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Some developments on Schur functors and dominant dimension

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Abstract

Schur functors have been playing crucial roles in representation theory to relate different algebras. Prominent examples include the remarkable Schur-Weyl duality which relates Schur algebras to group algebras of symmetric groups and Soergel's struktursatz which relates blocks of category $\mathcal{O}$ of a complex semisimple Lie algebra to coinvariant algebras of the corresponding flag manifold. This survey contains our recent investigation on homological properties of Schur functors in terms of dominant dimension as well as a number of new applications of dominant dimension in representation theory.

Keywords Schur functor, Dominant dimension, Schur algebra
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1 Introduction

General linear and symmetric groups are two fundamental objects in group theory. A hundred years ago, I. Schur determined the polynomial representations of the complex general linear group $GL_n(\mathbb{C})$ in his doctoral dissertation. An essential part of Schur's idea was to set up a correspondence between representations of $GL_n(\mathbb{C})$ of fixed homogenous degree r, and representations of the finite symmetric group Σ_r on r letters, and through this correspondence to apply C. Frobenius's discovery of the characters of Σ_r [14]. Such an idea was later publicized by H. Weyl and J. A. Green in the well-known *Schur Weyl duality*, and seemed to appear frequently in many other contexts, like Bernstein, Gelfand and Gelfand's theory on projective functors; Soergel's struktursatz which relates blocks of category $\mathcal{O}$ of a complex semisimple Lie algebra to coinvariant algebras and the theory of KZ functors which relate the category $\mathcal{O}$ of rational cherednik algebras to Hecke algebras. Nevertheless, Schur's original idea on relating representations of general linear and symmetric groups is very much subtle in modular situations since the correspondence above (via the Schur functor) is no longer an equivalence and cohomologies doubtlessly come in to play. As far as we have understood, there are two approaches to rescue Schur's idea, one by Hemmer, Kleshchev and Nakano on cohomologies of Schur functors

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[16, 19] and the other by Koenig, Slungard and Xi on double centralizer properties and dominant dimension [21].

From the elementary representation theory of symmetric groups, we know that every Young module of symmetric groups admits a filtration whose sub-quotients are isomorphic to Specht modules, whereas the filtration multiplicities are not well-defined. As typical examples, we have that when the characteristic p of the field equals 2, $S^{(2,0)} \cong S^{(1,1)}$ and when $p = 3$, there exists a short exact sequence of Specht modules, see [22]

$$0 \longrightarrow S^{(5,1,1)} \longrightarrow S^{(3,3,1)} \longrightarrow S^{(2,1^5)} \longrightarrow 0$$

However, based on Kleshchev and Nakano's comparison on cohomologies of general linear and symmetric groups in [19], Hemmer and Nakano proved that when $p \geq 5$, the Schur functor preserves extension groups between Weyl modules up to degree $p-3$ and as a result, the multiplicities of Specht modules in Young modules are well-defined [16], see also [17, 18] for some further advances of the same spirit. In [5], Doty, Erdmann and Nakano set up a general framework so as to generalize Hemmer, Kleshchev and Nakano's work to a broad context with the emphasis on cohomologies of Schur functors.

As the main ingredient in Koenig, Slungard and Xi's approach [21], dominant dimension was introduced to study QF-3 rings, one of the several generalizations of quasi-Frobenius rings, by Tachikawa and Morita [24, 25], see also [26, 28]. In representation theories, it behaves very subtle and was mainly used to study torsionless modules and *double centralizer properties*, the crucial point in Schur's technique. In [21], it was firstly applied in algebraic Lie theory to give a computation-free proof of both Schur-Weyl duality and Soergel's struktursatz by showing the dominant dimension of algebras in question at least 2.

In [7, 9], we combined two approaches above and considered the following setting. Let (A, e) be a QF-3 algebra where e is an idempotent in A such that Ae is a basic projective, injective and faithful A-module, see Section 2.4. Consider the Schur functor after [5, 14] afforded by e

$$f = eA \otimes_A - : A\text{-mod} \longrightarrow eAe\text{-mod}$$

Motivated by [16, 19], we shall assume that f induces natural isomorphisms $\mathrm{Hom}_A(P, Q) \cong \mathrm{Hom}_{eAe}(f(P), f(Q))$ (called *double centralizer property*) for all projective A-modules P and Q, just as in the (quantum) Schur-Weyl duality and Soergel's struktursatz, and consider the following questions

(a) find maximal $n \in \mathbb{N}$ such that f induces an isomorphism $\mathrm{Ext}^i_A(P, Q) \cong \mathrm{Ext}^i_{eAe}(eP, eQ)$ for any projective A-modules P and Q and $0 \leq i \leq n$.

(b) for a full subcategory $\mathscr{C}$ of A-mod, find maximal $n(\mathscr{C})$ such that f induces an isomorphism $\mathrm{Ext}^i_A(M, N) \cong \mathrm{Ext}^i_{eAe}(eM, eN)$ for any modules $M, N \in \mathscr{C}$ and $0 \leq i \leq n(\mathscr{C})$.

It turns out that the solutions to these questions are hinted around dominant dimensions. Indeed, (a) has been settled by Müller in terms of dominant dimension of the algebra [23], see also Section 2.4. As for (b), we will restrict ourselves to consider, say, $\mathscr{F}(\Delta)$ for quasi-hereditary algebras.

In particular, Kleshchev and Nakano's result for classical Schur algebras $S_k(n,r)$ can be restated as: assume that $n \geq r$ and k is an infinite field of characteristic $p \geq 5$, then $n(\mathscr{F}(\Delta)) \geq p-3$, where $\mathscr{F}(\Delta)$ consists of those $S_k(n,r)$-modules which are filtered by Weyl modules. Hemmer and Nakano's result for the quantized Schur algebra $S_q(n,r)$ can be restated as: assume that $n \geq r$ and q is an ℓ-th root of unity with $\ell \geq 4$, then $n(\mathscr{F}(\Delta)) \geq 1$. In [7], we computed $n(\mathscr{F}(\Delta))$ for some specific cases, namely $n(\mathscr{F}(\Delta)) = p-3$ for $S_k(p,p)$ where $p > 0$ is the characteristic of the field k; $n(\mathscr{F}(\Delta)) = 0$ for any non-semisimple block algebra A of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of a complex semisimple Lie algebra. Furthermore we proved a positive correlation between $n(\mathscr{F}(\Delta))$ and the dominant dimension of a QF-3 standardly stratified algebra, i.e., the larger the dominant dimension of the algebra, the larger $n(\mathscr{F}(\Delta))$.

In [9], we obtained a quantitative relation between dominant dimensions and $n(\mathscr{F}(\Delta))$ by restricting to a subclass of quasi-hereditary algebras, which includes (quantized, cyclotomic) Schur algebras, block algebras of BGG category as basic examples. As main results, we proved that for each algebra A in this class, its dominant dimension $\operatorname{dom.dim} A$ is exactly twice that of the characteristic tilting module, hence an even number; the Ringel-dual of A has dominant dimension the same as A; $n(\mathcal{P}_A) = \operatorname{dom.dim} A - 2$ and $n(\mathscr{F}(\Delta)) = \operatorname{dom.dim} A/2 - 2$ where $\mathcal{P}_A$ is the full subcategory of finitely generated projective A-modules; characterized how dominant dimension changes under truncation process; computed the dominant dimension of classical Schur algebras $S_k(n,r)$ for $n \geq r$. As a direct application, we recovered and strengthened the results above by Hemmer, Kleshchev and Nakano, see also Section 3 and Section 5, reproved and generalized James and Donkin's result on the torsionless property of (quantum) Weyl modules.

In [10, 11, 12], we aimed to develop new effective characterizations of dominant dimension as well as more applications. To do so, we enlarged the algebra class in [9] to introduce the so-called *generalized symmetric algebras.* By definition, this is an algebra class consists of endomorphism algebras of generators over symmetric algebras. It contains some quasi-hereditary algebras (finite global dimension) like Schur algebras, as well as all symmetric algebras (generally infinite global dimension) like Hecke algebras. To be precise, in [10], we obtained some equivalent characterizations of generalized symmetric algebras and a refinement of the Tachikawa-Morita correspondence. In particular, we extended a characterization of dominant dimension in [9] to generalized symmetric algebras. In [11], we made use of the property that every generalized symmetric algebra A admits a special coproduct (usually without a counit) to construct a Hochschild cocomplex and proved that the highest degree where the cocomplex is exact at each degree below, is $\operatorname{dom.dim} A + 2$. As a consequence, the computation of dominant dimensions in certain cases will be accessible to computer programmes once the above coproduct maps are explicitly known. As another application, we obtained for each generalized symmetric algebra A, the isomorphisms between the i-th Hochschild cohomologies of A and its centralizer subalgebra eAe for $0 \leq i \leq \operatorname{dom.dim} A - 2$. In [12], we gave an explicit construction of a multiplication on the dual space $A_k(n,r)$ of $S_k(n,r)$ and proved that the image of this multiplication map coincides with the Doty coalgebra $D_k(n,r)$. Moreover we proved that $D_k(n,r) = A_k(n,r)$ if and only if $r \leq n(p-1)$. By use of this multiplication map, we proved that $S_k(n,r)$ is generalized symmetric when $r \leq n(p-1)$, where p is the characteristic of the field and the dual of this multiplication map is a prerequisite coproduct map in [10]. Combined with the strategy developed in [10], we are able to compute the dominant dimension of these Schur algebras in principle.

This paper originates from the talk on the workshop "Topics in Combinatorial Representation Theory" (Kyoto, Oct. 2011) given by the author. To make it more accessible, we will include in Section 1 some basic theory on representations of general linear and symmetric groups, dominant dimensions etc. and omit all proofs to keep the size reasonable. The interested readers are invited to refer to another survey "Dominant dimension and almost relatively true versions of Schur's theorem" given by Koenig from a somewhat different point of view [20].

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2 Preliminaries

In this section, we fix the notation and recall some basic theory of representations of general linear groups and dominant dimension, etc.

2.1 Representations of general linear and symmetric group

Let n and r be natural numbers. Define $I(n,r)$ to be the set of sequences $\underline{i} = (i_1, \dots, i_r)$ of integers with $i_\rho \in \{1, \dots, n\}$ and $\Lambda^+(n,r)$ the set of sequences $\lambda = (\lambda_1, \dots, \lambda_n)$ of integers such that $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ and $\lambda_1 + \dots + \lambda_n = r$. Let $\Lambda^+(r) = \cup_{n\geq 1}\Lambda^+(n,r)$. Note that the symmetric group Σ_r on r-letters has a natural right action on $I(n,r)$ by place permutations: $(i_1, \dots, i_r) \cdot \sigma = (i_{1\sigma}, \dots, i_{r\sigma})$ for $\sigma \in \Sigma_r$. We define $\underline{i} \sim \underline{j}$ if they belong to the same Σ_r-orbit and define $\mathrm{Stab}(\underline{i})$ to be the stabilizer subgroup of $\underline{i}$ in Σ_r. Elements in $I(n,r)$ are called *multi-indices* and elements in $\Lambda^+(r)$ are called *partitions* of r. A partition λ is called *ℓ-restricted* for some number ℓ if $0 \leq \lambda_i - \lambda_{i+1} \leq \ell - 1$ for all i.

Let k be an infinite field of characteristic $p \geq 0$. For each partition λ of r, let S^λ denote the Specht module and S_λ the dual Specht module indexed by λ. It is well-known that if $p = 0$, then $S_\lambda \cong S^\lambda$ and $\{S_\lambda \mid \lambda \in \Lambda^+(r)\}$ is a complete set of non-isomorphic irreducible Σ_r-modules; if $p > 0$, then for each p-restricted partition λ, the dual Specht module S_λ has a unique irreducible quotient which we denoted by D_λ. Furthermore $\{D_\lambda \mid \lambda \in \Lambda^+(r)\ p\text{-restricted}\}$ forms a complete set of pairwise non-isomorphic irreducible $k\Sigma_r$-modules. Let Y^λ be the Young module indexed by λ. For more information about representations of symmetric groups, see [13] or [14, 22].

Let E be an n-dimensional k-vector space. Then the r-tensor space $E^{\otimes r}$ admits a natural right Σ_r action by place permutation, namely

$$e_{i_1} \otimes \dots \otimes e_{i_r} \cdot \sigma = e_{i_{1\sigma}} \otimes \dots \otimes e_{i_{r\sigma}}, \qquad i_1, \dots, i_r \in \{1, \dots, n\}, \sigma \in \Sigma_r$$

Definition 2.1. *The Schur algebra $S_k(n,r)$ is defined to be the endomorphism algebra* $\mathrm{End}_{\Sigma_r}(E^{\otimes r})$.

Remark. (1) If we replace the group algebra $k\Sigma_r$ by its quantization, the Hecke algebra $\mathcal{H}_q(r)$

and $E^{\otimes r}$ by the q-tensor space in the definition above , we will get the q-Schur algebras $S_q(n,r)$. For more details about Schur algebras or q-Schur algebras, see [4, 14, 22].

(2) Sometimes, it is convenient to work with the following equivalent definition (which also admits a quantized analogue, see [4]). Let $A_k(n)$ be the polynomial ring in n^2-indeterminants $\{c_{i,j} \mid 1 \leq i,j \leq n\}$ and $A_k(n,r)$ be the r-th homogenous subspace. $A_k(n)$ makes a bialgebra with the comultiplication Δ and counit ε defined by

$$\Delta(c_{i,j}) = \sum_{k=1}^{n} c_{i,k} \otimes c_{k,j}, \qquad \varepsilon(c_{i,j}) = \delta_{i,j}$$

and $A_k(n,r)$ makes a sub-coalgebra. Then $S_k(n,r) \cong \mathrm{Hom}_k(A_k(n,r),k)$ as k-algebras.

(3) The categories of finite dimensional $S_k(n,r)$-modules and polynomial representations of $GL_n(k)$ of homogeneous degree r are equivalent, see [14]. Since the symmetric power S^dE is a polynomial representation of $GL_n(k)$ of degree d, it follows that for any sequence $\alpha = (a_1, a_2, \ldots)$ with $a_1 + a_2 + \cdots = r$, the tensor product $S^\alpha E = S^{a_1}E \otimes S^{a_2}E \otimes \cdots$ is a $S_k(n,r)$-module. If $d \geq p > 0$, then $\{v^p \mid v \in E\}$ generates a $GL_n(k)$-submodule of S^dE. Let Γ^dE denote the quotient module (called *truncated symmetric power*), which is still homogenous of degree d. In particular, the truncated tensor symmetric power $\Gamma^\alpha E = \Gamma^{a_1}E \otimes \Gamma^{a_2}E \otimes \cdots$ is a $S_k(n,r)$-module.

(4) The categories of $S_k(n,r)$-modules and $A_k(n,r)$-comodules are canonically isomorphic. Given a $S_k(n,r)$-module, or equivalently an $A_k(n,r)$-comodule M, let $\delta : M \to M \otimes A_k(n,r)$ be the structure map. Choose a basis $\{m_i\}$ of M. Then $\delta(m_i) = \sum m_j \otimes f_{j,i}$ with $f_{j,i} \in A_k(n,r)$. The *coefficient space* $cf(M)$ is defined to be the k-span of all $f_{j,i}$. It is easy to check that $cf(M)$ is a sub-coalgebra of $A_k(n,r)$ and is independent of the choice of bases.

We collect necessary properties of Schur algebras (q-Schur algebras) as below, see also [4, 14].

(A) There exists an involution (anti-automorphism) of k-algebras $\omega : S_k(n,r) \to S_k(n,r)$ which preserves a complete set of orthogonal primitive idempotents. Call ω a duality later on.

(B) There is a bijection between the iso-classes of irreducible $S_k(n,r)$-modules and $\Lambda^+(n,r)$. For each $\lambda \in \Lambda^+(n,r)$, let $L(\lambda)$ be the corresponding irreducible module and $P(\lambda)$ the projective cover of $L(\lambda)$.

(C) $\Lambda^+(n,r)$ is a poset with respect to the dominance ordering:

$$\lambda \geq \mu \iff \lambda_1 + \cdots + \lambda_s \geq \mu_1 + \cdots + \mu_s, \qquad \forall s \geq 1$$

Moreover, $S_k(n,r)$ is quasi-hereditary in the sense of Cline, Parshall and Scott [2], see also Section 2.3. For each $\lambda \in \Lambda^+(n,r)$, there exists the *Weyl module* $\Delta(\lambda)$ and the *tilting module* $T(\lambda)$. Moreover, each $T(\lambda)$ admits a filtration by Weyl modules and the multiplicity $[T(\lambda) : \Delta(\mu)]$ is independent of choice of filtrations and thus well-defined.

(D) (Schur-Weyl duality) We have the surjective algebra morphism $k\Sigma_r \twoheadrightarrow \mathrm{End}_{S_k(n,r)}(E^{\otimes r})$. If $n \geq r$, then there exists an idempotent $e = e^2 \in S_k(n,r)$ such that $E^{\otimes r} \cong S_k(n,r)e$ and $k\Sigma_r \cong \mathrm{End}_{S_k(n,r)}(E^{\otimes r}) \cong eS_k(n,r)e$. Moreover, $E^{\otimes r}$ is a projective and injective faithful as a $S_k(n,r)$-module.

(E) Let $f : S_k(n,r)\text{-mod} \to k\Sigma_r\text{-mod}$ be the functor which sends a (left) $S_k(n,r)$-module M to $f(M) = \mathrm{Hom}_{S_k(n,r)}(E^{\otimes r}, M)$. Then f is exact when $n \geq r$ and in this case f sends $\Delta(\lambda)$ to S_λ, $P(\lambda)$ to Y^λ and $T(\lambda)$ to $Y^{\lambda'} \otimes \mathrm{sgn}$, where λ' is the transpose of λ.

Theorem 2.2. (Hemmer, Kleshchev and Nakano [16, 19]) *If $p \geq 5$ and $n \geq r$, then the functor f in* **(E)** *induces isomorphisms*

$$\mathrm{Ext}^i_{S_k(n,r)}(M,N) \cong \mathrm{Ext}^i_{k\Sigma_r}(f(M), f(N)), \qquad 0 \leq i \leq p-3$$

for any N which has a Weyl module filtration and all M. In particular, the filtration multiplicities of dual Specht modules in Young modules are well-defined. In the quantum case, the quantum Weyl modules and the functor f are defined similarly. Let ℓ be the quantum characteristic. If $n \geq r$ and $\ell \geq 4$, then f induces isomorphisms $\mathrm{Ext}^1_{S_q(n,r)}(M,N) \cong \mathrm{Ext}^1_{\mathcal{H}_q(r)}(f(M), f(N))$ for any N which has a quantum Weyl module filtration and all M.

2.2 Schur functor

Let A be a finite dimensional algebra over an arbitrary field k and denote by A-mod the category of finite dimensional left A-modules. Let $e = e^2$ be an idempotent in A. Then Ae is a projective left A-module with the endomorphism algebra $\mathrm{End}_A(Ae) \cong eAe$. Consider the exact functor

$$f: \quad A\text{-mod} \longrightarrow eAe\text{-mod} \qquad M \mapsto \mathrm{Hom}_A(Ae, M) \cong eM$$

and call it a general *Schur functor.* In [5], a Grothendieck spectral sequence is established for general Schur functors, namely for any A-module M and eAe-module N

$$E_2^{i,j} = \mathrm{Ext}^i_A(M, \mathrm{Ext}^j_{eAe}(eA, N)) \Rightarrow \mathrm{Ext}^{i+j}_{eAe}(eM, N)$$

2.3 Standardly stratified and quasi-hereditary algebra

Quasi-hereditary algebras were introduced by Cline, Parshall and Scott to study highest weight categories [2]. By definition, an algebra A is said to be *quasi-hereditary* over a poset X if there is a bijection from X to the iso-classes of simple A-modules and for each $x \in X$, there is a quotient module $\Delta(x)$ of $P(x)$, called a *standard module*, satisfying

(1) the kernel of the canonical morphism $\Delta(x) \twoheadrightarrow L(x)$ is filtered by $L(y)$ with $y < x$;

(2) the kernel of the canonical morphism $P(x) \twoheadrightarrow \Delta(x)$ is filtered by $\Delta(z)$ with $z > x$.

The costandard module $\nabla(x)$ is defined to be the k-dual of the standard module $\Delta_{A^{op}}(x)$ of the quasi-hereditary algebra (A^{op}, X). By $\mathscr{F}(\Delta)$ (resp. $\mathscr{F}(\nabla)$) we denote the full subcategory of A-mod whose objects are filtered by standard modules (resp. costandard modules).

Quasi-hereditary algebras have a ring-theoretical definition in terms of hereditary ideals, see [4, appendix]. The following contains some basic properties of these algebras.

(1) For each $x \in X$, there is a unique (up to isomorphism) indecomposable module $T(x)$ in $\mathscr{F}(\Delta) \cap \mathscr{F}(\nabla)$, called a *tilting module.* The direct sum $T := \oplus_{x \in X} T(x)$ is called the *characteristic tilting module* of (A, X) and $\mathrm{add}(T) = \mathscr{F}(\Delta) \cap \mathscr{F}(\nabla)$.

(2) Let Y be a *cosaturated* subset of X (i.e., $x > y \in Y$ implies $x \in Y$). Let e be the cosaturated idempotent of A corresponding to Y. Then $I = AeA$ is a hereditary ideal in A and $(A/I, X \backslash Y)$, (eAe, Y) are quasi-hereditary algebras. The standard modules of A/I are $\{\Delta(x) : x \in X \backslash Y\}$ and the tilting modules of A/I are $\{T(x) : x \in X \backslash Y\}$. The standard modules of eAe are $\{e\Delta(x) : x \in Y\}$ and the tilting modules of eAe are $\{eT(x) : x \in Y\}$.

Standardly stratified algebras were introduced to generalize quasi-hereditary algebras. We remark that there are some variant definitions of standardly stratified algebras, in Section 3, we shall adopt the definition due to Cline, Parshall and Scott. We will not give the definition here, and the interested readers are referred to [3] or [7].

2.4 Dominant dimension

Much of the theory of dominant dimension is due to Morita and Tachikawa, see [24, 25, 26, 28] for more information and [1] for a treatment of some of the topics to be discussed below in terms of homological algebra.

Let A be a finite dimensional algebra over an arbitrary field. The dominant dimension of a left A-module M, which we denote by $\mathrm{dom.dim}\, M$ is the maximal number t (or ∞) such that there exists an injective resolution $0 \to M \to I_0 \to I_1 \to \cdots \to I_t \to \cdots$ of M with I_j projective for all $j < t$ (or ∞). Here we set $\mathrm{dom.dim}\, M = 0$ if the injective hull I_0 of M is not projective or equivalently if M is not a submodule of a module that is both projective and injective. We write $\mathrm{dom.dim}\, A = \mathrm{dom.dim}\, {}_AA$. So, $\mathrm{dom.dim}\, A = 0$ if and only if A does not have a faithful module that is both projective and injective. Algebras of dominant dimension at least 1 are usually called QF-3 algebras (see [28]). If A is QF-3, then there exists a unique (up to conjugate) idempotent $e = e^2 \in A$ such that Ae is projective, injective and minimal faithful. To emphasize this idempotent, we shall denote a QF-3 algebra by (A, e).

If a QF-3 algebra (A, e) has dominant dimension at least two, then the classical results of Morita, Tachikawa and others (see [26]) imply that there is a double centralizer property between A and its centralizer subalgebra eAe: $\mathrm{End}_A(Ae) = eAe$ and $\mathrm{End}_{eAe}(Ae) = A$. In [21] it has been shown that classical Schur-Weyl duality between the Schur algebra $S_k(n, r)$ (for $n \geq r$) and the group algebra $k\Sigma_r$ of the symmetric group as well as Soergel's 'Struktursatz' for the Bernstein-Gelfand-Gelfand category $\mathcal{O}$, providing a double centraliser property between a block and a subalgebra of the coinvariant algebra, both are special cases of this situation. In particular, those Schur algebras and also blocks of $\mathcal{O}$ have dominant dimension at least two.

If A is self-injective, then obviously $\mathrm{dom.dim}\, A = \infty$. The Nakayama conjecture asserts that the converse also holds, i.e.,

$$\mathrm{dom.dim}\, A = \infty \iff A \text{ is self-injective.}$$

By use of the characterization of Schur algebras $S_k(p,p)$ in [27], we have $\operatorname{dom.dim} S(2,2) = 2$ when $p = 2$ and $\operatorname{dom.dim} S(3,3) = 2$, when $p = 2$. $\operatorname{dom.dim} S(3,3) = 4$ when $p = 3$. From the definition, it is generally very hard to compute the dominant dimension of a given algebra. Nevertheless, the following characterization due to Müller is rather applicable.

Theorem 2.3. **(Müller [23])** *Assume that* $\operatorname{dom.dim} A \geq 2$ *and* Ae *is a projective, injective and faithful module. Let* M *be an* A*-module. Then* $\operatorname{dom.dim} M \geq n \geq 2$ *if and only if* $M \cong \operatorname{Hom}_{eAe}(eA, eM)$ *canonically and* $\operatorname{Ext}^i_{eAe}(eA, eM) = 0$ *for* $1 \leq i \leq n-2$.

3 Connection between Schur functors and dominant dimension

Let (A, e) be a QF-3 algebra. We shall consider the Schur functor f afforded by the idempotent e. As we have seen, this setting has included the most interesting cases like Schur-Weyl duality, Soergel's struktursatz and so on.

Note that by Mülller's characterization, we have $n(\mathcal{P}_A) = \operatorname{dom.dim} A - 2$ following the notation from the introduction. This leads us to expect the connection between A-mod and eAe-mod to be all the better the larger the dominant dimension of A is. Indeed, we have

Theorem 3.1. *Let* (A, e) *be a QF-3 standardly stratified algebra whose simples are indexed by a quasi-poset* (X, θ). *If* $\operatorname{dom.dim} A \geq \ell(X) + 2 + s$ *for some* $s \geq 1$, *then there is a full embedding of categories:* $f : \mathscr{F}(\Delta) \hookrightarrow eAe$-mod, *where* $\ell(X)$ *denotes the diameter of* X. *Furthermore* $\operatorname{Ext}^i_A(M, N) \cong \operatorname{Ext}^i_{eAe}(eM, eN)$ *for* $0 \leq i \leq s$ *for any* $M \in A$*-mod and* $N \in \mathscr{F}(\Delta)$.

This theorem affirmatively hints at the importance of dominant dimension in the setting. However, it is still weak to say anything deep. In the following, we restrict ourselves further to a proper class of algebras, which is rich enough to contain all interesting examples mentioned already, say (quantum) Schur algebras, block algebras of BGG category $\mathcal{O}$, etc.

Definition 3.2. *The class* $\mathscr{A}$ *consists of finite dimensional* k*-algebras* A, *which are split over the field* k *and satisfy the following properties:*

(1) *A is a quasi-hereditary algebra over a poset X,*

(2) *A has a duality*[1],

(3) $\operatorname{dom.dim} A \geq 2$,

Section 2.1 **(A)-(D)** implies that $S_k(n,r)$ belongs to the class $\mathscr{A}$ when $n \geq r$. For simplicity, we shall denote an algebra A in the class $\mathscr{A}$ by a quadruple (A, X, ω, e) where e is an idempotent in A such that Ae is projective, injective and minimal faithful, X is a poset indexing simple A-modules and ω is the duality from the definition.

[1]See [9] for an exact definition of the duality here, see also Section 2.1 **(A)**.

Theorem 3.3. *Let (A, X, ω, e) be in the class $\mathscr{A}$ and M be any A-module. Then the Schur functor induces canonical isomorphisms for any projective A-module P and any $K \in \mathscr{F}(\Delta)$*

$$\mathrm{Ext}^i_A(M, P) \cong \mathrm{Ext}^i_{eAe}(eM, eP), \qquad 0 \leq i \leq \mathrm{dom.\,dim}\, A - 2$$
$$\mathrm{Ext}^i_A(M, K) \cong \mathrm{Ext}^i_{eAe}(eM, eK), \qquad 0 \leq i \leq \mathrm{dom.\,dim}\, T - 2.$$

Furthermore, there are equalities $n(\mathcal{P}_A) = \mathrm{dom.\,dim}\, A - 2$ and $n(\mathscr{F}(\Delta)) = \mathrm{dom.\,dim}\, T - 2$.

From Theorem 3.3, we see that the Schur functor f induces an equivalence from $\mathscr{F}(\Delta)$ to the full subcategory $\mathscr{F}(e\Delta)$ of eAe-mod which consists of the eAe-modules filtered by $\{e\Delta(x) : x \in X\}$ provided $\mathrm{dom.\,dim}\, T \geq 2$ (or equivalently $\mathrm{dom.\,dim}\, A \geq 4$ by Theorem 3.4(4) below). Moreover, this equivalence preserves Ext-groups up to degree $\mathrm{dom.\,dim}\, T - 2$. In this sense, Theorem 3.3 largely extends Hemmer, Kleshchev and Nakano's results for (quantized) Schur algebras to all algebras in $\mathscr{A}$ and attributes Theorem 2.2 to the computation of $\mathrm{dom.\,dim}\, T$. The following theorem reduces further the computation of $\mathrm{dom.\,dim}\, T$ to that of $\mathrm{dom.\,dim}\, A$. In particular, $n(\mathscr{F}(\Delta))$ is independent of the choice of quasi-hereditary structures.

Theorem 3.4. *Let (A, X, ω, e) be in the class $\mathscr{A}$ and T be the characteristic tilting module.*

(1) *The Ringel dual $R = \mathrm{End}_A(T)^{op}$ belongs to $\mathscr{A}$;*

(2) *R has characteristic tilting module $\mathrm{D}(T)$; $\mathrm{dom.\,dim}_A\, T \geq 1$ and $\mathrm{dom.\,dim}_R\, \mathrm{D}(T) \geq 1$;*

(3) $\mathrm{dom.\,dim}_R\, \mathrm{D}(T) = \mathrm{dom.\,dim}_A\, T$;

(4) $\mathrm{dom.\,dim}\, A = 2\,\mathrm{dom.\,dim}_A\, T$. *In particular,* $\mathrm{dom.\,dim}\, A = \mathrm{dom.\,dim}\, R$.

Corollary 3.5. *For any $x \in X$, the standard module $\Delta(x)$ has dominant dimension at least 1. In particular, all standard A-modules are torsionless (i.e., submodules of projectives).*

This generalizes results of James and Donkin [4] on the torsionless property of (quantum) Weyl modules, see also [16]. For blocks of category $\mathcal{O}$, it is well-known that all Verma modules are submodules of the projective Verma module. The following theorem describes the behavior of dominant dimension under truncation process.

Theorem 3.6. *Let (A, X, ω, e) be an indecomposable algebra in the class $\mathscr{A}$. Let $\lambda_{\max}$ be a maximal element in the poset X.*

(1) *If $T(\lambda_{\max}) = \Delta(\lambda_{\max})$, then A is simple and isomorphic to k.*

(2) *If $T(\lambda_{\max})/\Delta(\lambda_{\max})$ is tilting, then* $\mathrm{dom.\,dim}\, A = 2$ *and* $\mathrm{dom.\,dim}\, A/I = 2$ *or* ∞.

(3) *If* $\mathrm{dom.\,dim}\, A = n \geq 4$, *then* $\mathrm{dom.\,dim}_{A/I}\, A/I = n$ *or* $n - 2$.

There seems to be a close relation between global dimension and dominant dimension as the following corollary indicates.

Corollary 3.7. *Let A be an algebra in the class $\mathscr{A}$ and $i > 0$. Then $\mathrm{Ext}^i_A(\mathrm{D}(A), A) \neq 0$ implies* $\mathrm{dom.\,dim}\, A - 1 \leq i \leq \mathrm{gl.dim}A$; $\mathrm{Ext}^i_A(T, A) \neq 0$ *implies* $\mathrm{dom.\,dim}\, T - 1 \leq i \leq \mathrm{proj.dim}T$, *where* $\mathrm{D}(A)$ *is the k-dual of A.*

4 Generalized symmetric algebras and dominant dimension

In order to develop further new characterizations and applications of dominant dimension, we enlarged our algebra class $\mathscr{A}$ in Section 3 to introduce *generalized symmetric algebras.* The new class contains on the one side some quasi-hereditary algebras (finite global dimension), like (quantum) Schur algebras, block algebras of BGG category $\mathcal{O}$, and on the other side, all symmetric algebras (infinite global dimension), say Hecke algebras. The following theorem provides both a definition and equivalent characterizations of generalized symmetric algebras [10].

Theorem 4.1. *Let A be a finite dimensional k-algebra. The following statements are equivalent:*

(1) $\operatorname{dom.dim} A \geq 2$ *and* $\mathrm{D}(Ae) \cong eA$ *as* (eAe, A)*-bimodules, where* Ae *is a basic faithful projective and injective module;*

(2) $\mathrm{Hom}_A(\mathrm{D}(A), A) \cong A$ *as* (A, A)*-bimodules;*

(3) $\mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)$ *as* (A, A)*-bimodules;*

(4) *A is the endomorphism ring of a generator over a symmetric algebra.*

As an application, we obtained an refinement of the Tachikawa-Morita correspondence, i.e.,

$$\left\{ (\Lambda, M) \,\middle|\, \begin{array}{l} \Lambda \text{ finite dimensional symmetric} \\ \text{algebra, } M \text{ a generator in } \Lambda\text{-mod} \end{array} \right\} \longleftrightarrow \left\{ A \,\middle|\, \begin{array}{l} A \text{ finite dimensional} \\ \mathrm{Hom}_A(\mathrm{D}(A), A) \cong A \\ \text{as } (A, A)\text{-bimodules} \end{array} \right\}$$

which sends (Λ, M) to $\mathrm{End}_\Lambda(M)$. Here D denotes the duality over the ground field. We remark that generators over symmetric algebras are the same as cogenerators, therefore, no confusion will arise when talking about generators instead of generator-cogenerators for symmetric algebras. Another feature of generalized symmetric algebras is that they admit a new characterization of dominant dimension [10].

Proposition 4.2. *Let A be a generalized symmetric k-algebra. Then for any left A-module M, we have* $\operatorname{dom.dim} M \geq n$ *if and only if* $\mathrm{Hom}_A(\mathrm{D}(A), M) \cong M$ *and* $\mathrm{Ext}^i_A(\mathrm{D}(A), M) = 0$ *for* $i = 1, 2, \ldots, n-2$, *where n is an integer no less than* 2.

In [11], we construct a Hochschild cocomplex for every generalized symmetric algebras and study its exactness. As a byproduct, we obtained a relation between dominant dimension and Hochschild cohomology group. Indeed, by Theorem 4.1(3), every generalized symmetric algebra A satisfies $\mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)$ as (A, A)-bimodules. Thus the composite

$$\mathrm{D}(A) \otimes_k \mathrm{D}(A) \xrightarrow{m} \mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)$$

defines an associative multiplication on $\mathrm{D}(A)$. We remark that $\mathrm{D}(A)$ with this multiplication m usually has no unit and we can prove that $(\mathrm{D}(A), m)$ has a unit if and only if A is symmetric. Consider the bar complex of the algebra $(\mathrm{D}(A), m)$.

$$\cdots \xrightarrow{m_3} \mathrm{D}(A) \otimes \mathrm{D}(A) \otimes \mathrm{D}(A) \xrightarrow{m_2} \mathrm{D}(A) \otimes \mathrm{D}(A) \xrightarrow{m_1 := m} \mathrm{D}(A) \to 0$$

where $m_i(f_0 \otimes \cdots \otimes f_i) = \sum_{j=0}^{i}(-1)^j f_0 \otimes \cdots \otimes f_{j-1} \otimes m(f_i, f_{i+1}) \otimes f_{i+2} \otimes \cdots \otimes f_i$ for $f_1, \ldots, f_i \in$ D(A). Dualizing the complex, we obtain a Hochschild cocomplex

$$0 \to A \to A \otimes_k A \to A \otimes_k A \otimes_k A \to \cdots$$

We note that the cocomplex is not canonically determined by A for the choice of the isomorphism $\mathrm{D}(A) \otimes_A \mathrm{D}(A) \cong \mathrm{D}(A)$. However, different choices will result in isomorphic cocomplexes. Note also that if $(\mathrm{D}(A), m)$ has a unit, by the usual arguments, the cocomplex or its dual above is exact, whereas generally the cocomplex is not necessarily exact by the following theorem [11].

Theorem 4.3. *Let A be a generalized symmetric algebra. Let n be a natural number. Then* $\mathrm{dom.dim}\, A \geq n$ *iff the sequence* $0 \to A \to A^{\otimes 2} \to \cdots \to A^{\otimes n} \to A^{\otimes(n+1)}$ *is exact.*

By definition, $\mathrm{dom.dim}\, A^{op} = \mathrm{dom.dim}\, A$. Regarding A as an (A, A)-bimodule, we have

Theorem 4.4. *Let A be a generalized symmetric algebra. Then* $\mathrm{dom.dim}\, A^e = \mathrm{dom.dim}\, A$ *and* $\mathrm{dom.dim}_{A^e} A = \mathrm{dom.dim}\, A$.

This theorem combined with Theorem 3.3 yields

Theorem 4.5. *Let A be a generalized symmetric algebra. Let Ae be a faithful projective and injective module. Then* $\mathrm{HH}^i(A) \cong \mathrm{HH}^i(eAe)$ *for* $0 \leq i \leq \mathrm{dom.dim}\, A - 2$.

5 Computation of dominant dimension

With the techniques developed in Section 4 and Müller's theorem, we are able to compute the dominant dimension of a number of algebras. To be more precise, we have in [9]

Theorem 5.1. *Let k be an infinite field of characteristic $p > 0$. Let n and r be two natural numbers. If $n \geq r \geq p$, then* $\mathrm{dom.dim}\, S_k(n, r) = 2(p-1)$ *and* $n(\mathscr{F}(\Delta)) = p - 3$.

Note that if $p = 0$ or $p > r$, the Schur algebra $S_k(n, r)$ is semisimple and hence has ∞ dominant dimension. We remark that there are now three different proofs available for this theorem and a quantum analogue of the theorem also exists. Namely, let ℓ be the quantum characteristic and assume that $n \geq r \geq \ell$. Then $\mathrm{dom.dim}\, S_q(n, r) = 2(\ell - 1)$. In [7], we computed the dominant dimension of block algebras of BGG category $\mathcal{O}$ of semisimple Lie algebras, i.e.,

Theorem 5.2. *Let A be a non-seimsimple block algebra of the category $\mathcal{O}$ of a semisimple Lie algebra. Then* $\mathrm{dom.dim}\, A = 2$.

In [12], we made an attempt to compute the dominant dimension of the Schur algebra $S_k(n, r)$ for general n, r. Note that the tensor space $E^{\otimes r}$ becomes a projective, injective faithful $S_k(n, r)$-module when $n \geq r$, see Section 2.1 **(D)**, but fails to be projective when $n < r$. Even worse, we don't know a clear candidate of a projective, injective and faithful $S_k(n, r)$-module in this latter case. As a result, we are left almost no chance to apply Müller's characterization directly. Our strategy below is to make an essential use of the theory of dominant dimension newly developed in Section 4.

Let R be a commutative ring with a unit. Let $\mathrm{M}_n(R)$ be the algebraic monoid of $n \times n$ matrices over R and let $A_R(n,r)$ be the space of homogenous polynomials of degree r in n^2-indeterminants $c_{i,j}$. Clearly $A_R(n,r)$ is a free R-module of finite rank. Let $S_R(n,r) = \mathrm{Hom}_R(A_R(n,r), R)$. Then we have by Section 2.1 (see also [14]) that, $A_R(n,r) \otimes_{S_R(n,r)} A_R(n,r) \cong A_R(n,r) \otimes_{\mathrm{M}_n(R)} A_R(n,r)$ and $A_k(n,r) \cong A_{\mathbb{Z}}(n,r) \otimes_{\mathbb{Z}} k, S_k(n,r) \cong S_{\mathbb{Z}}(n,r) \otimes_{\mathbb{Z}} k$ for any field k.

Lemma 5.3. *Let k be any infinite field. Then the Schur algebra $S_k(n,r)$ with $n \geq r$ is generalized symmetric. In particular, $A_k(n,r) \otimes_{S_k(n,r)} A_k(n,r) \cong A_k(n,r)$ as $S_k(n,r)$-bimodules.*

With the aid of this lemma, we have that for all n, r

Theorem 5.4. *$A_R(n,r) \otimes_{\mathrm{M}_n(R)} A_R(n,r)$ is a free R-module.*

Note that $A_R(n,r)$ is spanned by the monomial $c_{\underline{i},\underline{j}} = c_{i_1,j_1} \cdots c_{i_r,j_r}$ with $\underline{i}, \underline{j} \in I(n,r)$.

Lemma 5.5. *In $A_R(n,r) \otimes_{\mathrm{M}_n(R)} A_R(n,r)$, we have that $c_{\underline{i},\underline{j}} \overline{\otimes} c_{\underline{k},\underline{l}} \neq 0$ implies $\underline{i} \sim \underline{l}$.*

Theorem 5.6. *Let R be a commutative ring with a unit. Let n and r be two natural numbers. Then the map $\Theta = \Theta_{n,r,R} : A_R(n,r) \otimes_{\mathrm{M}_n(R)} A_R(n,r) \to A_R(n,r)$ defined by*

$$\Theta(c_{\underline{i},\underline{j}} \otimes c_{\underline{k},\underline{i}}) = \sum_{\sigma \in \mathrm{Stab}(\underline{i})} c_{\underline{k},\underline{j}\sigma}$$

is a $S_R(n,r)$-bimodule homomorphism. If $n \geq r$, then $\Theta_{n,r,R}$ is an isomorphism.

Combined with Lemma 5.5, to establish the morphism Θ, we only need to check that it is well-defined. We note that $\Theta_{n,r,R}$ is combinatorially defined, but fails to be either injective or surjective for general n and r. Let k be an infinite field of characteristic $p = 2$ and $n = 2, r = 3$. Then the image of $\Theta_{2,3,k}$ is spanned by $\{c_{11}^2c_{22} + c_{11}c_{12}c_{21}, c_{11}c_{12}c_{22} + c_{12}^2c_{21}, c_{11}c_{22}^2 + c_{12}c_{21}c_{22}, c_{11}c_{21}c_{22} + c_{12}c_{21}^2\}$, which is definitely not $A_k(2,3)$. Indeed, as we shall see below, the image of $\Theta_{n,r,k}$ coincides with the Doty coalgebra $D_{n,r,p}$.

Definition 5.7. *Let k be a field of characteristic $p > 0$. The Doty coalgebra $D_k(n,r) := D_{n,r,p}$ is defined to be the sum of the coefficient space of the truncated tensor symmetric power $\Gamma^\alpha E$ for all sequences $\alpha = (a_1, a_2, \ldots)$ with $a_i \geq 0$ and $a_1 + a_2 + \cdots = r$, see Section 2.1.*

Doty coalgebras were introduced by Doty and Walker in order to study the decomposition matrix of general linear groups [6]. Recently, Heaton made a systematic investigation on when a Doty coalgebra is quasi-hereditary (i.e., the dual algebra is quasi-hereditary) [15]. In [12], we proved the following relation between the map $\Theta_{n,r,k}$ and the Doty coalgebra $D_{n,r,p}$.

Proposition 5.8. *Let n and r be two natural numbers and $\Theta_{n,r,k}$ be the map from Theorem 5.6. Then $\mathrm{Im}(\Theta_{n,r,k}) = D_{n,r}$.*

Theorem 5.9. *Let k be an infinite field of characteristic $p > 0$. Let n and r be two natural numbers. Let $\Theta := \Theta_{n,r,k}$ be the multiplication map on $A_k(n,r)$ from Theorem 5.6. Then the following statements are equivalent*

(1) Θ *is an isomorphism;*

(2) $D_{n,r} = A_k(n,r)$*;*

(3) $r \leq n(p-1)$.

We remark that (1) the proof of this theorem involves straightening bideterminants of boson type following the line in [8]; (2) the theorem supplements Heaton's result and combined with Theorem 4.1 yields

Corollary 5.10. *Let k be an infinite field of characteristic $p > 0$. Let n and r be two natural numbers. Then $S_k(n,r)$ is generalized symmetric and hence has dominant dimension at least* 2 *whenever $r \leq n(p-1)$.*

With this corollary and Theorem 4.3 in Section 4, we are able to compute $\mathrm{dom.dim}\, S_k(n,r)$ using computer programme in case $r \leq n(p-1)$. On the other hand, the condition in this corollary is only sufficient but not necessary as the following example illustrates, see also [22]. Let k be a field of characteristic 2. Then the Schur algebra $S_k(2,7)$ is Morita equivalent to $S_k(2,2) \times k \times k$. In particular $\mathrm{dom.dim}\, S_k(2,7) = \mathrm{dom.dim}\, S_k(2,2) = 2$.

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