Y_\phi)\) is an \(\mathfrak{sl}_2\)-triple. The list of weighted Dynkin diagrams of \(\mathcal{O}^{G_C}_{\text{min}}\) for all simple \(\mathfrak{g}_C\) can be found in Collingwood–McGovern [4, Ch.5.4 and 8.4].
Recall that our concern in this paper is in real simple Lie algebras $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n - 1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_6(-26)$ and $\mathfrak{f}_4(-20)$. The complexifications of such algebras are $\mathfrak{sl}(2k, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{sp}(p + q, \mathbb{C})$, $\mathfrak{e}_{6, \mathbb{C}}$ and $\mathfrak{f}_{4, \mathbb{C}}$, respectively. For the convenience of the reader, we give a list of weighted Dynkin diagrams of complex minimal nilpotent orbits in such complex simple Lie algebras.

<table>
<thead>
<tr>
<th>$\mathfrak{g}_C$</th>
<th>$\dim_{\mathbb{C}}\mathcal{O}<em>{\text{min}}^{G</em>{\mathbb{C}}}$</th>
<th>Weighted Dynkin diagram of $\mathcal{O}<em>{\text{min}, \mathfrak{g}}^{G</em>{\mathbb{C}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{sl}(n, \mathbb{C})$</td>
<td>$2n$</td>
<td>$\begin{array}{ccccccccccc} 1 &amp; 0 &amp; 0 &amp; 0 &amp; \cdots &amp; 0 &amp; 0 &amp; 0 &amp; 1 \end{array}$ (n ≥ 2)</td>
</tr>
<tr>
<td>$\mathfrak{so}(n, \mathbb{C})$</td>
<td>$2n - 6$</td>
<td>$\begin{array}{cccccccc} 0 &amp; 1 &amp; 0 &amp; \cdots &amp; 0 &amp; 0 \end{array}$ (n is odd, n ≥ 7)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{array}{cccccccc} 0 &amp; 1 \end{array}$ (n = 5)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{array}{cccccccc} 0 &amp; 1 &amp; 0 &amp; \cdots &amp; 0 &amp; 0 \end{array}$ (n is even, n ≥ 6)</td>
</tr>
<tr>
<td>$\mathfrak{sp}(n, \mathbb{C})$</td>
<td>$2n$</td>
<td>$\begin{array}{cccccccc} 1 &amp; 0 &amp; 0 &amp; 0 &amp; \cdots &amp; 0 &amp; 0 \end{array}$ (n ≥ 2)</td>
</tr>
<tr>
<td>$\mathfrak{e}_{6, \mathbb{C}}$</td>
<td>22</td>
<td>$\begin{array}{cccc} 0 &amp; 0 &amp; 0 &amp; 0 \end{array}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\begin{array}{c} 1 \end{array}$</td>
</tr>
<tr>
<td>$\mathfrak{f}_{4, \mathbb{C}}$</td>
<td>16</td>
<td>$\begin{array}{cccc} 1 &amp; 0 &amp; 0 &amp; 0 \end{array}$</td>
</tr>
</tbody>
</table>

Table 2: List of weighted Dynkin diagrams of $\mathcal{O}_{\text{min}}^{G_{\mathbb{C}}}$ for $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{sp}(n, \mathbb{C})$, $\mathfrak{e}_{6, \mathbb{C}}$ and $\mathfrak{f}_{4, \mathbb{C}}$.

### 3 Outline of a proof of Theorem 1.1

Let $\mathfrak{g}_C$ be a complex simple Lie algebra and $\mathfrak{g}$ a non-compact real form of $\mathfrak{g}$ with a Cartan decomposition $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{p}$. In this section, we describe an idea of the proof of Theorem 1.1.

We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ (such $\mathfrak{a}$ is called a maximally split abelian subspace of $\mathfrak{g}$) and write $\Sigma(\mathfrak{g}, \mathfrak{a})$ for the restricted root system
for \((g, a)\). For any restricted root \(\xi\) of \(\Sigma(g, a)\), we define \(A_{\xi^\vee} \in a\) by

\[
\eta(A_{\xi^\vee}) = \frac{2(\xi, \eta)}{\langle \xi, \xi \rangle} \quad (\forall \eta \in a^*)
\]

(where ( , ) is the inner product on \(a^*\) induced by the Killing form on \(g\)). Namely, \(A_{\xi^\vee}\) is the element of \(a\) corresponding to the coroot \(\xi^\vee\) of \(\xi\). Then, the fact below holds:

**Fact 3.1.** For any restricted root \(\xi\) of \(\Sigma(g, a)\) and any non-zero root vector \(X_\xi\) in \(g_\xi\), there exists \(Y_\xi \in g_{-\xi}\) such that \((A_{\xi^\vee}, X_\xi, Y_\xi)\) is an \(sl_2\)-triple in \(g\).

We fix an ordering on \(a\) and write \(\Sigma^+(g, a)\) for the positive system of \(\Sigma(g, a)\) corresponding to the ordering on \(a\). We denote by \(\lambda\) the highest root of \(\Sigma^+(g, a)\) with respect to the ordering on \(a\). Next two lemmas give characterizations of the highest root \(\lambda\) of \(\Sigma^+(g, a)\) (we omit proofs of the two lemmas in this paper):

**Lemma 3.2.** The highest root \(\lambda\) of \(\Sigma^+(g, a)\) is a unique dominant longest root of \(\Sigma(g, a)\).

**Lemma 3.3.** Let \(\xi\) be a root of \(\Sigma(g, a)\). If \(\xi\) is not the highest root \(\lambda\), then for any non-zero root vector \(X_\xi\) in \(g_\xi\), there exists a positive root \(\eta\) in \(\Sigma^+(g, a)\) and a root vector \(X_\eta \in g_\eta\) such that \([X_\xi, X_\eta] \neq 0\). In particular, \(\xi = \lambda\) if and only if \(\xi + \eta \in a^*\) is not a root of \(\Sigma(g, a)\) for any \(\eta \in \Sigma^+(g, a)\).

We write \(G_C\) for the inner automorphism group of \(g_C\). Then, the following two propositions hold:

**Proposition 3.4.** For any non-zero real nilpotent orbit \(O'_0\) in \(g\). Then, there exists a non-zero highest root vector \(X_\lambda\) in \(g_\lambda\) such that \(X_\lambda\) is in the closure of \(O'_0\) in \(g\).

**Proposition 3.5.** For any two highest root vectors \(X_\lambda, X'_\lambda\) in \(g_\lambda\), there exists \(g_C \in G_C\) such that \(g_C X_\lambda = X'_\lambda\).

**Proof of Proposition 3.4.** There is no loss of generality in assuming that the ordering on \(a\) is lexicographic. Let us put \(m = Z_t(a)\). Then, \(g\) can be decomposed as

\[
g = m \oplus a \oplus \bigoplus_{\xi \in \Sigma(g, a)} g_\xi.
\]
For any $X' \in g$, we denote by

$$X' = X'_m + X'_a + \sum_{\xi \in \Sigma(g,a)} X'_{\xi} \quad (X'_m \in m, X'_a \in a, X'_{\xi} \in g_{\xi}).$$

We put $\overline{O'_0}$ to the closure of $O'_0$ in $g$ and fix an element $X'$ in $\overline{O'_0}$. Let us denote by $\lambda'$ the highest one of

$$\Sigma_{X'} := \{ \xi \in \Sigma(g,a) \mid X'_{\xi} \neq 0 \}$$

with respect to the ordering on $a$ (if $X' \neq 0$, then $\Sigma_{X'}$ is not empty since $X'$ is nilpotent element in $g$). As a first step of the proof, we shall prove that the root vector $X'_{\lambda'}$ is also in $\overline{O'_0}$. We take $A' \in a$ satisfying

$$\xi(A') < \lambda'(A') \quad (\forall \xi \in \Sigma_{X'} \setminus \{\lambda'\}).$$

(such $A'$ exists since $\lambda'$ is highest in $\Sigma_{X'}$ with respect to the lexicographic ordering on $a$). Let us put

$$X'_k := \frac{1}{e^{k\lambda'(A')}} \exp(ad_{\mathfrak{g}}kA')X' \quad \text{(for } k \in \mathbb{N})$$

Then, $X'_k$ is in $\overline{O'_0}$ for any $k$ since $\overline{O'_0}$ is stable by positive scalars. Furthermore,

$$\lim_{k \to \infty} X'_k = \lim_{k \to \infty} \sum_{\xi \in \Sigma_{X'}} e^{k(\xi(A') - \lambda'(A'))} X'_{\xi} = X'_{\lambda'}.$$

This means that $X'_{\lambda'}$ is in $\overline{O'_0}$. To complete the proof, we only need to show that there exists $X' \in \overline{O'_0}$ such that $\lambda' = \lambda$ (where $\lambda'$ is the highest one of $\Sigma_{X'}$). Let $\lambda_0$ be the highest one of

$$\Sigma_{\overline{O'_0}} := \{ \xi \in \Sigma(g,a) \mid \exists X' \in \overline{O'_0} \text{ such that } X'_{\xi} \neq 0 \}$$

(namely, $\Sigma_{\overline{O'_0}} = \bigcup_{X' \in \overline{O'_0}} \Sigma_{X'}$) with respect to the ordering on $a$. Then, we can find a root vector $X'_{\lambda_0}$ in $g_{\lambda_0} \cap \overline{O'_0}$ by the argument above. We assume that $\lambda_0 \neq \lambda$. Then, by Lemma 3.3, there exists $\eta \in \Sigma^+(g,a)$ and $X_{\eta} \in g_{\eta}$ such that $[X_{\eta}, X'_{\lambda_0}] \neq 0$. In particular, for the element $X'' := \exp(ad_{\mathfrak{g}}(X_{\eta}))X'_{\lambda_0}$ in $\overline{O'_0}$, we obtain that

$$\lambda_0 + \eta \in \Sigma_{X''} \subset \Sigma_{\overline{O'_0}}.$$

This contradicts the definition of $\lambda_0$. Thus, $\lambda_0 = \lambda$. \qed
Proof of Proposition 3.5. Let $A_{\lambda^\vee}$ be the element in $\mathfrak{a}$ corresponding to the coroot $\lambda^\vee$ of the highest root $\lambda$. We put

$$(\mathfrak{g}_C)_2 = \{ X \in \mathfrak{g}_C \mid [A_{\lambda^\vee}, X] = 2X \}.$$ 

Then, $\mathfrak{g}_\lambda$ is included in $(\mathfrak{g}_C)_2$. We note that there exists $X, Y \in \mathfrak{g}_C$ such that $(A_{\lambda^\vee}, X, Y)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_C$ (in fact, we can find such $X, Y$ in $\mathfrak{g}_\lambda$ by Fact 3.1). Therefore, we can use Malcev's theorem. Namely, for any two non-zero vectors $X$ and $X'$ in $(\mathfrak{g}_C)_2$, there exists $g_C \in G_C$ such that $g_C X = X'$. Since $\mathfrak{g}_\lambda \subset (\mathfrak{g}_C)_2$, the proof is completed. 

By using Proposition 3.4 and Proposition 3.5, Theorem 1.1 follows by taking $\mathcal{O}_{\min,\mathfrak{g}}^{G_C}$ as

$$\mathcal{O}_{\min,\mathfrak{g}}^{G_C} := G_C \cdot \mathfrak{g}_\lambda \setminus \{0\}.$$ 

4 Outline of a proof of Theorem 1.2

Let us consider the same setting in §3. Recall that $\mathcal{O}_{\min,\mathfrak{g}}^{G_C}$ is not the complex minimal nilpotent orbit $\mathcal{O}_{\min}^{G_C}$ if and only if $\mathcal{O}_{\min}^{G_C}$ does not meet $\mathfrak{g}$. The proposition below give a characterization of $\mathfrak{g}$ for which $\mathcal{O}_{\min}^{G_C}$ is not $\mathcal{O}_{\min,\mathfrak{g}}^{G_C}$ (see Proposition 5.6 for another characterizations of it).

Proposition 4.1. The following conditions on $\mathfrak{g}$ are equivalent:

1. $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$.
2. $\mathcal{O}_{\min}^{G_C} \cap \mathfrak{g} = \emptyset$.

We can prove the proposition without any classification, but we omit it in this paper.

Here, we put $\mathfrak{m} := Z(\mathfrak{a})$ and denote by $M_0, A$ to the analytic subgroups of $G$ corresponding to $\mathfrak{m}, \mathfrak{a}$, respectively. Then, the connected Lie group $M_0 A$ (which is the analytic subgroup of $G$ corresponding to $\mathfrak{m} \oplus \mathfrak{a}$) acts on $\mathfrak{a}$. Furthermore, the following proposition holds:

Proposition 4.2. If $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$, then $\mathfrak{g}_\lambda \setminus \{0\}$ is a single $M_0 A$-orbit.

Combining Proposition 3.4, Proposition 4.1 with Proposition 4.2, we obtain Theorem 1.2.

We will use the next lemma to prove Proposition 4.2.
Lemma 4.3. Suppose that $\mathfrak{g}$ has real rank one (i.e. $\dim_{\mathbb{R}} \mathfrak{a} = 1$) and $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$. Then, $\mathfrak{g}_{\lambda} \setminus \{0\}$ is a single $M_{0}A$-orbit.

Proof of Lemma 4.3. Let $A_{\lambda}$ be the element of $\mathfrak{a}$ corresponding to the coroot $\lambda^{\vee}$ of the highest root $\lambda$ in $\Sigma^{+}(\mathfrak{g}, \mathfrak{a})$ (see §3). Since $\mathfrak{g}$ has real rank one, we have $\mathfrak{a} = \mathbb{R}A_{\lambda}$, and $\mathfrak{g}$ can be written by

$$\mathfrak{g} = \mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_{-\frac{\lambda}{2}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{g}_{\frac{\lambda}{2}} \oplus \mathfrak{g}_{\lambda}$$

($\mathfrak{g}_{\pm\frac{\lambda}{2}}$ can be zero). Let us denote by $\mathfrak{g}_{\mathbb{C}}, \mathfrak{m}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}}, (\mathfrak{g}_{\pm\lambda})_{\mathbb{C}}, (\mathfrak{g}_{\pm\frac{\lambda}{2}})_{\mathbb{C}}$ the complexification of $\mathfrak{g}$, $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{g}_{\pm\lambda}$, $\mathfrak{g}_{\pm\frac{\lambda}{2}}$, respectively. We set

$$(\mathfrak{g}_{\mathbb{C}})_i = \{ X \in \mathfrak{g}_{\mathbb{C}} \mid [A_{\lambda}, X] = iX \} \quad \text{(for } i \in \mathbb{Z}).$$

Then,

$$(\mathfrak{g}_{\mathbb{C}})_0 = \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}}, \quad (\mathfrak{g}_{\mathbb{C}})_{\pm 1} = (\mathfrak{g}_{\pm\frac{\lambda}{2}})_{\mathbb{C}}, \quad (\mathfrak{g}_{\mathbb{C}})_{\pm 2} = (\mathfrak{g}_{\pm\lambda})_{\mathbb{C}}.$$ 

By Fact 3.1, for any non-zero highest root vector $X_{\lambda}$ in $\mathfrak{g}_{\lambda}$, there exists $Y_{\lambda} \in \mathfrak{g}_{-\lambda}$ such that $(A_{\lambda}, X_{\lambda}, Y_{\lambda})$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_{\mathbb{C}}$. By the theory of representations of $\mathfrak{sl}(2, \mathbb{C})$, we obtain that $[(\mathfrak{g}_{\mathbb{C}})_0, X_{\lambda}] = (\mathfrak{g}_{\mathbb{C}})_2$. In particular, we have

$$[\mathfrak{m} \oplus \mathfrak{a}, X_{\lambda}] = \mathfrak{g}_{\lambda}.$$ 

Therefore, for the $M_{0}A$-orbit $O^{M_{0}A}(X_{\lambda})$ in $\mathfrak{g}_{\lambda}$ through $X_{\lambda}$, we obtain that

$$\dim_{\mathbb{R}} O^{M_{0}A}(X_{\lambda}) = \dim_{\mathbb{R}} \mathfrak{g}_{\lambda}.$$ 

This means that the $M_{0}A$-orbit $O^{M_{0}A}(X_{\lambda})$ is open in $\mathfrak{g}_{\lambda}$ for any non-zero root vector $X_{\lambda}$ in $\mathfrak{g}_{\lambda}$. Recall that we are assuming that $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$. Hence, $\mathfrak{g}_{\lambda} \setminus \{0\}$ is connected. Therefore, $\mathfrak{g}_{\lambda} \setminus \{0\}$ is a single $M_{0}A$-orbit.

We are ready to prove Proposition 4.2.

Sketch of a proof of Proposition 4.2. Let $\mathfrak{h}' := [\mathfrak{g}_{\lambda}, \mathfrak{g}_{-\lambda}] \subset \mathfrak{m} \oplus \mathfrak{a}$. Then $\mathfrak{g}' := \mathfrak{g}_{-\lambda} \oplus \mathfrak{h}' \oplus \mathfrak{g}_{\lambda}$ becomes a subalgebra of $\mathfrak{g}$ (since $\pm 2\lambda$ is not a root). Furthermore, one can prove that $\mathfrak{g}'$ is a real rank one simple Lie algebra with a maximally split abelian subspace $\mathfrak{a}' := \mathbb{R}A_{\lambda}$, where $A_{\lambda}$ is the element of $\mathfrak{a}$ corresponding to the coroot $\lambda^{\vee}$ of the highest root $\lambda$ in $\Sigma^{+}(\mathfrak{g}, \mathfrak{a})$ (see §3). We put $\mathfrak{m}' \oplus \mathfrak{a}' := Z_{\mathfrak{g}'}(\mathfrak{a}')$ and denote by $M_{0}'A'$ the analytic subgroup of $G$ corresponding to $\mathfrak{m}' \oplus \mathfrak{a}'$. Then, by Lemma 4.3, we obtain that $\mathfrak{g}_{\lambda} \setminus \{0\}$ is a single $M_{0}'A'$-orbit. Since $M_{0}'A'$ is a subgroup of $M_{0}A$, the proof is completed.
5 Determination of \( \mathcal{O}_{\text{min,g}}^{G_{\mathbb{C}}} \)

In this section, we determine \( \mathcal{O}_{\text{min,g}}^{G_{\mathbb{C}}} \) by describing the weighted Dynkin diagram of \( \mathcal{O}_{\text{min,g}}^{G_{\mathbb{C}}} \). Recall that Proposition 4.1 claims that \( \mathcal{O}_{\text{min}}^{G_{\mathbb{C}}} = \mathcal{O}_{\text{min,g}}^{G_{\mathbb{C}}} \) if and only if \( \dim_{\mathbb{R}} g_{\lambda} = 1 \). Thus, our concern is in the cases where \( \dim_{\mathbb{R}} g_{\lambda} \geq 2 \) (i.e. \( g \) is isomorphic to \( \mathfrak{su}^*(2k), \mathfrak{so}(n - 1, 1), \mathfrak{sp}(p, q), \mathfrak{e}_{6(-26)} \) or \( \mathfrak{f}_{4(-20)} \)).

5.1 Satake diagrams and weighted Dynkin diagrams

In order to determine the weighted Dynkin diagram of our \( \mathcal{O}_{\text{min,g}}^{G_{\mathbb{C}}} \), we describe some lemmas of relationship between weighted Dynkin diagrams of \( g_{\mathbb{C}} \) and Satake diagrams of \( g \) in this subsection.

Let \( g_{\mathbb{C}} \) be a semisimple Lie algebra and \( g \) a real form of it through this subsection. First, we recall briefly the definition of Satake diagram of a real form \( g \) of a complex semisimple Lie algebra \( g_{\mathbb{C}} \) (see also [1] for more details). Fix a Cartan decomposition \( g = \mathfrak{k} \oplus \mathfrak{p} \) of \( g \). We take a maximal abelian subspace \( \mathfrak{a} \) in \( \mathfrak{p} \), and extend it to a maximal abelian subspace \( \mathfrak{h} = \sqrt{-1}\mathfrak{k} \oplus \mathfrak{a} \) in \( \sqrt{-1}\mathfrak{k} \oplus \mathfrak{p} \). Then, the complexification, denoted by \( \mathfrak{h}_{\mathbb{C}} \), of \( \mathfrak{h} \) is a Cartan subalgebra of \( g_{\mathbb{C}} \), and \( \mathfrak{h} \) coincide with the real form

\[
\{ X \in \mathfrak{h}_{\mathbb{C}} \mid \alpha(X) \in \mathbb{R} \ (\forall \alpha \in \Delta(g_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})) \}
\]

of \( \mathfrak{h}_{\mathbb{C}} \), where \( \Delta(g_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \) is the root system of \( (g_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \). Let us denote by

\[
\Sigma(g, \mathfrak{a}) := \{ \alpha|_{\mathfrak{a}} \mid \alpha \in \Delta(g_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \} \setminus \{0\} \subset \mathfrak{a}^* 
\]

the restricted root system of \( (g, \mathfrak{a}) \). We will denote by \( W(g, \mathfrak{a}), W(g_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \) the Weyl group of \( \Sigma(g, \mathfrak{a}) \). \( \Delta(g_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \) and \( \Delta(g_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \), respectively. Fix an ordering on \( \mathfrak{a} \) and extend it to an ordering on \( \mathfrak{h} \). We write \( \Sigma^+(g, \mathfrak{a}), \Delta^+(g_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \) for the positive system of \( \Sigma(g, \mathfrak{a}), \Delta(g_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \) corresponding to the ordering on \( \mathfrak{a}, \mathfrak{h} \), respectively. Then, \( \Sigma^+(g, \mathfrak{a}) \) can be written by

\[
\Sigma^+(g, \mathfrak{a}) = \{ \alpha|_{\mathfrak{a}} \mid \alpha \in \Delta^+(g_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \} \setminus \{0\}. 
\]

We denote by \( \Pi \) the fundamental system of \( \Delta^+(g_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \). Then,

\[
\overline{\Pi} := \{ \alpha|_{\mathfrak{a}} \mid \alpha \in \Pi \} \setminus \{0\}
\]

is the simple system of \( \Sigma^+(g, \mathfrak{a}) \). Let \( \Pi_0 \) be the set of all simple roots in \( \Pi \) whose restriction to \( \mathfrak{a} \) is zero. The Satake diagram \( S_\mathfrak{a} \) of \( g \) consists of the
following data: The Dynkin diagram of $g_C$ with nodes $\Pi$; black nodes $\Pi_0$ in $S_g$; and arrows joining $\alpha \in \Pi \setminus \Pi_0$ and $\beta \in \Pi \setminus \Pi_0$ in $S_g$ whose restrictions to $a$ are the same.

Second, we define that a weighted Dynkin diagram $\Psi_H \in \text{Map}(\Pi, \mathbb{R})$ "matches" the Satake diagram $S_g$ of $g$ as follows:

**Definition 5.1.** Let $\Psi_H \in \text{Map}(\Pi, \mathbb{R})$ be a weighted Dynkin diagram (see §2) and $S_g$ the Satake diagram of $g$ with nodes $\Pi$. We say that $\Psi_H$ matches $S_g$ if all the weights on black nodes are zero and any pair of nodes joined by an arrow has the same weights.

**Remark 5.2.** The concept of "match" defined above is same as "weighted Satake diagrams" in Djocovic [5] and the condition described in Sekiguchi [11, Proposition 1.16].

Recall that $\Psi$ is a linear isomorphism from $\mathfrak{h}$ to $\text{Map}(\Pi, \mathbb{R})$ (see §2). Then, the next two lemmas hold (we omit proofs of the two lemmas in this paper):

**Lemma 5.3.** $\Psi: \mathfrak{h} \rightarrow \text{Map}(\Pi, \mathbb{R})$ induces a linear isomorphism below:

$$a \rightarrow \{ \Psi_H \in \text{Map}(\Pi, \mathbb{R}) \mid \Psi_H \text{ matches } S_g \}.$$  

**Lemma 5.4.** For each simple root $\alpha$ of $\Pi$, we denote by $H_{\alpha^\vee}$ the element in $\mathfrak{h}$ corresponding to the coroot $\alpha^\vee$ of the simple root $\alpha$. Then, the set

$$\{ H_{\alpha^\vee} \mid \alpha \text{ is black in } S_g \} \cup \{ H_{\alpha^\vee} - H_{\beta^\vee} \mid \alpha \text{ and } \beta \text{ are joined by an arrow in } S_g \}$$

is a basis of $\sqrt{-1}t$.

Lemma 5.3 and Lemma 5.4 will be used to compute the weighted Dynkin diagrams of $O^{G_C}_{\min, g}$ for the cases where $O^{G_C}_{\min, g}$ is not the complex minimal nilpotent orbit $O^{G_C}_{\min}$.

Recall that our concern in this paper is in real simple Lie algebras $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{e}_6(-26)$ and $\mathfrak{f}_4(-20)$. For the convenience of the reader, we give a list of Satake diagrams of such simple Lie algebras.

<table>
<thead>
<tr>
<th>$g$</th>
<th>Satake diagrams of $g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathfrak{su}^*(2k)$</td>
<td>![Diagram SU*(2k)]</td>
</tr>
</tbody>
</table>

![Diagram SU*(2k)]
so\((n-1,1)\) 
\[
\begin{array}{c}
\bullet - \bullet - \bullet - \cdots - \bullet - \bullet - \Rightarrow \bullet
\end{array}
\] 
(n is odd, \(n \geq 5\))

\[
\begin{array}{c}
\bullet - \bullet - \bullet - \Rightarrow \bullet
\end{array}
\] 
(n is even, \(n \geq 6\))

\[
\begin{array}{c}
\bullet - \bullet - \bullet - \Rightarrow \bullet
\end{array}
\] 
(p \(\geq q \geq 1\))

\[
\begin{array}{c}
\bullet - \bullet - \bullet - \Rightarrow \bullet
\end{array}
\] 
\(\mathfrak{sp}(p,q)\)

\[
\begin{array}{c}
\bullet - \bullet - \bullet - \Rightarrow \bullet
\end{array}
\] 
\(\mathfrak{e}_{6(-26)}\)

\[
\begin{array}{c}
\bullet - \bullet - \bullet - \Rightarrow \bullet
\end{array}
\] 
\(f_{4(-20)}\)

| Table 3: List of Satake diagrams of \(su^*(2k), so(n-1,1),\) \(\mathfrak{sp}(p,q), \mathfrak{e}_{6(-26)}\) and \(f_{4(-20)}\). |

\[
\begin{array}{c}
\bullet - \bullet - \bullet - \Rightarrow \bullet
\end{array}
\] 
\(\mathfrak{e}_{6(-26)}\)

\[
\begin{array}{c}
\bullet - \bullet - \bullet - \Rightarrow \bullet
\end{array}
\] 
\(f_{4(-20)}\)

5.2 Computation of weighted Dynkin diagrams of \(\mathcal{O}_{\text{min, } g}^{G_{\mathbb{C}}}\)

We consider the same setting on §5.1 and suppose that \(g_{\mathbb{C}}\) is simple and \(g\) is non-compact. Let us denote by

\[
a_{+} := \{ A \in a \mid \xi(A) \geq 0 \ (\forall \xi \in \Sigma^+(g, a)) \}.
\]

Then \(a_{+}\) is a fundamental domain of \(a\) under the action of \(W(g, a)\). Since

\[
\Sigma^+(g, a) = \{ \alpha|_{a} \mid \alpha \in \Delta(g_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \} \setminus \{0\},
\]

the domain \(a_{+}\) coincide with \(\mathfrak{h}_{+} \cap a\). Recall that \(\lambda\) is dominant (by Lemma 3.2) and \(\mathcal{O}_{\text{min, } g}^{\mathbb{C}}\) contains \(g_{\lambda} \setminus \{0\}\) (by the proof of Theorem 1.1). Thus, \(A_{\lambda^{\vee}}\) is the hyperbolic element in \(a_{+}\) corresponding to \(\mathcal{O}_{\text{min, } g}^{\mathbb{C}}\) (see §2) since we can find \(X_{\lambda} \in g_{\lambda}, Y_{\lambda} \in g_{-\lambda}\) such that the triple \((A_{\lambda^{\vee}}, X_{\lambda}, Y_{\lambda})\) is an \(\mathfrak{sl}_2\)-triple in \(g_{\mathbb{C}}\) by Lemma 3.1 (then, \(X_{\lambda}, Y_{\lambda} \in \mathcal{O}_{\text{min, } g}^{\mathbb{C}}\)). Therefore, to determine the weighted Dynkin diagram of \(\mathcal{O}_{\text{min, } g}^{\mathbb{C}}\), we shall compute the weighted Dynkin diagram corresponding to \(A_{\lambda^{\vee}}\).

Let \(\phi\) be the highest root of \(\Delta^+(g_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})\). Recall that the complex minimal nilpotent orbit \(\mathcal{O}_{\text{min}}^{g_{\mathbb{C}}}\) contains the root space \((g_{\mathbb{C}})_{\phi}\) without zero, and the weighted Dynkin diagram of \(\mathcal{O}_{\text{min}}^{g_{\mathbb{C}}}\) is the weighted Dynkin diagram corresponding to \(H_{\phi^{\vee}}\) (see §2). The next lemma gives a formula for \(A_{\lambda^{\vee}}\) by \(H_{\phi^{\vee}}\) (we omit a proof of the lemma):
Lemma 5.5. We denote by \( \tau \) the anti-\( \mathbb{C} \)-linear involution corresponding to \( g_{\mathbb{C}} = g \oplus \sqrt{-1}g \) (i.e. \( \tau \) is the complex conjugation of \( g_{\mathbb{C}} \) with respect to the real form \( g \)). Then, \( H_{\phi^\vee} \) is in \( a \) if and only if \( \dim_{\mathbb{R}} g_\lambda \geq 2 \) and

\[
A_\lambda^\vee = \begin{cases} 
H_{\phi^\vee} & (\text{if } \dim_{\mathbb{R}} g_\lambda = 1), \\
H_{\phi^\vee} + \tau H_{\phi^\vee} & (\text{if } \dim_{\mathbb{R}} g_\lambda \geq 2). 
\end{cases}
\]

In particular, we have another characterizations of \( g \) for which \( \mathcal{O}_{\text{min},g}^{G_{\mathbb{C}}} \) is not \( \mathcal{O}_{\text{min}}^{G_{\mathbb{C}}} \) from Proposition 4.1.

Proposition 5.6. The following conditions on \( g \) are equivalent:

1. \( \mathcal{O}_{\text{min},g}^{G_{\mathbb{C}}} \neq \mathcal{O}_{\text{min}}^{G_{\mathbb{C}}} \).
2. \( \mathcal{O}_{\text{min}}^{G_{\mathbb{C}}} \cap g = \emptyset \).
3. \( \dim_{\mathbb{R}} g_\lambda \geq 2 \).
4. The highest root \( \phi \) in \( \Delta^+(g_{\mathbb{C}}, h_{\mathbb{C}}) \) is not a real root.
5. The weighted Dynkin diagram of \( \mathcal{O}_{\text{min}}^{G_{\mathbb{C}}} \) matches the Satake diagram \( S_g \) of \( g \) (see Definition §5.1).
6. \( g \) is isomorphic to \( \mathfrak{su}^*(2k), \mathfrak{so}(n-1,1), \mathfrak{sp}(p,q), \mathfrak{e}_6(-26) \) or \( \mathfrak{f}_4(-20) \), where \( k \geq 2, n \geq 5 \) and \( p \geq q \geq 1 \).

We now determine the weighted Dynkin diagram of \( \mathcal{O}_{\text{min},g}^{G_{\mathbb{C}}} \) for the cases where \( g \) is isomorphic to \( \mathfrak{su}^*(2k), \mathfrak{so}(n-1,1), \mathfrak{sp}(p,q), \mathfrak{e}_6(-26) \) or \( \mathfrak{f}_4(-20) \). By Lemma 5.5, our purpose is to compute the weighted Dynkin diagram corresponding to \( A_\lambda^\vee = H_{\phi^\vee} + \tau H_{\phi^\vee} \). We only give the computation for the case \( g = \mathfrak{e}_6(-26) \) below. For the other \( g \) with \( \dim_{\mathbb{R}} g_\lambda \geq 2 \), we can compute the weighted Dynkin diagram corresponding to \( A_\lambda^\vee \) by the same way.

Example 5.7. Let \( (g_{\mathbb{C}}, g) = (\mathfrak{e}_6, \mathfrak{e}_6(-26)) \). We denote the Satake diagram of \( \mathfrak{e}_6(-26) \) by

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\
\circ & \bullet & \bullet & \bullet & \circ \\
\bullet & \alpha_6
\end{array}
\]
By Table 2, the weighted Dynkin diagram corresponding to $H_{\phi^\vee}$ is

$$
\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
\circ & \circ & \circ & \circ & 1 \\
\end{array}
$$

We now compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee} = H_{\phi^\vee} + \tau H_{\phi^\vee}$. By Lemma 5.3, the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$ matches the Satake diagram of $\mathfrak{e}_6(-26)$. Thus, we can put the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$ as

$$
\begin{array}{cc}
a & 0 & 0 & b \\
\circ & \circ & \circ & \circ \\
0 & \\
\end{array}
$$

(a, b \in \mathbb{R}).

To determine $a, b \in \mathbb{R}$, we also put

$$H_{\phi^\vee}^{im} = H_{\phi^\vee} - \tau H_{\phi^\vee} \in \sqrt{-1}t.$$

Since $A_{\lambda^\vee} + H_{\phi^\vee}^{im} = 2H_{\phi^\vee}$, the weighted Dynkin diagram corresponding to $H_{\phi^\vee}^{im}$ can be written by

$$
\begin{array}{cccc}
-a & 0 & 0 & -b \\
\circ & \circ & \circ & \circ \\
2 & \\
\end{array}
$$

Namely, we have

$$
\begin{align*}
\alpha_1(H_{\phi^\vee}^{im}) &= -a, \\
\alpha_2(H_{\phi^\vee}^{im}) &= \alpha_3(H_{\phi^\vee}^{im}) = \alpha_4(H_{\phi^\vee}^{im}) = 0, \\
\alpha_5(H_{\phi^\vee}^{im}) &= -b, \\
\alpha_6(H_{\phi^\vee}^{im}) &= 2.
\end{align*}
$$

By Lemma 5.4, the set \{ $H_{\alpha_2^\vee}, H_{\alpha_3^\vee}, H_{\alpha_4^\vee}, H_{\alpha_6^\vee}$ \} is a basis of $\sqrt{-1}t$. Thus, $H_{\phi^\vee}^{im} \in \sqrt{-1}t$ can be written by

$$H_{\phi^\vee}^{im} = c_2 H_{\alpha_2^\vee} + c_3 H_{\alpha_3^\vee} + c_4 H_{\alpha_4^\vee} + c_6 H_{\alpha_6^\vee} \quad (c_2, c_3, c_4, c_6 \in \mathbb{R}).$$

By the Dynkin diagram of $\mathfrak{e}_{6,C}$, we can compute

$$\alpha_i(H_{\alpha_j^\vee}) = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}$$
for each $i, j$. Thus, we also have

\[
\begin{align*}
\alpha_1(H_{\phi^\vee}^{im}) &= -c_2, \\
\alpha_2(H_{\phi^\vee}^{im}) &= 2c_2 - c_3, \\
\alpha_3(H_{\phi^\vee}^{im}) &= -c_2 + 2c_3 - c_4 - c_6, \\
\alpha_4(H_{\phi^\vee}^{im}) &= -c_3 + 2c_4, \\
\alpha_5(H_{\phi^\vee}^{im}) &= -c_4, \\
\alpha_6(H_{\phi^\vee}^{im}) &= -c_3 + 2c_6.
\end{align*}
\]

Then, we obtain that $a = b = 1$. Therefore, the weighted Dynkin diagram of $\mathcal{O}_{\min, g}^{G_{\mathbb{C}}}$ for $g = \iota_6(-26)$ is

\[
\begin{array}{cccccc}
1 & 0 & 0 & 0 & 1 \\
\hline
& & & & 0
\end{array}
\]

The result of our computation for all $g$ with $\dim_{\mathbb{R}} g_\lambda \geq 2$ is Table 1 in §1.

References


