数理解析研究所講究録
第 1795 巻 2012 年 1-44

On some combinatorial and algebraic properties of Dunkl elements

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To Volodya Korepin, my friend and co-author, on the occasion of his sixtieth birthday, with grateful and admiration

Abstract

We introduce and study a certain class of nonhomogeneous quadratic algebras together with the special set of mutually commuting elements inside of each, the so-called *Dunkl elements.* We describe relations among the Dunkl elements. This result is a further generalization of similar results obtained in [10],[32] and [21]. As an application we describe explicitly the set of relations among the Gaudin elements in the group ring of the symmetric group, cf [30].

Yet another objective of our paper is to describe several combinatorial properties of some special elements, the so-called *Coxeter element* and the *longest element*, in the associative quasi-classical Yang–Baxter algebra. In the case of *Coxeter element* we relate the corresponding reduced polynomials introduced in [40], with the β-Grothendieck polynomials for some special permutations $\pi_k^{(n)}$. Moreover, we show that the specialization $\mathfrak{G}_{\pi_k^{(n)}}^{(\beta)}(1)$ of the β-Grothendieck polynomial $\mathfrak{G}_{\pi_k^{(n)}}^{(\beta)}(X_n)$ counts the number of *k-dissections* of a convex $(n+k+1)$-gon according to the number of diagonals involved. When the number of diagonals in a k-dissection is the maximal possible, we recover the well-known fact that the number of k-triangulations of a convex $(n+k+1)$-gon is equal to the value of a certain Catalan-Hankel determinant, see e.g. [36]. We also show that for a certain 5-parameters family of vexillary permutations, the specialization $x_i=1, \forall i \geq 1$, of the corresponding $\underline{\beta\text{-Schubert}}$ polynomials $\mathfrak{S}_w^{(\beta)}(1)$ coincides with some Fuss-Narayana polynomials and their generalizations. We also point out on a conjectural connection between the sets of *maximal* compatible sequences for the permutation $\sigma_{n,2n,2,0}$ and that $\sigma_{n,2n+1,2,0}$ from one side, and the set of $VSASM(n)$ and that of $CSTCPP(n)$ correspondingly, from the other, see Comments 3.6 for details. Finally, in Section 3 we introduce and study a multiparameter generalization of reduced polynomials introduced in [40], that of Catalan, Narayana and Schröder numbers.

In the case of the *longest element* we relate the corresponding reduced polynomial with the Ehrhart polynomial of the Chan–Robbins polytope.

Introduction

The Dunkl operators have been introduced in the later part of 80's of the last century by Charles Dunkl [7], [8] as a powerful mean to study of harmonic and orthogonal polynomials related with finite Coxeter groups. In the present paper we don't need the definition of Dunkl operators for arbitrary (finite) Coxeter groups, see e.g. [7], but only for the special case of the symmetric group $\mathbb{S}_n$.

Definition 0.1 *Let $P_n = \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials in variables $x_1, \ldots, x_n$. The type A_{n-1} (additive) rational Dunkl operators $D_1, \ldots, D_n$ are the differential-difference operators of the following form*

$$D_i = \lambda \, \frac{\partial}{\partial x_i} + \sum_{j \neq i} \frac{1 - s_{ij}}{x_i - x_j}, \tag{0.1}$$

Here s_{ij}, $1 \leq i < j \leq n$, denotes the exchange (or permutation) operator, namely,

$$s_{ij}(f)(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n);$$

$\frac{\partial}{\partial x_i}$ stands for the derivative w.r.t. the variable x_i; $\lambda \in \mathbb{C}$ is a parameter.

The key property of the Dunkl operators is the following result.

Theorem 0.1 *(C.Dunkl [7]) For any finite Coxeter group (W, S), where $S = \{s_1, \ldots, s_l\}$ denotes the set of simple reflections, the Dunkl operators $D_i := D_{s_i}$ and $D_j := D_{s_j}$ commute: $D_i \, D_j = D_j \, D_i$, $1 \leq i, j \leq l$.*

Another fundamental property of the Dunkl operators which finds a wide variety of applications in the theory of integrable systems, see e.g. [15], is the following statement: the operator

$$\sum_{i=1}^{l} (D_i)^2$$

"essentially" coincides with the Hamiltonian of the rational Calogero–Moser model related to the finite Coxeter group (W, S).

Definition 0.2 *Truncated (additive) Dunkl operator (or the Dunkl operator at critical level), denoted by $\mathcal{D}_i$, $i = 1, \ldots, l$, is an operator of the form (0.1) with parameter $\lambda = 0$.*

For example, the type A_{n-1} rational truncated Dunkl operator has the following form

$$\mathcal{D}_i = \sum_{j \neq i} \frac{1 - s_{ij}}{x_i - x_j}.$$

Clearly the truncated Dunkl operators generate a commutative algebra.
The important property of the truncated Dunkl operators is the following result discovered and proved by C.Dunkl [8]; see also [1] for a more recent proof.

Theorem 0.2 *(C.Dunkl [8], Y.Bazlov [1]) For any finite Coxeter group (W,S) the algebra over $\mathbb{Q}$ generated by the truncated Dunkl operators $\mathcal{D}_1,\ldots,\mathcal{D}_l$ is canonically isomorphic to the coinvariant algebra of the Coxeter group (W,S).*

Example 0.1 In the case when $W=\mathbb{S}_n$ is the symmetric group, Theorem 0.2 states that the algebra over $\mathbb{Q}$ generated by the truncated Dunkl operators $\mathcal{D}_i=\sum_{j\neq i}\frac{1-s_{ij}}{x_i-x_j}$, $i=1,\ldots,n$, is canonically isomorphic to the cohomology ring of the full flag variety $\mathcal{F}l_n$ of type A_{n-1}

$$\mathbb{Q}[\mathcal{D}_1,\ldots,\mathcal{D}_n]\cong\mathbb{Q}[x_1,\ldots,x_n]/J_n, \tag{0.2}$$

where J_n denotes the ideal generated by elementary symmetric polynomials $\{e_k(X_n),\ 1\le k\le n\}$.

Recall that the symmetric polynomials $e_i(X_n)$, $i=1,\ldots,n$, are defined through the generating function

$$1+\sum_{i=1}^{n}e_i(X_n)\,t^i=\prod_{i=1}^{n}(1+t\,x_i),$$

where we set $X_n:=(x_1,\ldots,x_n)$. It is well-known that in the case $W=\mathbb{S}_n$, the isomorphism (0.2) can be defined over the ring of integers $\mathbb{Z}$. ∎

Theorem 0.2 by C.Dunkl has raised a number of questions:

(**A**) What is the algebra generated by the truncated

- trigonometric,
- elliptic,
- super, matrix, ...,

(a) additive Dunkl operators ?

(b) Ruijsenaars–Schneider–Macdonald operators ?

(c) Gaudin operators ?

(**B**) Describe commutative subalgebra generated by the Jucys–Murphy elements in

- the group ring of the symmetric group;
- the Hecke algebra ;
- the Brauer algebra, BMV algebra, ...

(**C**) Does there exist an analogue of Theorem 0.2 for

- Classical and quantum equivariant cohomology and equivariant K-theory rings of the flag varieties ?
- Cohomology and K-theory rings of affine flag varieties ?
- Diagonal coinvariant algebras of finite Coxeter groups ?
- Complex reflection groups ?

The present paper is a short Introduction to a few items from Section 5 of [18].

The main purpose of my paper "On some quadratic algebras, II" is to give some partial answers on the above questions in the case of the symmetric group $\mathbb{S}_n$.

The purpose of the present paper is to draw attention to an interesting class of nonhomogeneous quadratic algebras closely connected (still mysteriously !) with different branches of Mathematics such as

Classical and Quantum Schubert and Grothendieck Calculi,

Low dimensional Topology,

Classical, Basic and Elliptic Hypergeometric functions,
Algebraic Combinatorics and Graph Theory,
Integrable Systems,
. . . .

What we try to explain in [18] is that upon passing to *a suitable representation* of the quadratic algebra in question, the subjects mentioned above, are a manifestation of certain general properties of that quadratic algebra.

From this point of view, we treat the commutative subalgebra generated by the additive (resp. multiplicative) truncated Dunkl elements in the algebra $3T_n(\beta)$, see Definition 2.1, as *universal cohomology* (resp. *universal K-theory*) ring of the complete flag variety $\mathcal{F}l_n$. The classical or quantum cohomology (resp. the classical or quantum K-theory) rings of the flag variety $\mathcal{F}l_n$ are certain quotients of the *universal ring.*

For example, in [20] we have computed relations among the (truncated) Dunkl elements $\{\theta_i,\ i = 1,\ldots,n\}$ in the *elliptic representation* of the algebra $3T_n(\beta = 0)$. We **expect** that the commutative subalgebra obtained is isomorphic to (yet not defined, but see [14]) the *elliptic cohomology ring* of the flag variety $\mathcal{F}l_n$.

Another example from [18]. Consider the algebra $3T_n(\beta = 0)$.
One can prove [18] the following <u>*identities*</u> in the algebra $3T_n(\beta = 0)$

(A) Summation formula

$$\sum_{j=1}^{n-1}\Big(\prod_{b=j+1}^{n-1} u_{b,b+1}\Big)\, u_{1,n}\, \Big(\prod_{b=1}^{j-1} u_{b,b+1}\Big) = \prod_{a=1}^{n-1} u_{a,a+1}.$$

(B) Duality transformation formula Let $m \leq n$, then

$$\sum_{j=m}^{n-1}\Big(\prod_{b=j+1}^{n-1} u_{b,b+1}\Big)\Big[\prod_{a=1}^{m-1} u_{a,a+n-1}\, u_{a,a+n}\Big]\, u_{m,m+n-1}\, \Big(\prod_{b=m}^{j-1} u_{b,b+1}\Big) =$$

$$\sum_{j=1}^{m}\Big[\prod_{a=1}^{m-j} u_{a,a+n}\, u_{a+1,a+n}\Big]\Big(\prod_{b=m}^{n-1} u_{b,b+1}\Big)\Big[\prod_{a=1}^{j-1} u_{a,a+n-1}\, u_{a,a+n}\Big] -$$

$$\sum_{j=2}^{m}\Big[\prod_{a=j}^{m-1} u_{a,a+n-1}\, u_{a,a+n}\Big]\, u_{m,n+m-1}\, \Big(\prod_{b=m}^{n-1} u_{b,b+1}\Big)\, u_{1,n}.$$

One can check that upon passing to the *elliptic representation* of the algebra $3T_n(\beta = 0)$, *see* [18], Section 5.1.7, or [20] for the definition of *elliptic representation*, the above identities (**A**) and (**B**) finally end up correspondingly, to be a Summation formula and a Duality transformation formula for multiple elliptic hypergeometric series (of type A_{n-1}), see e.g. [16] for definition of the latter.

After passing to the so-called *Fay representation* [18], the identities (**A**) and (**B**) become correspondingly to be the Summation formula and Duality transformation formula for the Riemann theta functions of genus $g > 0$, [18]. These formulas in the case $g \geq 2$ seems to be new.

■

A few words about the content of the present paper.

In the first section I introduce the so-called *dynamical classical Yang–Baxter algebra* as "a natural quadratic algebra" in which the Dunkl elements form a pair-wise commuting family. It is the study of the algebra generated by the (truncated) Dunkl elements that is the main objective of our investigation in [18] and the present paper.

In Section 2, see Definition 2.1, we introduce the algebra $3HT_n(\beta)$, which seems to be the most general (noncommutative) deformation of the (even) Orlik–Solomon algebra, such that it's still possible to describe relations among the Dunkl elements, see Theorem 2.1. As an application we describe explicitly a set of relations among the (additive) Gaudin / Dunkl elements, cf [30].

In Section 3 we describe some combinatorial properties of special elements in the associative quasi-classical Yang–Baxter algebra [1] $\widehat{ACYB}_n$. The results of Section 3.1, see Proposition 3.1, items (**1**)–(**5**), are more or less known among the specialists in the subject, while those of the item (**6**) seem to be new. Namely, we show that the polynomial $Q_n(x_{ij} = t_i)$ from [40], (**6.C8**), (c), essentially coincides with the β-deformation [11] of the Lascoux- Schützenberger Grothendieck polynomial [25] for some particular permutation. The results of Proposition 3.1, (**6**), point out on a deep connection between reduced forms of monomials in the algebra $\widehat{ACYB}_n$ and the Schubert and Grothendieck Calculi. This observation was the starting point for the study of some combinatorial properties of certain specializations of the Schubert, β-Grothendieck [12] and double β-Grothendieck polynomials in Section 3.2 . One of the main results of Section 3.2 can be stated as follows.

Theorem 0.3

(1) *Let $w \in \mathbb{S}_n$ be a permutation, consider the specialization $x_1 := q, x_i = 1, \ \forall i \geq 2$, of the β-Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}(X_n)$. Then*

$$\mathcal{R}_w(q, \beta + 1) := \mathfrak{G}_w^{(\beta)}(x_1 = q, x_i = 1, \ \forall i \geq 2) \in \mathbb{N}[q, 1 + \beta].$$

In other words, the polynomial $\mathcal{R}_w(q, \beta)$ has non-negative integer coefficients [2].
For late use we define *polynomials*

$$\mathfrak{R}_w(q, \beta) := q^{1-w(1)} \ \mathcal{R}_w(q, \beta).$$

(2) *Let $w \in \mathbb{S}_n$ be a permutation, consider the specialization $x_i := q, y_i = t, \ \forall i \geq 1$, of the double β-Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}(X_n, Y_n)$. Then*

$$\mathfrak{G}_w^{(\beta-1)}(x_i := q, y_i := t, \forall i \geq 1) \in \mathbb{N}[q, t, \beta].$$

(3) *Let w be a permutation, then*

$$\mathfrak{R}_w(1, \beta) = \mathfrak{R}_{1 \times w}(0, \beta).$$

Note that $\mathcal{R}_w(1, \beta) = \mathcal{R}_{w^{-1}}(1, \beta)$, but $\mathcal{R}_w(t, \beta) \neq \mathcal{R}_{w^{-1}}(t, \beta)$, in the general case.

[1] The algebra $\widehat{ACYB}_n$ can be treated as "one-half" of the algebra $3T_n(\beta)$. It appears, see Lemma 3.1, that the basic relations among the Dunkl elements, which do **not** mutually commute anymore, are still valid.

[2] For a more general result see Appendix

For the reader convenience we collect some basic definitions and results concerning the β-Grothendieck polynomials in Appendix.

Let us observe that $\mathfrak{R}_w(1,1) = \mathfrak{S}_w(1)$, where $\mathfrak{S}_w(1)$ denotes the specialization $x_i := 1$, $\forall i \geq 1$, of the Schubert polynomial $\mathfrak{S}_w(X_n)$ corresponding to permutation w. Therefore, $\mathfrak{R}_w(1,1)$ is equal to the number of *compatible sequences* [4] (or *pipe dreams*, see e.g. [36]) corresponding to permutation w.

Problem 0.1

Let $w \in \mathbb{S}_n$ be a permutation and $l := \ell(w)$ be its length. Denote by $CS(w) = \{\mathbf{a} = (a_1 \leq a_2 \leq \ldots \leq a_l) \in \mathbb{N}^l\ \}$ the set of compatible sequences [4] corresponding to permutation w.

- *Define statistics $r(\mathbf{a})$ on the set of all compatible sequences $CS_n := \coprod_{w \in \mathbb{S}_n} CS(w)$ in a such way that*

$$\sum_{\mathbf{a} \in CS(w)} q^{a_1}\ \beta^{r(\mathbf{a})} = \mathcal{R}_w(q,\beta).$$

- *Find and investigate a geometric interpretation, combinatorial and algebra-geometric properties of polynomials $\mathfrak{S}_w^{(\beta)}(X_n)$,*

where for a permutation $w \in \mathbb{S}_n$ we denoted by $\mathfrak{S}_w^{(\beta)}(X_n)$ the β-Schubert polynomial defined as follows

$$\mathfrak{S}_w^{(\beta)}(X_n) = \sum_{\mathbf{a} \in CS(w)} \beta^{r(\mathbf{a})} \prod_{i=1}^{l:=\ell(w)} x_{a_i}.$$

We **expect** that polynomial $\mathfrak{S}_w^{(\beta)}(1)$ coincides with the Hilbert polynomial of a certain graded commutative ring naturally associated to permutation w.

Remark 0.1 *It should be mentioned that, in general, the principal specialization*

$$\mathfrak{G}_w^{(\beta-1)}(x_i := q^{i-1},\ \forall i \geq 1)$$

of the $(\beta-1)$-Grothendieck polynomial may have negative coefficients. ■

Our main objective in Section 3.2 is to study the polynomials $\mathfrak{R}_w(q,\beta)$ for a special class of permutations in the symmetric group $\mathbb{S}_\infty$. Namely, in Section 3.2 we study some combinatorial properties of polynomials $\mathfrak{R}_{\varpi_{\lambda,\phi}}(q,\beta)$ for the five parameters family of *vexillary* permutations $\{\varpi_{\lambda,\phi}\}$ which have the shape

$\lambda := \lambda_{n,p,b} = (p(n-i+1)+b,\ i = 1,\ldots,n+1)$ and flag

$\phi := \phi_{k,r} = (k + r(i-1),\ i = 1,\ldots,n+1)$.

This class of permutations is notable for many reasons, including that the specialized value of the Schubert polynomial $\mathfrak{S}_{\varpi_{\lambda,\phi}}(1)$ admits a nice product formula, see Theorem 3.6. Moreover, we describe also some interesting connections of polynomials $\mathfrak{R}_{\varpi_{\lambda,\phi}}(q,\beta)$ with the Fuss-Catalan numbers [3] and Fuss-Narayana polynomials, k-

[3]We define the (generalized) Fuss-Catalan numbers to be $FC_n^{(p)}(b) := \frac{1+b}{1+b+(n-1)p}\binom{np+b}{n}$. Connection of the Fuss-Catalan numbers with the *p-ballot numbers* $Bal_p(m,n) := \frac{n-mp+1}{n+m+1}\binom{n+m+1}{m}$ and the *Rothe numbers* $R_n(a,b) := \frac{a}{a+bn}\binom{a+bn}{n}$ can be described as follows

$$FC_n^{(p)}(b) = R_n(b+1,p) = Bal_{p-1}(n,(n-1)p+b).$$

triangulations and k-dissections of a convex polygon, as well as a connection with two families of ASM. For example, if $\lambda = (b^n)$ and $\phi = (k^n)$, then the polynomial $\mathfrak{R}_{\varpi_{\lambda,\phi}}(q,\beta)$ defines a (q,β)-deformation of the number of (descending) plane partitions sitting in the box $b \times k \times n$. It seems an interesting **problem** to find an algebra-geometric interpretation of polynomials $\mathfrak{R}_w(q,\beta)$ in the general case.

In Section 3.3 we give a partial answer on the question **6.C8**(d) by R.Stanley [40].

Almost all results in Section 3 state that some two specific sets have the same number of elements. Our proofs of these results are pure algebraic. It is an interesting **problem** to find *bijective proofs* of results from Section 3 which generalize and extend the bijective proofs presented in [44], [36], [41] to the case of β-Grothendieck polynomials, the Schröder numbers and k-dissections of a convex $(n+k+1)$-gon. We are planning to treat and present these bijections in (a) separate publication(s). ■

At the end of Introduction I want to add two remarks.

(**a**) After a suitable modification of the algebra $3HT_n$, see [22], and the case $\beta \neq 0$ in [18], one can compute the set of relations among the (additive) Dunkl elements (defined in Section 1, (1.3)). In the case $\beta = 0$ and $q_{ij} = q_i \; \delta_{j-i,1}$, $1 \le i < j \le n$, where $\delta_{a,b}$ is the Kronecker delta, the commutative algebra generated by additive Dunkl elements (1.3) appears to be "almost" isomorphic to the equivariant quantum cohomology ring of the flag variety $\mathcal{F}l_n$, see [22] for details. Using the *multiplicative* version of Dunkl elements (1.3), one can extend the results from [22] to the case of equivariant quantum K-theory of the flag variety $\mathcal{F}l_n$, see [18].

(**b**) In fact one can construct an analogue of the algebra $3HT_n$ and a commutative subalgebra inside it, for any graph $\Gamma = (V,E)$ on n vertices, possibly with loops and multiple edges, [18]. We denote this algebra by $3T_n(\Gamma)$, and denote by $3T_n^{(0)}(\Gamma)$ its *nil-quotient*, which may be considered as a "classical limit of the algebra $3T_n(\Gamma)$".
The case of the complete graph $\Gamma = K_n$ reproduces the results of the present paper and those of [18], i.e. the case of the full flag variety $\mathcal{F}l_n$. The case of the *complete multipartite graph* $\Gamma = K_{n_1,\ldots,n_r}$ reproduces the analogues of results stated in the present paper for the full flag variety $\mathcal{F}l_n$, in the case of the partial flag variety $\mathcal{F}_{n_1,\ldots,n_r}$, see [18] for details.
We **expect** that in the case of the complete graph with all edges of multiplicity m, $\Gamma = K_n^{(m)}$, the commutative subalgebra generated by the Dunkl elements in the algebra $3T_n^{(0)}(\Gamma)$ is related to the algebra of coinvariants of the diagonal action of the symmetric group $\mathbb{S}_n$ on the ring of polynomials $\mathbb{Q}[X_n^{(1)},\ldots,X_n^{(m)}]$.

Example 0.2 *Take $\Gamma = K_{2,2}$. The algebra $3T^{(0)}(\Gamma)$ is generated by four elements $\{a = u_{13}, b = u_{14}, c = u_{23}, d = u_{24}\}$ subject to the following set of (defining) relations*

- $a^2 = b^2 = c^2 = d^2 = 0, \quad c\; b = b\; c, \quad a\; d = d\; a,$
- $a\; b\; a + b\; a\; b = 0 = a\; c\; a + c\; a\; c, \quad b\; d\; b + d\; b\; d = 0 = c\; d\; c + d\; c\; d,$
  $a\; b\; d - b\; d\; c - c\; a\; b + d\; c\; a = 0 = a\; c\; d - b\; a\; c - c\; d\; b + d\; b\; a,$
- $a\; b\; c\; a + a\; d\; b\; c + b\; a\; d\; b + b\; c\; a\; d + c\; a\; d\; c + d\; b\; c\; d = 0.$

It is not difficult to see that

$$Hilb(3T^{(0)}(K_{2,2}),t) = [3]_t^2\; [4]_t^2, \quad Hilb(3T^{(0)}(K_{2,2})^{ab},t) = (1,4,6,3).$$

Here for any algebra A we denote by A^{ab} its abelization.

The commutative subalgebra in $3T^{(0)}(K_{2,2})$, which corresponds to the intersection $3T^{(0)}(K_{2,2}) \bigcap \mathbb{Z}[\theta_1, \theta_2, \theta_3, \theta_4]$, is generated by the elements $c_1 := \theta_1 + \theta_2 = (a+b+c+d)$ and $c_2 := \theta_1\ \theta_2 = (ac+ca+bd+db+ad+bc)$. The elements c_1 and c_2 commute and satisfy the following relations

$$c_1^3 - 2\ c_1\ c_2 = 0, \quad c_2^2 - c_1^2\ c_2 = 0.$$

The ring of polynomials $\mathbb{Z}[c_1, c_2]$ is isomorphic to the cohomology ring $H^*(Gr(2,4), \mathbb{Z})$ of the Grassmannian variety $Gr(2,4)$.

This example is illustrative of the similar results valid for the graphs $K_{n_1,\ldots,n_r}$, i.e. for the partial flag varieties [18]. The meaning of the algebra $3T_n^{(0)}(\Gamma)$ and the corresponding commutative subalgebra inside it for a general graph Γ, is still unclear. ■

Conjecture 1

Let $\Gamma = (V, E)$ be a connected subgraph of the complete graph K_n on n vertices. Then

$$Hilb(3T_n^{(0)}(\Gamma)^{ab}, t) = t^{|V|-1}\ \ T(\Gamma; 1+t^{-1}, 0),$$

where for any graph Γ the symbol $T(\Gamma; x, y)$ denotes the **Tutte polynomial** corresponding to this graph.

Acknowledgments

I thank Professor Toshiaki Maeno for many years fruitful collaboration. I'm also grateful to Professors Y. Bazlov, A. Buch, I. Burban, B. Feigin, S. Fomin, A. Isaev, M. Ishikawa, B. Shapiro and Dr. Evgeny Smirnov for fruitful discussions on different stages of writing [18].

1 Dunkl elements

Let $\mathcal{A}_n$ be the free associative algebra over $\mathbb{Z}$ with the set of generators $\{u_{ij},\ 1 \le i, j \le n\}$. We set $x_i := u_{ii},\ i = 1, \ldots, n$.

Definition 1.1 *Define (additive) Dunkl elements $\theta_i, i = 1, \ldots, n$, in the algebra $\mathcal{A}_n$ to be*

$$\theta_i = x_i + \sum_{\substack{j=1 \\ j \ne i}}^{n} u_{ij}. \tag{1.3}$$

We are interested in to find "natural relations" among the generators $\{u_{ij}\}$ such that the Dunkl elements (1.3) are pair-wise commute. One natural condition which is the commonly accepted in the theory of integrable systems, is

- (Locality condition)

$$[x_i, x_j] = 0, \quad u_{ij}\ u_{kl} = u_{kl}\ u_{ij}, \quad if \quad \{i,j\} \cap \{k,l\} = \emptyset. \tag{1.4}$$

Lemma 1.1 *Assume that elements* $\{u_{ij}\}$ *satisfy the locality conditions* (1.4). *Then*

$$[\theta_i, \theta_j] = \Big[x_i + \sum_{k \neq i,j} u_{ik},\ u_{ij} + u_{ji}\Big] + \Big[u_{ij}, \sum_{k=1}^{n} x_k\Big] + \sum_{k \neq i,j} w_{ijk},$$

where

$$w_{ijk} = [u_{ij}, u_{ik} + u_{jk}] + [u_{ik}, u_{jk}] + [x_i, u_{jk}] + [u_{ik}, x_j] + [x_k, u_{ij}]. \qquad (1.5)$$

Therefore in order to ensure that the Dunkl elements form a pair-wise commuting family, it's natural to assume that the following conditions hold

- (Unitarity)

$$[u_{ij} + u_{ji}, u_{kl}] = 0 = [u_{ij} + u_{ji}, x_k] \quad for \quad all \quad i, j, k, l, \qquad (1.6)$$

i.e. the elements $u_{ij} + u_{ji}$ are central.

- (Crossing relations)

$$[\sum_{k=1}^{n} x_k\ ,\ u_{ij}] = 0 \ for \quad all \quad i, j. \qquad (1.7)$$

- **(Dynamical classical Yang–Baxter relations)**

$$[u_{ij}, u_{ik} + u_{jk}] + [u_{ik}, u_{jk}] + [x_i, u_{jk}] + [u_{ik}, x_j] + [x_k, u_{ij}] = 0, \qquad (1.8)$$

if i, j, k are pair-wise distinct.

We denote by $DCYB_n$ the quotient of the algebra $\mathcal{A}_n$ by the two-sided ideal generated by relations (1.4),(1.6),(1.7) and (1.8). Clearly, the Dunkl elements (1.3) generate a commutative subalgebra inside the algebra $CDYB_n$, and the sum $\sum_{i=1}^{n} \theta_i = \sum_{i=1}^{n} x_i$ belongs to the center of the algebra $DCYB_n$.

Example 1.1 *(A representation of the algebra* $DCYB_n$*, cf [9])*

Given a set $q_1, \ldots, q_{n-1}$ of mutually commuting parameters, define $q_{ij} = \prod_{a=i}^{j-1} q_a$, if $i < j$ and set $q_{ij} = q_{ji}$ in the case $i > j$. Clearly, that if $i < j < k$, then $q_{ij} q_{jk} = q_{ik}$.

Let $z_1, \ldots, z_n$ be a set of variables. Denote by $P_n := \mathbb{Z}[z_1, \ldots, z_n]$ the corresponding ring of polynomials. We consider the variable z_i, $i = 1, \ldots, n$, also as the operator acting on the ring of polynomials P_n by multiplication on z_i..

Let $s_{ij} \in \mathbb{S}_n$ be a transposition. We consider the transposition s_{ij} also as the operator which acts on the ring P_n by interchanging z_i and z_j, and fixes all other variables. We denote by

$$\partial_{ij} = \frac{1 - s_{ij}}{z_i - z_j}$$

the divided difference operator corresponding to the transposition s_{ij}. Finally we define operator (cf [9])

$$\partial_{(ij)} := \partial_i \cdots \partial_{j-1} \partial_j \partial_{j-1} \cdots \partial_i, \quad if \quad i < j.$$

The operators $\partial_{(ij)}, 1 \leq i < j \leq n$ satisfy (among others) the following set of relations (cf [9])

- $[z_j, \partial_{(ik)}] = 0, \ \ if \ \ j \notin [i,k], \qquad [\partial_{(ij)}, \sum_{a=i}^{j} z_a] = 0,$
- $[\partial_{(ij)}, \partial_{(kl)}] = \delta_{jk} \ [z_j, \partial_{(il)}] + \delta_{il} \ [\partial_{(kj)}, z_i], \ \text{if} \ \ i < j, \ \ k < l.$

Therefore, if we set $u_{ij} = q_{ij} \ \partial_{(ij)}, \ \ if \ \ i < j,$ and $u_{(ij)} = -u_{(ji)}, \ \ if \ \ i > j$, then for a triple $i < j < k$ we will have

$$[u_{ij}, u_{ik}+u_{jk}]+[u_{ik}, u_{jk}]+[z_i, u_{jk}]+[u_{ik}, z_j]+[z_k, u_{jk}] = q_{ij}q_{jk}[\partial_{(ij)}, \partial_{(jk)}]+q_{ik}[\partial_{(ik)}, z_j] = 0.$$

Thus the elements $z_i, i = 1, \ldots, n$ and $\{u_{ij}, 1 \le i < j \le n\}$ define a representation of the algebra $DCYB_n$, and therefore the Dunkl elements

$$\theta_i := z_i + \sum_{j \ne i} u_{ij} = z_i - \sum_{j<i} q_{ji}\partial_{(ji)} + \sum_{j>i} q_{ij}\partial_{(ij)}$$

form a pairwise commuting family of operators acting on the ring of polynomials $\mathbb{Z}[q_1, \ldots, q_{n-1}][z_1, \ldots, z_n]$, cf [9].

■

Comments 1.1

(Non-unitary dynamical classical Yang–Baxter algebra) Let $\widetilde{\mathcal{A}_n}$ be the quotient of the algebra $\mathcal{A}_n$ by the two-sided ideal generated by the relations (1.4), (1.7) and (1.8). Consider elements

$$\theta_i = x_i + \sum_{a \ne i} u_{ia}, \quad and \quad \bar{\theta}_j = -x_j + \sum_{b \ne j} u_{bj}. \quad 1 \le i < j \le n.$$

Then

$$[\theta_i, \bar{\theta}_j] = [\sum_{k=1}^{n} x_k \ , u_{ij}] + \sum_{k \ne i,j} w_{ikj}.$$

Therefore the elements θ_i and $\bar{\theta}_j$ commute in the algebra $\widetilde{A}_n$.

In the case when $x_i = 0$ for all $i = 1, \ldots, n$, the relations $w_{ijk} = 0$, assuming that i, j, k are all distinct, are well-known as the (non-unitary) classical Yang-Baxter relations. Note that for a given triple (i, j, k) we have in fact 6 relations. These six relations imply that $[\theta_i, \bar{\theta}_j] = 0$. However,

$$[\theta_i, \theta_j] = \Big[\sum_{k \ne i,j} u_{ik} \ , \ u_{ij} + u_{ji}\Big] \ne 0.$$

In order to ensure the commutativity relations among the Dunkl elements, i.e. $[\theta_i, \theta_j] = 0$ for all i, j, one needs to impose on the elements $\{u_{ij}, 1 \le i \ne j \le n\}$ the "twisted" classical Yang–Baxter relations, namely

$$[u_{ij} + u_{ik}, u_{jk}] + [u_{ik}, \mathbf{u_{ji}}] = 0, \quad if \quad i, j, k \ \ are \ \ all \ \ distinct. \tag{1.9}$$

Contrary to the case of non-unitary classical Yang–Baxter relations, it is easy to see that in the case of twisted classical Yang–Baxter relations, for a given triple (i, j, k) one has only 3 relations.

2 Algebra $3HT_n$

Consider the twisted classical Yang–Baxter relation

$$[u_{ij}+u_{ia},u_{ja}]+[u_{ia},u_{ji}]=0, \quad where \ \ i,j,k \ \ are \ \ distinct.$$

Having in mind applications of the Dunkl elements, we split the above relation on two relations

$$\underline{u_{ij}\ u_{jk}=u_{jk}\ u_{ik}-u_{ik}\ u_{ji}} \quad and \quad \underline{u_{jk}\ u_{ij}=u_{ik}\ u_{jk}-u_{ji}\ u_{ik}},$$

and impose the unitarity constraints

$$u_{ij}+u_{ji}=\beta,$$

where β is a central element. Summarizing, we come to the following definition.

Definition 2.1 *Define algebra $3T_n(\beta)$ to be the quotient of the free associative algebra $\mathbb{Z}[\beta]\ \langle\ u_{ij},\ \ 1\le i<j\le n\ \ \rangle$ by the set of relations*

- *(Locality)* $u_{ij}\ u_{kl}=u_{kl}\ u_{ij}$, *if* $\{i,j\}\cap\{k,l\}=\emptyset$,
- $u_{ij}\ u_{jk}=u_{ik}\ u_{ij}+u_{jk}\ u_{ik}-\beta\ u_{ik}, \qquad u_{jk}\ u_{ij}=u_{ij}\ u_{ik}+u_{ik}\ u_{jk}-\beta\ u_{ik}$,

if $1\le i<j<k\le n$.

For each pair $i<j$, we define element $q_{ij}:=u_{ij}^2-\beta\ u_{ij}\in 3T_n(\beta)$.

Lemma 2.1

(1) *The elements* $\{q_{ij},\ 1\le i<j\le n\}$ *satisfy the Kohno- Drinfeld relations (known also as the horizontal four term relations)*

$$q_{ij}\ q_{kl}=q_{kl}\ q_{ij}, \quad if \quad \{i,j\}\cap\{k,l\}=\emptyset,$$
$$[q_{ij},q_{ik}+q_{jk}]=0, \quad [q_{ij}+q_{ik},q_{jk}]=0, \quad if \quad i<j<k.$$

(2) *For a triple $(i<j<k)$ define $u_{ijk}:=u_{ij}-u_{ik}+u_{jk}$. Then*

$$u_{ijk}^2=\beta\ u_{ijk}+q_{ij}+q_{ik}+q_{jk}.$$

(3) *(Deviation from the Yang-Baxter and Coxeter relations)*

$u_{ij}\ u_{ik}\ u_{jk}-u_{jk}\ u_{ik}\ u_{ij}=[u_{ik},q_{ij}]=[q_{jk},u_{ik}]$,

$u_{ij}\ u_{jk}\ u_{ij}-u_{jk}\ u_{ij}\ u_{jk}=q_{ij}\ u_{ik}-u_{ik}\ q_{jk}$.

Comments 2.1 It is easy to see that the horizontal 4-term relations listed in Lemma 2.1, (1), are equivalent to the locality condition among the generators $\{q_{ij}\}$, together with the commutativity conditions among the Jucys–Murphy elements

$$d_i:=\sum_{j=i+1}^{n} q_{ij}, \quad i=2,\ldots,n,$$

namely, $[d_i,d_j]=0$. In [18] we describe some properties of a commutative subalgebra generated by the Jucys-Murphy elements in the Kohno– Drinfeld algebra. It is well-known that the Jucys–Murphy elements generate a maximal commutative subalgebra in the group ring of the symmetric group $\mathbb{S}_n$. It is an open problem to describe defining relations among the Jucys–Murphy elements in the group ring $\mathbb{Z}[\mathbb{S}_n]$. ■

Finally we introduce the "Hecke quotient" of the algebra $3T_n(\beta)$, denoted by $3HT_n$.

Definition 2.2 *Define algebra $3HT_n$ to be the quotient of the algebra $3T_n(\beta)$ by the set of relations*

$$q_{ij}\ q_{kl} = q_{kl}\ q_{ij}, \quad for \quad all \quad i,j,k,l.$$

In other words we assume that the all elements $\{q_{ij},\ \ 1 \le i < j \le n\}$ are <u>central</u> in the algebra $3T_n(\beta)$. From Lemma 2.1 follows immediately that in the algebra $3HT_n$ the elements $\{u_{ij}\}$ satisfy the multiplicative (or quantum) Yang–Baxter relations

$$u_{ij}\ u_{ik}\ u_{jk} = u_{jk}\ u_{ik}\ u_{ij}, \quad if \quad i < j < k. \tag{2.10}$$

Therefore one can define multiplicative analogues $\Theta_i,\ 1 \le i \le n,$ of the Dunkl elements θ_i. Namely, to start with, we define elements

$$h_{ij} := h_{ij}(t) = 1 + t\ u_{ij}, \quad i \ne j.$$

We consider $h_{ij}(t)$ as an element of the algebra $\widetilde{3HT_n} := 3HT_n \otimes \mathbb{Z}[[\beta, q_{ij}^{\pm 1}, t, x, y, \ldots]]$, where we assume that all parameters $\{\beta, q_{ij}, t, x, y, \ldots\}$ are <u>central</u> in the algebra $\widetilde{3HT_n}$.

Lemma 2.2

(1a) $h_{ij}(x)\ h_{ij}(y) = h_{ij}(x + y + \beta\ xy) + q_{ij}\ xy,$

(1b) $h_{ij}(x)\ h_{ji}(y) = h_{ij}(x - y) + \beta\ y - q_{ij}\ x\ y,\ \ if\, i < j.$

It follows from (1b) that $h_{ij}(t)\ h_{ji}(t) = 1 + \beta\ t - t^2\ q_{ij},$ if $i < j$, and therefore the elements $\{h_{ij}\}$ are invertible in the algebra $\widetilde{3HT_n}$.

(2) $h_{ij}(x)\ h_{jk}(y) = h_{jk}(y)\ h_{ik}(x) + h_{ik}(y)\ h_{ij}(x) - h_{ik}(x + y + \beta\ xy).$

(3) *(Multiplicative Yang-Baxter relations)*

$$h_{ij}\ h_{ik}\ h_{jk} = h_{jk}\ h_{ik}\ h_{ij}, \quad if \quad i < j < k.$$

(4) *Define multiplicative Dunkl elements (in the algebra $\widetilde{3HT_n}$) as follows*

$$\Theta_j := \Theta_j(t) = \Big(\prod_{a=j-1}^{1} h_{aj}^{-1} \Big) \ \Big(\prod_{a=n}^{j+1} h_{ja} \Big), \quad 1 \le j \le n. \tag{2.11}$$

Then the multiplicative Dunkl elements pair-wise commute.

Clearly

$$\prod_{j=1}^{n} \Theta_j = 1, \quad \Theta_j = 1 + t\ \theta_j + t^2(\ldots), \quad and \quad \Theta_I \prod_{\substack{i \notin I, j \in I \\ i<j}} (1 + t\beta - t^2\ q_{ij}) \in 3HT_n.$$

Here for a subset $I \subset [1,n]$ we use notation $\Theta_I = \prod_{a \in I} \Theta_a$,

Our main result of this Section is a description of relations among the multiplicative Dunkl elements.

Theorem 2.1

In the algebra $3HT_n$ the following relations hold true

$$\sum_{\substack{I\subset[1,n]\\|I|=k}} \Theta_I \prod_{\substack{i\notin I, j\in J\\ i<j}} (1+t\,\beta-t^2\,q_{ij}) = \begin{bmatrix} n\\ k\end{bmatrix}_{1+t\beta}.$$

Here $\begin{bmatrix} n\\ k\end{bmatrix}_q$ denotes the q-Gaussian polynomial.

Corollary 2.1

Assume that $q_{ij}\neq 0$ for all $1\le i<j\le n$. Then the all elements $\{u_{ij}\}$ are invertible and $u_{ij}^{-1}=q_{ij}^{-1}(u_{ij}-\beta)$ Now define elements $\Phi_i\in\widetilde{3HT_n}$ as follows

$$\Phi_i=\left\{\prod_{a=i-1}^{1} u_{ai}^{-1}\right\}\left\{\prod_{a=n}^{i+1} u_{ia}\right\},\quad i=1,\dots,n.$$

Then we have

(1) *(Relationship among Θ_j and Φ_j)*

$$t^{n-2j+1}\,\Theta_j(t^{-1})\,|_{t=0}=(-1)^j\,\Phi_j.$$

(2) *The elements $\{\Phi_i,\ 1\le i\le n,\}$ generate a commutative subalgebra in the algebra $\widetilde{3HT_n}$.*

(3) *For each $k=1,\dots,n$, the following relation in the algebra $3HT_n$ among the elements $\{\Phi_i\}$ holds*

$$\sum_{\substack{I\subset[1,n]\\|I|=k}} \prod_{\substack{i\notin I,\ j\in I\\ i<j}} (-q_{ij})\,\Phi_I=\beta^{k(n-k)},$$

where $\Phi_I:=\prod_{a\in I}\Phi_a$.

In fact the element Φ_i admits the following "reduced expression" which is useful for proofs and applications

$$\Phi_i=\left\{\overrightarrow{\prod_{j\in I}}\left\{\overrightarrow{\prod_{\substack{i\in I_+^c\\ i<j}}} u_{ij}^{-1}\right\}\right\}\left\{\overrightarrow{\prod_{j\in I_+^c}}\left\{\overrightarrow{\prod_{\substack{i\in I\\ i<j}}} u_{ij}\right\}\right\}. \tag{2.12}$$

Let us explain notations. For any (totally) ordered set $I=(i_1<i_2<\dots<i_k)$ we denote by I_+ the set I with the opposite order, i.e. $I_+=(i_k>i_{k-1}>\dots>i_1)$; if $I\subset[1,n]$, then $I^c=[1,n]\setminus I$. For any (totally) ordered set I we denote by $\overrightarrow{\prod_{i\in I}}$ the ordered product according to the order of the set I.

Note that the total number of terms in the RHS of (2.12) is equal to $k(n-k)>$

Finally, from the "reduced expression" (2.12) for the element Φ_i one can see that

$$\prod_{\substack{i\notin I, j\in I\\ i<j}} (-q_{ij})\, \Phi_I = \Big\{\overrightarrow{\prod_{j\in I}}\Big\{\overrightarrow{\prod_{\substack{i\in I_+^c\\ i<j}}} (\beta - u_{ij})\Big\}\Big\} \; \Big\{\overrightarrow{\prod_{j\in I_+^c}}\Big\{\overrightarrow{\prod_{\substack{i\in I\\ i<j}}} u_{ij}\Big\}\Big\} := \widetilde{\Phi_I} \in 3HT_n.$$

Therefore the identity

$$\sum_{\substack{I\subset[1,n]\\ |I|=k}} \widetilde{\Phi_I} = \beta^{k(n-k)}$$

is true in the algebra $3HT_n$ for arbitrary set of parameters $\{q_{ij}\}$.

Comments 2.2

(I) In fact from our proof of Theorem 2.1 we can deduce more general statement, namely, consider integers m and k such that $1 \le k \le m \le n$. Then

$$\sum_{\substack{I\subset[1,m]\\ |I|=k}} \Theta_I \prod_{\substack{i\in[1,m]\setminus I, j\in J\\ i<j}} (1+t\,\beta - t^2\, q_{ij}) = \begin{bmatrix} m\\ k\end{bmatrix}_{1+t\beta} + \sum_{\substack{A\subset[1,n], B\subset[1,n]\\ |A|=|B|=r}} u_{A,B}, \tag{2.13}$$

where , by definition, for two sets $A=(i_1,\ldots,i_r)$ and $B=(j_1,\ldots,j_r)$ the symbol $u_{A,B}$ is equal to the (ordered) product $\prod_{a=1}^r u_{i_a,j_a}$. Moreover, the elements of the sets A and B have to satisfy the following conditions:

- for each $a=1,\ldots,r$ one has $1\le i_a \le m < j_a \le n$, and $k \le r \le k(n-k)$.

Even more, if $r=k$, then sets A and B have to satisfy the following additional conditions:

- $B=(j_1\le j_2 \le \ldots \le j_k)$ and the elements of the set A are pair-wise distinct.

In the case $\beta=0$ and $r=k$, i.e. in the case of additive (truncated) Dunkl elements, the above statement, also known as the quantum Pieri formula, has been stated as Conjecture in [10], and has been proved later in [32].

Corollary 2.2 *([21])*

In the case when $\beta=0$ and $q_{ij}=q_i\,\delta_{j-i,1}$, the algebra over $\mathbb{Z}[q_1,\ldots,q_{n-1}]$ generated by the multiplicative Dunkl elements $\{\Theta_i$ and Θ_i^{-1}, $1\le i\le n\}$ is canonically isomorphic to the quantum K-theory of the complete flag variety $\mathcal{F}l_n$ of type A_{n-1}.

It is still an open problem to describe explicitly the set of monomials $\{u_{A,B}\}$ which appear in the RHS of (2.13) when $r>k$.

(II) (**Truncated Gaudin operators**) Let $\{p_{ij}\ 1\le i\ne j\le n\}$ be a set of mutually commuting parameters. We assume that parameters $\{p_{ij}\}$ are invertible and satisfy the Arnold relations

$$\frac{1}{p_{ik}} = \frac{1}{p_{ij}} + \frac{1}{p_{jk}}, \quad i<j,k.$$

For example one can take $p_{ij}=(z_i-z_j)^{-1}$, where $z=(z_1,\ldots,z_n)\in(\mathbb{C}\backslash 0)^n$.

Definition 2.3 *Truncated (rational) Gaudin operator corresponding to the set of parameters* $\{p_{ij}\}$, *is defined to be*

$$G_i = \sum_{j \neq i} p_{ij}^{-1} \, s_{ij}, \quad 1 \leq i \leq n,$$

where s_{ij} *denotes the exchange operator which switches variables* x_i *and* x_j, *and fixes parameters* $\{p_{ij}\}$.

We consider the Gaudin operator G_i *as an element of the group ring* $\mathbb{Z}[\{p_{ij}^{\pm 1}\}][\mathbb{S}_n]$, *call this element* $G_i \in \mathbb{Z}[\{p_{ij}^{\pm 1}\}][\mathbb{S}_n]$, $i = 1, \ldots, n$. *by Gaudin element and denoted it by* $\theta_i^{(n)}$.

It is easy to see that the elements $u_{ij} := p_{ij}^{-1} \, s_{ij}$, $1 \leq i \neq j \leq n$, define a representation of the algebra $3HT_n$ with parameters $\beta = 0$ and $q_{ij} = u_{ij}^2 = p_{ij}^2$.

Therefore one can consider the (truncated) Gaudin elements as a special case of the (truncated) Dunkl elements. Now one can rewrite the relations among the Dunkl elements, as well as the quantum Pieri formula [10] , [32], in terms of the Gaudin elements.

The key observation which allows to rewrite the quantum Pieri formula as a certain relation among the Gaudin elements is the following one: parameters $\{p_{ij}^{-1}\}$ satisfy the *Plücker* relations

$$\frac{1}{p_{ik} \, p_{jl}} = \frac{1}{p_{ij} \, p_{kl}} + \frac{1}{p_{il} \, p_{jk}}, \quad if \ \ i < j < k < l.$$

To describe relations among the Gaudin elements $\theta_i^{(n)}$, $i = 1, \ldots, n$, we need a bit of notation. Let $\{p_{ij}\}$ be a set of invertible parameters as before.

Define polynomials in the variables $\mathbf{h} = (h_1, \ldots, h_n)$

$$G_{m,k,r}(\mathbf{h}, \{p_{ij}\}) = \sum_{\substack{I \subset [1,n-1] \\ |I| = r}} \frac{1}{\prod_{i \in I} p_{in}} \sum_{\substack{J \subset [1,n] \\ |I|+m=|J|+k}} \binom{n - |I \bigcup J|}{n - m - |I|} \tilde{h}_J, \tag{2.14}$$

where

$$\tilde{h}_J = \sum_{\substack{K \subset J, \ L \subset J, \\ |K|=|L|, \ K \bigcap L = \emptyset}} \prod_{j \in J \setminus (K \bigcup L)} h_j \prod_{k_a \in K, \ l_a \in L} p_{k_a, l_a}^2,$$

and summation runs over subsets $K = \{k_1, k_2 < \ldots < k_r\} \subset J$, and $L = \{l_a \in J, \ a = 1, \ldots, r\}$, such that $k_a < l_a$, $1 \leq a \leq r$, and $l_1, \ldots, l_r$ are pairwise distinct.

Theorem 2.2 *(Relations among the* **Gaudin** *elements, [18], cf [30])*
Under the assumption that elements $\{p_{ij}, \ 1 \leq i < j \leq n\}$ *are invertible, mutually commute and satisfy the Arnold relations, one has*

- $$G_{m,k,r}(\theta_1^{(n)}, \ldots, \theta_n^{(n)}, \{p_{ij}\}) = 0, \quad if \ \ m > k, \tag{2.15}$$

- $G_{0,0,k}(\theta_1^{(n)}, \ldots, \theta_n^{(n)}, \{p_{ij}\}) = e_k(d_2, \ldots, d_n)$, *where* $d_2, \ldots, d_n$ *denote the Jucys–Murphy elements in the group ring* $\mathbb{Z}[\mathbb{S}_n]$ *of the symmetric group* $\mathbb{S}_n$.

It is well-known that the elementary symmetric polynomials $e_k(d_2,\ldots,d_n) := C_k,\ k = 1,\ldots,n,$ generate the center of the group ring $\mathbb{Z}[p_{ij}^{\pm 1}][\mathbb{S}_n]$, whereas the Gaudin elements $\{\theta_i^{(n)},\ i = 1,\ldots,n\}$, generate a maximal commutative subalgebra $\mathcal{B}(p_{ij})$, the so-called Bethe subalgebra, in $\mathbb{Z}[p_{ij}^{\pm 1}][\mathbb{S}_n]$. It is well-known, see e.g. [30], that $\mathcal{B}(p_{ij}) = \bigoplus_{\lambda\vdash n}\mathcal{B}_\lambda(p_{ij})$, where $\mathcal{B}_\lambda(p_{ij})$ is the $\lambda-$isotypic component of $\mathcal{B}(p_{ij})$. On each $\lambda-$isotypic component the value of the central element C_k is the explicitly known constant $c_k(\lambda)$. It follows from [30] that the relations (2.15) together with relations

$$G_{0,0,k}(\theta_1^{(n)},\ldots,\theta_n^{(n)},\{p_{ij}\}) = c_k(\lambda),$$

are the defining relations for the algebra $\mathcal{B}_\lambda(p_{ij})$.

Let us remark that in the definition of the Gaudin elements we can use *any* set of mutually commuting, invertible elements $\{p_{ij}\}$ which satisfies the Arnold conditions. For example, we can take

$$p_{ij} := \frac{q^{j-2}(1-q)}{1-q^{j-i}},\quad 1 \le i < j \le n.$$

It is not difficult to see that in this case

$$\lim_{q\to 0}\frac{\theta_j^{(n)}}{p_{1j}} = -d_j = -\sum_{a=1}^{j-1} s_{aj},$$

where d_j denotes the Jucys–Murphy element in the group ring $\mathbb{Z}[\mathbb{S}_n]$ of the symmetric group $\mathbb{S}_n$. Basically from relations (2.15) one can deduce the relations among the Jucys–Murphy elements $d_2,\ldots,d_n$ after plugging in (2.15) the values $p_{ij} := \frac{q^{j-2}(1-q)}{1-q^{j-i}}$ and passing to the limit $q \to 0$. However the real computations are rather involved.

Finally we note that the multiplicative Dunkl / Gaudin elements $\{\Theta_i,\ 1,\ldots,n\}$ also generate a maximal commutative subalgebra in the group ring $\mathbb{Z}[p_{ij}^{\pm 1}][\mathbb{S}_n]$. Some relations among the elements $\{\Theta_l\}$ follow from Theorem 2.1, but we don't know an analogue of relations (2.13) for the multiplicative Gaudin elements, but see [30].

(III) Shifted Dunkl elements $\mathfrak{d}_i$ and $\mathfrak{D}_i$

As it was stated in Corollary 2.2, the *truncated* additive and multiplicative Dunkl elements in the algebra $3HT_n(0)$ generate over the ring of polynomials $\mathbb{Z}[q_1,\ldots,q_{n-1}]$ correspondingly the *quantum cohomology* and *quantum $K-theory$* rings of the full flag variety $\mathcal{F}l_n$. In order to describe the corresponding *equivariant* theories, we will introduce the *shifted* additive and multiplicative Dunkl elements. To start with we need at first to introduce an extension of the algebra $3HT_n(\beta)$.

Let $\{z_1,\ldots,z_n\}$ be a set of mutually commuting elements and $\{\beta,h,t,q_{ij}=q_{ji},\ 1 \le i,j \le n\}$ be a set of parameters.

Definition 2.4 *Define algebra $\overline{3TH_n(\beta)}$ to be the semi-direct product of the algebra $3TH_n(\beta)$ and the ring of polynomials $\mathbb{Z}[h,t][z_1,\ldots,z_n]$ with respect to the crossing relations*

(1) $z_i\ u_{kl} = u_{kl}\ z_i$ *if* $i \notin \{k,l\}$,

(2) $z_i\ u_{ij} = u_{ij}\ z_j + \beta\ z_i + h,\ \ z_j\ u_{ij} = u_{ij}\ z_i - \beta\ z_i - h,$ *if* $1 \le i < j < k \le n.$

Now we set as before $h_{ij} := h_{ij}(t) = 1 + t\ u_{ij}$.

Definition 2.5

- *Define shifted additive Dunkl elements to be*

$$\mathfrak{d}_i = z_i - \sum_{i<j} u_{ij} + \sum_{i<j} u_{ji}.$$

- *Define shifted multiplicative Dunkl elements to be*

$$\mathfrak{D}_i = \Big(\prod_{a=i-1}^{1} h_{ai}^{-1}\Big)\ (1+z_i)\ \Big(\prod_{a=n}^{i+1} h_{ia}\Big).$$

Lemma 2.3

$$[\mathfrak{d}_i, \mathfrak{d}_j] = 0, \quad [\mathfrak{D}_i, \mathfrak{D}_j] = 0 \quad for \quad all \quad i, j.$$

Now we stated an analogue of Theorem 2.1. for shifted multiplicative Dunkl elements. As a preliminary, for any subset $I \subset [1,n]$ let us set $\mathfrak{D}_I = \prod_{a \in I} \mathfrak{D}_a$. It is clear that

$$\mathfrak{D}_I \prod_{\substack{i \notin I,\ j \in I \\ i<j}} (1 + t\ \beta - t^2\ q_{ij}) \in \overline{3HT_n(\beta)}.$$

Theorem 2.3
In the algebra $\overline{3HT_n(\beta)}$ *the following relations hold true*

$$\sum_{\substack{I \subset [1,n] \\ |I|=k}} \mathfrak{D}_I \prod_{\substack{i \notin I, j \in J \\ i<j}} (1 + t\ \beta - t^2\ q_{ij}) = \begin{bmatrix} n \\ k \end{bmatrix}_{1+t\beta} +$$

$$\sum_{\substack{I \subset [1,n] \\ I=\{i_1,\ldots,i_k\}}} \prod_{a=1}^{k} \Big[z_a (1+\beta t)^{n-k} + h\ \frac{(1+\beta t)^{n-k} - (1+\beta t)^{i_a - a}}{\beta} \Big].$$

In particular, if $\beta = 0$, we will have

Corollary 2.3 *In the algebra* $\overline{3HT_n(0)}$ *the following relations hold*

$$\sum_{\substack{I \subset [1,n] \\ |I|=k}} \mathfrak{D}_I \prod_{\substack{i \notin I, j \in J \\ i<j}} (1 - t^2\ q_{ij}) = \binom{n}{k} + \sum_{\substack{I \subset [1,n] \\ I=\{i_1,\ldots,i_k\}}} \prod_{a=1}^{k} \prod_{a=1}^{n} \Big(z_a + t\ h\ (n - k - i_a + a) \Big).$$

One of the main steps in our proof of Theorem 2.3. is the following explicit formula for the elements $\mathfrak{D}_I$.

Lemma 2.4 *One has*

$$\widetilde{\mathfrak{D}_I} := \mathfrak{D}_I\ (1 + t\ \beta - t^2\ q_{ij}) = \prod_{b \in I}^{\nearrow} \Big(\prod_{\substack{a \notin I \\ a<b}}^{\searrow} h_{ba} \Big) \prod_{a \in I}^{\nearrow} \Big((1+z_a) \prod_{\substack{b \notin I \\ a<b}}^{\searrow} h_{ab} \Big).$$

Note that if $a < b$, then $h_{ba} = 1 + \beta t - u_{ab}$. Here we have used the symbol

$$\prod_{b \in I}^{\nearrow} \Big(\prod_{\substack{a \notin I \\ a<b}}^{\searrow} h_{ba}\Big)$$

to denote the following product. At first, for a given element $b \in I$ let us define the set $I(b) := \{a \in [1,n] \backslash I,\ a < b\} := (a_1^{(b)} < \ldots < a_p^{(b)})$ for some p (depending on b). If $I = (b_1 < b_2 \ldots < b_k)$, then we set

$$\prod_{b \in I}^{\nearrow} \Big(\prod_{\substack{a \notin I \\ a<b}}^{\searrow} h_{ba}\Big) = \prod_{j=1}^{k} \Big(u_{b_j,a_s}\ u_{b_j,a_{s-1}} \cdots u_{b_j,a_1}\Big).$$

For example, let us take $n = 6$ and $I = (1,3,5)$. Then
$\widetilde{\mathfrak{D}}_I = h_{32}h_{54}h_{52}(1+z_1)h_{16}h_{14}h_{12}(1+z_3)h_{36}h_{34}(1+z_5)h_{56}$.

■

3 Combinatorics of associative quasi-classical Yang–Baxter algebras

Let β be a parameter.

Definition 3.1 *([18]) The associative quasi-classical Yang–Baxter algebra of weight β, denoted by $\widehat{ACYB}_n(\beta)$, is an associative algebra, over the ring of polynomials $\mathbb{Z}[\beta]$, generated by the set of elements $\{x_{ij},\ 1 \le i < j \le n\}$, subject to the set of relations*

(a) $x_{ij}\ x_{kl} = x_{kl}\ x_{ij}$, *if* $\{i,j\} \cap \{k,l\} = \emptyset$,

(b) $x_{ij}\ x_{jk} = x_{ik}\ x_{ij} + x_{jk}\ x_{ik} + \beta\ x_{ik}$, *if* $1 \le 1 < i < j \le n$,

Comments 3.1 The algebra $3T_n(\beta)$, see Definition 2.1, is the quotient of the algebra $\widehat{ACYB}_n(-\beta)$, by the "dual relations"

$$x_{jk}x_{ij} - x_{ij}\ x_{ik} - x_{ik}\ x_{jk} + \beta\ x_{ik} = 0, \quad i < j < k.$$

The (truncated) Dunkl elements $\theta_i = \sum_{j \ne i} x_{ij}$, $i = 1, \ldots, n$, do not commute in the algebra $\widehat{ACYB}_n(\beta)$. However a certain version of noncommutative elementary polynomial of degree $k \ge 1$, still is equal to zero after the substitution of Dunkl elements instead of variables, [18]. We state here the corresponding result only "in classical case", i.e. if $\beta = 0$ and $q_{ij} = 0$ *for all* i, j.

Lemma 3.1 *([18]) Define noncommutative elementary polynomial $L_k(x_1, \ldots, x_n)$ as follows*

$$L_k(x_1, \ldots, x_n) = \sum_{I=(i_1<i_2<\ldots<i_k) \subset [1,n]} x_{i_1}\ x_{i_2} \cdots x_{i_k}.$$

Then $L_k(\theta_1, \theta_2, \ldots, \theta_n) = 0$.

Moreover, if $1 \le k \le m \le n$, then one can show that the value of the noncommutative polynomial $L_k(\theta_1, \ldots, \theta_m)$ in the algebra $\widehat{ACYB}_n(\beta)$ is given by the Pieri formula, see [10], [32].

3.1 Combinatorics of Coxeter element

Consider the "Coxeter element" $w \in \widehat{ACYB}_n(\beta)$ which is equal to the ordered product of "simple generators": $w := w_n = \prod_{a=1}^{n-1} x_{a,a+1}$. Let us bring the element w to the reduced form in the algebra $\widehat{ACYB}_n(\beta)$, that is, let us consecutively apply the defining relations (a) and (b) to the element w in any order until unable to do so. Denote the resulting (noncommutative) polynomial by $P(x_{ij};\beta)$. In principal, the polynomial itself can depend on the order in which the relations (a) and (b) are applied.

Proposition 3.1 *(Cf [40],* **8.C5**, *(c); [28])*

(1) *Apart from applying the relation (a) (commutativity), the polynomial $P(x_{ij};\beta)$ does not depend on the order in which relations (a) and (b) have been applied, and can be written in a unique way as a linear combination:*

$$P_n(x_{ij};\beta) = \sum_{s=1}^{n-1} \beta^{n-s-1} \sum_{\{i_a\}} \prod_{a=1}^{s} x_{i_a,j_a},$$

where the second summation runs over all sequences of integers $\{i_a\}_{a=1}^{s}$ such that $n-1 \ge i_1 \ge i_2 \ge \ldots \ge i_s = 1$, and $i_a \le n-a$ for $a = 1,\ldots,s-1$; moreover, the corresponding sequence $\{j_a\}_{a=1}^{n-1}$ can be defined uniquely by that $\{i_a\}_{a=1}^{n-1}$.

• It is clear that the polynomial $P(x_{ij};\beta)$ also can be written in a unique way as a linear combination of monomials $\prod_{a=1}^{s} x_{i_a,j_a}$ such that $j_1 \ge j_2 \ldots \ge j_s$.

(2) *Denote by $T_n(k,r)$ the number of degree k monomials in the polynomial $P(x_{ij};\beta)$ which contain exactly r factors of the form $x_{*,n}$. (Note that $1 \le r \le k \le n-1$). Then*

$$T_n(k,r) = \frac{r}{k}\binom{n+k-r-2}{n-2}\binom{n-2}{k-1}.$$

In particular, $T_n(k,k) = \binom{n-2}{k-1}$, and $T_n(k,1) = T(n-2,k-1)$, where $T(n,k) := \frac{1}{k+1}\binom{n+k}{k}\binom{n}{k}$ is equal to the number of Schröder paths (i.e. consisting of steps $U=(1,1)$, $D=(1,-1)$, $H=(2,0)$ and never going below the x-axis) from $(0,0)$ to $(2n,0)$, having k U's, see [37], A088617.

Moreover, $T_n(n-1,r) = Tab(n-2,r-1)$, where $Tab(n,k) := \frac{k+1}{n+1}\binom{2n-k}{n}$ is equal to the number of standard Young tableaux of the shape $(n,n-k)$, see [37], A009766.

(3) *After the specialization $x_{ij} \longrightarrow 1$ the polynomial $P(x_{ij})$ is transformed to the polynomial*

$$P_n(\beta) := \sum_{k=0}^{n-1} N(n,k)\,(1+\beta)^k,$$

where $N(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$, $k=0,\ldots,n-1$, stand for the Narayana numbers. Furthermore, $P_n(\beta) = \sum_{d=0}^{n-1} s_n(d)\,\beta^d$, where $s_n(d) = \frac{1}{n+1}\binom{2n-d}{n}\binom{n-1}{d}$ is the number of ways to draw $n-1-d$ diagonals in a convex $(n+2)$-gon, such that no two diagonals intersect their interior.

Therefore, the number of (nonzero) terms in the polynomial $P(x_{ij};\beta)$ *is equal to the* n*-th little Schröder number* $s_n := \sum_{d=0}^{n-1} s_n(d)$, *also known as the* n*-th super-Catalan number, see e.g. [37], A001003.*

(4) *After the specialization* $x_{1j} \longrightarrow t$, $1 \le j \le n$, *and that* $x_{ij} \longrightarrow 1$, *if* $2 \le i < j \le n$, *the polynomial* $P(x_{ij};\beta)$ *is transformed to the polynomial*

$$P_n(\beta,t) = t\sum_{k=1}^{n} (1+\beta)^{n-k} \sum_{\pi} t^{p(\pi)},$$

where the second summation runs over the set of Dick paths π *of length* $2n$ *with exactly* k *picks (UD-steps), and* $p(\pi)$ *denotes the number of valleys (DU-steps) that touch upon the line* $x=0$.

(5) *The polynomial* $P(x_{ij};\beta)$ *is invariant under the action of anti-involution* $\phi\circ\tau$, *see Section 5.1.1 [18] for definitions of* ϕ *and* τ.

(6) *Follow [40],* **6.C8**, *(c), consider the specialization*

$$x_{ij} \longrightarrow t_i, \quad 1 \le i < j \le n,$$

and define $P_n(t_1,\ldots,t_{n-1};\beta) = P_n(x_{ij}=t_i;\beta)$.

One can show, ibid , that

$$P_n(t_1,\ldots,t_{n-1};\beta) = \sum \beta^{n-k} \ t_{i_1}\cdots t_{i_k}, \tag{3.16}$$

where the sum runs over all pairs $\{(a_1,\ldots,a_k),(i_1,\ldots,i_k)\in \mathbb{Z}_{\ge 1}\times\mathbb{Z}_{\ge 1}\}$ *such that* $1 \le a_1 < a_2 < \ldots < a_k$, $1 \le i_1 \le i_2 \ldots \le i_k \le n$ *and* $i_j \le a_j$ *for all* j.

Now we are ready to state our main result about polynomials $P_n(t_1,\ldots,t_n;\beta)$.

Let $\pi \in \mathbb{S}_n$ *be the permutation* $\pi = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & n & n-1 & \ldots & 2 \end{pmatrix}$. *Then*

$$P_n(t_1,\ldots,t_{n-1};\beta) = \left(\prod_{i=1}^{n-1} t_i^{n-i}\right) \mathfrak{G}_{\pi}^{(\beta)}(t_1^{-1},\ldots,t_{n-1}^{-1}),$$

where $\mathfrak{G}_w^{(\beta)}(x_1,\ldots,x_{n-1})$ *denotes the* β*-Grothendieck polynomial corresponding to a permutation* $w\in\mathbb{S}_n$, *[11].*

In particular,

$$\mathfrak{G}_{\pi}^{(\beta)}(x_1=1,\ldots,x_{n-1}=1) = \sum_{k=0}^{n-1} N(n,k)\ (1+\beta)^k,$$

where $N(n,k)$ *denotes the Narayana numbers, see item* **(3)** *of Proposition 3.1.*

• Note that if $\beta=0$, then one has $\mathfrak{G}_w^{(\beta=0)}(x_1,\ldots,x_{n-1})=\mathfrak{S}_w(x_1,\ldots,x_{n-1})$, that is the β-Grothendieck polynomial at $\beta=0$, is equal to the Schubert polynomial corresponding to the same permutation w. Therefore, if $\pi=\begin{pmatrix}1 & 2 & 3 & \ldots & n\\ 1 & n & n-1 & \ldots & 2\end{pmatrix}$, then

$$\mathfrak{S}_\pi(x_1=1,\ldots,t_{n-1}=1)=C_{n-1}, \tag{3.17}$$

where C_m denotes the $m-$th Catalan number. Using the formula (3.16) it is not difficult to check the following formula for the principal specialization of the Schubert polynomial $\mathfrak{S}_\pi$

$$\mathfrak{S}_\pi(1,q,\ldots,q^{n-1})=q^{\binom{n-1}{3}}\ C_{n-1}(q), \tag{3.18}$$

where $C_m(q)$ denotes the Carlitz - Riordan q-analogue of the Catalan numbers, see e.g. [38]. The formula (3.17) has been proved in [13] using the observation that π is a *vexillary* permutation, see [26] for the a definition of the latter. A combinatorial/bijective proof of the formula (3.18) is is due to A.Woo [44].

Comments 3.2

The Grothendieck polynomials defined by A. Lascoux and M.-P. *Schützenberger*, see e.g. [25], correspond to the case $\beta=-1$. In this case $P_n(-1)=1$, if $n\geq 0$, and therefore the specialization $\mathfrak{G}_w^{(-1)}(x_1=1,\ldots,x_{n-1}=1)=1$ for all $w\in\mathbb{S}_n$.

■

3.1.1 Multiparameter deformation of Catalan, Narayana and Schröder numbers

Let $\mathfrak{b}=(\beta_1,\ldots,\beta_{n-1})$ be a set of mutually commuting parameters. We define a multi-parameter analogue of the associative quasi-classical Yang–Baxter algebra $\widehat{MACYB}_n(\mathfrak{b})$ as follows.

Definition 3.2 *The multiparameter associative quasi-classical Yang–Baxter algebra of weight* $\mathfrak{b}$, *denoted by* $\widehat{MACYB}_n(\mathfrak{b})$, *is an associative algebra, over the ring of polynomials* $\mathbb{Z}[\beta_1,\ldots,\beta_{n-1}]$, *generated by the set of elements* $\{x_{ij},\ 1\leq i<j\leq n\}$, *subject to the set of relations*

(a) $x_{ij}\ x_{kl}=x_{kl}\ x_{ij}$, *if* $\{i,j\}\cap\{k,l\}=\emptyset$,

(b) $x_{ij}\ x_{jk}=x_{ik}\ x_{ij}+x_{jk}\ x_{ik}+\beta_i\ x_{ik}$, *if* $1\leq 1<i<j\leq n$,

Consider the "Coxeter element" $w_n\in\widehat{MACYB}_n(\mathfrak{b})$ which is equal to the ordered product of "simple generators":

$$w_n:=\prod_{a=1}^{n-1}x_{a,a+1}.$$

Now we can use the same method as in [40], **8.C5**, (c) , see Section 3.1. to define the **reduced form** of the Coxeter element w_n. Namely, let us bring the element w_n to the reduced form in the algebra $\widehat{MACYB}_n(\beta)$, that is, let us consecutively apply the defining relations (a) and (b) to the element w_n in any order until unable to do so. Denote the resulting (noncommutative) polynomial by $P(x_{ij};\mathfrak{b})$. In principal, the polynomial itself can depend on the order in which the relations (a) and (b) are applied.

Proposition 3.2 *(Cf [40],* **8.C5**, *(c); [28])*

Apart from applying the relation (a) (commutativity), the polynomial $P(x_{ij};\mathfrak{b})$ does not depend on the order in which relations (a) and (b) have been applied.

To state our main result of this Section, let us define polynomials

$$Q(\beta_1,\ldots,\beta_{n-1}):=P(x_{ij}=1,\forall i,j\ ;\beta_1-1,\beta_2-1,\ldots,\beta_{n-1}-1).$$

Example 3.1

$Q(\beta_1,\beta_2)=1+2\ \beta_1+\beta_2+\beta_1^2,$
$Q(\beta_1,\beta_2,\beta_3)=1+2(\beta_1+\beta_2+\beta_3)+3\beta_1^2+\beta_1\beta_2+\beta_1\beta_3+\beta_2^2+\beta_1^3,$
$Q(\beta_1,\beta_2,\beta_3,\beta_4)=1+7\beta_1+\beta_2+\beta_3+\beta_1(9\beta_1+3\beta_3+5\beta_4)+\beta_2(\beta_3+\beta_4)+\beta_3^2+$
$\beta_1^2\ (4\beta_1+\beta_2+\beta_3+\beta_4)+\beta_1(\beta_2^2+\beta_3^2)+\beta_2^3+\beta_1^4.$

Theorem 3.1

Polynomial $Q(\beta_1,\ldots,\beta_{n-1})$ has non-negative integer coefficients.

It follows from [40] and Proposition 3.1 that

$$Q(\beta_1,\ldots,\beta_{n-1})\Big|_{\beta_1=1,\ldots,\beta_{n-1}=1}=Cat_n.$$

Polynomials $Q(\beta_1,\ldots,\beta_{n-1})$ and $Q(\beta_1+1,\ldots,\beta_{n-1}+1)$ can be considered as a multiparameter deformation of the Catalan and Schröder numbers correspondingly, and the homogeneous degree k part of $Q(\beta_1,\ldots,\beta_{n-1})$ as a multiparameter analogue of Narayana numbers. We **expect** that the polynomial

$$t_1^n t_2^{n-1}\cdots t_n\ P(x_{ij}=t_i^{-1}\ ;\mathfrak{b})$$

coincides with a multiparameter deformation $\mathfrak{G}^{(\mathfrak{b})}_{\pi_k^{(n)}}(t_1,\ldots,t_{n-1})$ of the corresponding Grothendieck polynomial.

3.2 Grothendieck and q-Schröder polynomials

3.2.1 Schröder paths and polynomials

Definition 3.3 *A Schröder path of the length n is an over diagonal path from $(0,0)$ to (n,n) with steps $(1,0)$, $(0,1)$ and steps $D=(1,1)$ without steps of type D on the diagonal $x=y$.*

If p is a Schröder path, we denote by $d(p)$ the number of the diagonal steps resting on the path p, and by $a(p)$ the number of unit squares located between the path p and the diagonal $x=y$. For each (unit) diagonal step D of a path p we denote by $i(D)$ the x-coordinate of the column which contains the diagonal step D. Finally, define the index $i(p)$ of a path p as the some of the numbers $i(D)$ for all diagonal steps of the path p.

Definition 3.4 *Define q-Schröder polynomial $S_n(q;\beta)$ as follows*

$$S_n(q;\beta) = \sum_{p} q^{a(p)+i(p)}\ \beta^{d(p)}, \tag{3.19}$$

where the sum runs over the set of all Schröder paths of length n.

Example 3.2

$S_1(q;\beta) = 1,\ S_2(q;\beta) = 1+q+\beta\, q,\ S_3(q;\beta) = 1+2\,q+q^2+q^3+\beta\,(q+2q^2+2q^3)+\beta^2\,q^3,$
$S_4(q;\beta) = 1+3q+3q^2+3q^3+2q^4+q^5+q^6+\beta(q+3q^2+5q^3+6q^4+3q^5+3q^6)+\beta^2(q^3+2q^4+3q^5+3q^6)+\beta^3\,q^6.$

Comments 3.3

The q-Schröder polynomials defined by the formula (3.19) are <u>*different*</u> from the q-analogue of Schröder polynomials which has been considered in [5]. It seems that there are no simple connections between the both. ■

Proposition 3.3 (**Recurrence relations for q-Schröder polynomials**)
The Schröder polynomials satisfy the following relations

$$S_{n+1}(q;\beta) = (1+q^n+\beta\,q^n)\,S_n(q;\beta)+\sum_{k=1}^{k=n-1}(q^k+\beta\,q^{n-k})\,S_k(q;q^{n-k}\,\beta)\,S_{n-k}(q;\beta), \tag{3.20}$$

and the initial condition $S_1(q;\beta) = 1.$

Note that $P_n(\beta) = S_n(1;\beta)$ and in particular, the polynomials $P_n(\beta)$ satisfy the following recurrence relations

$$P_{n+1}(\beta) = (2+\beta)\,P_n(\beta)+(1+\beta)\sum_{k=1}^{n-1}P_k(\beta)\,P_{n-k}(\beta). \tag{3.21}$$

Theorem 3.2 (**Evaluation of the Schröder - Hankel Determinant**)
Consider permutation

$$\pi_k^{(n)} = \begin{pmatrix} 1 & 2 & \dots & k & k+1 & k+2 & \dots & n \\ 1 & 2 & \dots & k & n & n-1 & \dots & k+1 \end{pmatrix}.$$

Let as before

$$P_n(\beta) = \sum_{j=0}^{n-1} N(n,j)\,(1+\beta)^j, \quad n \ge 1, \tag{3.22}$$

denotes the Narayana-Schröder polynomials. <u>*Then*</u>

$$(1+\beta)^{\binom{k}{2}}\,\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}(x_1=1,\dots,x_{n-k}=1) = Det\,|P_{n+k-i-j}(\beta)\,|_{1\le i,j\le k}. \tag{3.23}$$

Proof is based on an observation that the permutation $\pi_k^{(n)}$ is a *vexillary* one and the recurrence relations (3.21).

Comments 3.4

(1) In the case $\beta = 0$, i.e. in the case of Schubert polynomials Theorem 3.1 has been proved in [13].

(2) In the cases when $\beta = 1$ and $0 \le n-k \le 2$, the value of the determinant in the RHS(3.22) is known, see e.g. [5], or M. Ichikawa talk *Hankel determinants of Catalan, Motzkin and Schröder numbers and its q-analogues*, http:/denjoy.ms.u-tokyo.ac.jp. One can check that in the all cases mentioned above, the formula (3.22) gives the same results.

(3) **Grothendieck and Narayana polynomials**

It follows from the expression (3.22) for the Narayana-Schröder polynomials that $P_n(\beta-1) = \mathfrak{N}_n(\beta)$, where

$$\mathfrak{N}_n(\beta) := \sum_{j=0}^{n-1} \frac{1}{n} \binom{n}{j} \binom{n}{j+1} \beta^j,$$

denotes the n-th Narayana polynomial. Therefore, $P_n(\beta-1) = \mathfrak{N}_n(\beta)$ is a symmetric polynomial in β with non-negative integer coefficients. Moreover. the value of the polynomial $P_n(\beta-1)$ at $\beta = 1$ is equal to the n-th Catalan number $C_n := \frac{1}{n+1}\binom{2n}{n}$.

It is well-known, see e.g. [42], that the Narayana polynomial $\mathfrak{N}_n(\beta)$ is equal to the generating function of the statistics $\pi(\mathfrak{p}) =$ (*number of peaks of a Dick path* $\mathfrak{p}$) $- 1$ on the set $Dick_n$ of Dick paths of the length $2n$

$$\mathfrak{N}_n(\beta) = \sum_{\mathfrak{p}} \beta^{\pi(\mathfrak{p})}.$$

Moreover, using the Lindström–Gessel–Viennot lemma see e.g.
http://en.wikipedia.org/wiki/Lindström–Gessel–Viennot lemma,
one can see that

$$DET\ |\mathfrak{N}_{n+k-i-j}(\beta)|_{1\le i,j\le k} = \beta^{\binom{k}{2}} \sum_{(\mathfrak{p}_1,\ldots,\mathfrak{p}_k)} \beta^{\pi(\mathfrak{p}_1)+\ldots+\pi(\mathfrak{p}_k)}, \tag{3.24}$$

where the sum runs over k-tuple of non-crossing Dick paths $(\mathfrak{p}_1,\ldots,\mathfrak{p}_k)$ such that the path $\mathfrak{p}_i$ starts from the point $(i-1,0)$ and has length $2(n-i+1)$, $i = 1,\ldots,k$.

We denote the sum in the RHS(3.24) by $\mathfrak{N}_n^{(k)}(\beta)$. Note that $\mathfrak{N}_{k-1}^{(k)}(\beta) = 1$ for all $k \ge 2$.

Thus, $\mathfrak{N}_n^{(k)}(\beta)$ is a symmetric polynomial in β with non-negative integer coefficients, and

$$\mathfrak{N}_n^{(k)}(\beta = 1) = C_n^{(k)} = \prod_{1\le i<j\le n-k+2} \frac{2k+i+j-1}{i+j-1}.$$

As a corollary we obtain the following statement

Proposition 3.4 *Let $n \ge k$, then*

$$\mathfrak{G}_{\pi_k^{(n)}}^{(\beta-1)}(x_1 = 1,\ldots,x_n = 1) = \mathfrak{N}_n^{(k)}(\beta).$$

Summarizing, the specialization $\mathfrak{G}^{(\beta-1)}_{\pi_k^{(n)}}(x_1=1,\ldots,x_n=1)$ is a symmetric polynomial in β with non-negative integer coefficients. and coincides with the generating function of the statistics $\sum_{i=1}^{k}\pi(\mathfrak{p}_i)$ on the set k-$Dick_n$ of k-tuple of non-crossing Dick paths $(\mathfrak{p}_1,\ldots,\mathfrak{p}_k)$. ■

Example 3.3 *Take $n=5$, $k=1$.Then $\pi_1^{(5)}=(15432)$ and one has*

$$\mathfrak{G}^{(\beta)}_{\pi_1^{(5)}}(1,q,q^2,q^3)=q^4(1,3,3,3,2,1,1)+q^5\ (1,3,5,6,3,3)\ \beta+q^7(1,2,3,3)\beta^2+q^{10}\beta^3.$$

It is easy to compute the Carlitz-Riordan q-analogue of the Catalan number C_5, namely, $C_5(q)=(1,3,3,3,2,1,1)$.

(4) **Grothendieck polynomials $\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}(x_1,\ldots,x_n)$ and k-dissections**

Let $k\in\mathbb{N}$ and $n\geq k-1$, be a integer, define *a k-dissection* of a convex $(n+k+1)$-gon to be a collection $\mathcal{E}$ of diagonals in $(n+k+1)$-gon not containing $(k+1)$-subset of pairwise crossing diagonals and such that at least $2(k-1)$ diagonals are coming from each vertex of the $(n+k+1)$-gon in question. One can show that the number of diagonals in any k-dissection $\mathcal{E}$ of a convex $(n+k+1)$-gon contains at least $(n+k+1)(k-1)$ and at most $n(2k-1)-1$ diagonals. We define the *index* of a k-dissection $\mathcal{E}$ to be $i(\mathcal{E})=n(2k-1)-1-\#|\mathcal{E}|$. Dnote by

$$\mathcal{T}_n^{(k)}(\beta)=\sum_{\mathcal{E}}\beta^{i(\mathcal{E})}$$

the generating function for the number of k-dissections with a fixed index, where the above sum runs over the set of all k-dissections of a convex $(n+k+1)$-gon.

Theorem 3.3

$$\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}(x_1=1,\ldots,x_n=1)=\mathcal{T}_n^{(k)}(\beta).$$

A k-dissection of a convex $(n+k+1)$-gon with the maximal number of diagonals (which is equal to $n(2k-1)-1$), is called **k-triangulation**. It is well-known that the number of k-triangulations of a convex $(n+k+1)$-gon is equal to the Catalan-Hankel number $C_{n-1}^{(k)}$. Explicit bijection between the set of k-triangulations of a convex $(n+k+1)$-gon and the set of k-tuple of non-crossing Dick paths $(\gamma_1,\ldots,\gamma_k)$ such that the Dick path γ_i connects points $(i-1,0)$ and $(2n-i-1,0)$, has been constructed in [36], [41].

(5) **Polynomials $\mathfrak{F}_w(\beta)$, $\mathfrak{H}_w(\beta)$, $\mathfrak{H}_w(q,t;\beta)$ and $\mathfrak{R}_w(q;\beta)$**

Let $w\in\mathbb{S}_n$ be a permutation and $\mathfrak{G}_w^{(\beta)}(X_n)$ and $\mathfrak{G}_w^{(\beta)}(X_n,Y_n)$ be the coressponding β-Grothendieck and double β-Grothendieck polynomials. We denote by $\mathfrak{G}_w^{(\beta)}(1)$ and by $\mathfrak{G}_w^{(\beta)}(1;1)$ the specializations $X_n:=(x_1=1,\ldots,x_n=1)$, $Y_n:=(y_1=1,\ldots,y_n=1)$ of the β-Grothendieck polynomials introduced above.

Theorem 3.4 *Let $w\in\mathbb{S}_n$ be a permutation. Then*

(*i*) *The polynomials $\mathfrak{F}_w(\beta):=\mathfrak{G}_w^{(\beta-1)}(1)$ and $\mathfrak{H}_w(\beta):=\mathfrak{G}_w^{(\beta-1)}(1;1)$*

have both non-negative integer coefficients.

(*ii*) *One has*

$$\mathfrak{H}_w(\beta) = (1+\beta)^{\ell(w)}\ \mathfrak{F}_w(\beta^2).$$

(*iii*) *Let $w \in \mathbb{S}_n$ be a permutation, define polynomials*

$$\mathfrak{H}_w(q,t;\beta) := \mathfrak{G}_w^{(\beta)}(x_1 = q, x_2 = q, \ldots, x_n = q, y_1 = t, y_2 = t, \ldots, y_n = t)$$

to be the specialization $\{x_i = q, y_i = t,\ \forall i\}$, of the double β-Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}(X_n, Y_n)$. Then

$$\mathfrak{H}_w(q,t;\beta) = (q+t+\beta\ q\ t)^{\ell(w)}\ \mathfrak{F}_w((1+\beta\ q)(1+\beta\ t)).$$

In particular, $\mathfrak{H}_w(1,1;\beta) = (2+\beta)^{\ell(w)}\ \mathfrak{F}_w((1+\beta)^2)$.

(*iv*) *Let $w \in \mathbb{S}_n$ be a permutation, define polynomial*

$$\mathcal{R}_w(q;\beta) := \mathfrak{G}_w^{(\beta-1)}(x_1 = q, x_2 = 1, x_3 = 1, \ldots)$$

to be the specialization $\{x_1 = q, x_i = 1,\ \forall i \geq 2\}$, of the $(\beta-1)$-Grothendieck polynomial $\mathfrak{G}_w^{(\beta-1)}(X_n)$. Then

$$\mathcal{R}_w(q;\beta) = q^{w(1)-1}\ \mathfrak{R}_w(q;\beta),$$

where $\mathfrak{R}_w(q;\beta)$ is a polynomial in q and β with non-negative integer coefficients, and $\mathfrak{R}_w(0;\beta=0) = 1$.

■

Remark 3.1

One can show, cf [26], p. 89, that if $w \in \mathbb{S}_n$, then $\mathcal{R}_w(1,\beta) = \mathcal{R}_{w^{-1}}(1,\beta)$. However, the equality $\mathfrak{R}_w(q,\beta) = \mathfrak{R}_{w^{-1}}(q,\beta)$ can be violated, and it seems that in general, there are no simple connections between polynomials $\mathfrak{R}_w(q,\beta)$ and $\mathfrak{R}_{w^{-1}}(q,\beta)$, if so.

From this point we shell use the notation $(a_0, a_1, \ldots, a_r)_\beta := \sum_{j=0}^r\ a_j\ \beta^j$, *etc.*

Example 3.4 Let us take $w = [1,3,4,6,7,9,10,2,5,8]$. Then $\mathfrak{R}_w(q,\beta) =$
$(1,6,21,36,51,48,26)_\beta + q\beta\ (6,36,126,216,306,288,156)_\beta +$
$q^2\beta^3\ (20,125,242,403,460,289)_\beta + q^3\beta^5\ (6,46,114,204,170)_\beta$. Moreover,
$\mathfrak{R}_w(q,1) = (189,1134,1539,540)_q$. On the other hand,
$w^{-1} = [1,8,2,3,9,4,5,10,6,7]$, and $\mathfrak{R}_{w^{-1}}(q,\beta) = (1,6,21,36,51,48,26)_\beta +$
$q\beta\ (1,6,31,56,96,110,78)_\beta + q^2\beta\ (1,6,27,58,92,122,120,78)_\beta +$
$q^3\beta\ (1,6,24,58,92,126,132,102,26)_\beta + q^4\beta\ (1,6,22,57,92,127,134,105,44)_\beta +$
$q^5\beta\ (1,6,21,56,91,126,133,104,50)_\beta + q^6\beta\ (1,6,21,56,91,126,133,104,50)_\beta$.
Moreover, $\mathfrak{R}_{w^{-1}}(q,1) = (189,378,504,567,588,588,588)_q$.

Notice that $w = 1 \times u$, where $u = [2,3,5,6,8,9,1,4,7]$. One can show that
$\mathfrak{R}_u(q,\beta) = (1,6,11,16,11)_\beta + q\beta^2\ (10,20,35,34)_\beta + q^2\beta^4\ (5,14,26)_\beta$. On the other hand,
$u^{-1} = [7,1,2,8,3,4,9,5,6]$ and $\mathfrak{R}_{u^{-1}}(q,\beta) = (1,6,21,36,51,48,26)_\beta = \mathfrak{R}_u(1,\beta)$.

■

[Hereinafter we have used the notation $(a_0, a_1, \ldots, a_r)_\beta := \sum_{j=0}^{r} \; a_j \; \beta^j$, *etc.*]

Problems 3.1

(1) *Define a bijection between monomials of the form* $\prod_{a=1}^{s} x_{i_a, j_a}$ *involved in the polynomial* $P(x_{ij}; \beta)$, *and dissections of a convex* $(n+2)$*-gon by* s *diagonals, such that no two diagonals intersect their interior.*

(2) *Describe permutations* $w \in \mathbb{S}_n$ *such that the Grothendieck polynomial* $\mathfrak{G}_w(t_1, \ldots, t_n)$ *is equal to the "reduced polynomial" for a some monomial in the associative qasi-classical Yang–Baxter algebra* $\widehat{ACYB_n}(\beta)$. *?*

(3) *Study "reduced polynomial" corresponding to the monomials*

$u_{12}u_{23} \cdots u_{n-1,n}u_{n-2,n-1} \cdots u_{23}u_{12}, \qquad (u_{12}u_{23} \cdots u_{n-1,n})^k$

in the algebra $\widehat{ACYB_n}(\beta)^{ab}$.

(4) *Construct a bijection between the set of k-dissections of a convex* $(n+k+1)$*-gon and " pipe dreams" corresponding to the Grothendieck polynomial* $\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}(x_1, \ldots, x_n)$. *As for a definition of "pipe greams" for Grothendieck polynomials, see [23]; see also [11].*

3.2.2 Principal specialization of Grothendieck polynomials, and q-Schröder polynomials

Let $\pi_k^{(n)} = 1^k \times w_0^{(n-k)} \in \mathbb{S}_n$ be the vexillary permutation as before, see Theorem 3.1. Recall that

$$\pi_k^{(n)} = \begin{pmatrix} 1 & 2 & \ldots & k & k+1 & k+2 & \ldots & n \\ 1 & 2 & \ldots & k & n & n-1 & \ldots & k+1 \end{pmatrix}.$$

(A) Principal specialization of the Schubert polynomial $\mathfrak{S}_{\pi_k^{(n)}}$

Note that $\pi_k^{(n)}$ is a vexillary permutation of the staircase shape $\lambda = (n-k-1, \ldots, 2, 1)$ and has the staircase flag $\phi = (k+1, k+2, \ldots, n-1)$. It is known, see e.g. [43], [26], that for a vexillary permutation $w \in \mathbb{S}_n$ of the shape λ and flag $\phi = (\phi_1, \ldots, \phi_r)$, $r = \ell(\lambda)$, the corresponding Schubert polynomial $\mathfrak{S}_w(X_n)$ is equal to the multi-Schur polynomial $s_\lambda(X_\phi)$, where X_ϕ denotes the flagged set of variables , namely, $X_\phi = (X_{\phi_1}, \ldots, X_{\phi_r})$ and $X_m = (x_1, \ldots, x_m)$. Therefore we can write the folloing determinantal formula for the principal specialization of the Schubert polynomial corresponding to the vexillary permutation $\pi_k^{(n)}$

$$\mathfrak{S}_{\pi_k^{(n)}}(1, q, , q^2, \ldots) = DET\Big(\begin{bmatrix} n-i+j-1 \\ k+i-1 \end{bmatrix}_q \Big)_{1 \le i,j \le n-k},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the q-binomial coefficient.

Let us observe that the Carlitz–Riordan q-analogue $C_n(q)$ of the Catalan number C_n is equal to the value of the q-Schröder polynomial at $\beta = 0$, namely, $C_n(q) = S_n(q, 0)$.

Lemma 3.2 *Let k, n be integers and $n > k$, then*

$$(1) \qquad DET\Big(\begin{bmatrix} n-i+j-1 \\ k+i-1 \end{bmatrix}_q \Big)_{1 \le i,j \le n-k} = q^{\binom{n-k}{3}} \; C_n^{(k)}(q),$$

$$(2) \qquad DET\Big(C_{n+k-i-j}(q)\Big)_{1\le i,j\le k} = q^{k(k-1)(6n-2k-5)/6}\ C_n^{(k)}(q).$$

(B) Principal specialization of the Grothendieck polynomial $\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}$

Theorem 3.5

$$q^{\binom{n-k+1}{3}-(k-1)\binom{n-k}{2}}\ DET|S_{n+k-i-j}(q;q^{i-1}\beta)|_{1\le i,j\le k} =$$

$$q^{k(k-1)(4k+1)/6}\prod_{a=1}^{k-1}(1+q^{a-1}\beta)\ \ \mathfrak{G}_{\pi_k^{(n)}}(1,q,q^2,\ldots).$$

Corollary 3.1 (1) *If* $k=n-1$, *then*

$$DET|S_{2n-1-i-j}(q;q^{i-1}\beta)|_{1\le i,j\le n-1} = q^{(n-1)(n-2)(4n-3)/6}\prod_{a=1}^{n-2}(1+q^{a-1}\beta)^{n-a-1},$$

(2) *If* $k=n-2$, *then*

$$q^{n-2}\ DET|S_{2n-2-i-j}(q;q^{i-1}\beta)|_{1\le i,j\le n-2} =$$

$$q^{(n-2)(n-3)(4n-7)/6}\prod_{a=1}^{n-3}(1+q^{a-1}\beta)^{n-a-2}\ \left\{\frac{(1+\beta)^{n-1}-1}{\beta}\right\}.$$

- **Generalization**

Let $\mathbf{n}=(n_1,\ldots,n_p)\in\mathbb{N}^p$ be a composition of n so that $n=n_1+\cdots+n_p$. We set $n^{(j)}=n_1+\cdots+n_j$, $j=1,\ldots,p$, $n^{(0)}=0$.

Now consider the permutation $w^{(\mathbf{n})}=w_0^{(n_1)}\times w_0^{(n_2)}\times\cdots\times w_0^{(n_p)}\in\mathbb{S}_n$,

where $w_0^{(m)}\in\mathbb{S}_m$ denotes the longest permutation in the symmetric group $\mathbb{S}_m$. In other words,

$$w^{(\mathbf{n})}=\begin{pmatrix} 1 & 2 & \ldots & n_1 & n^{(2)} & \ldots & n_1+1 & \ldots & n^{(p-1)} & \ldots & n \\ n_1 & n_1-1 & \ldots & 1 & n_1+1 & \ldots & n^{(2)} & \ldots & n & \ldots & n^{(p-1)}+1\end{pmatrix}.$$

For the permutation $w^{(\mathbf{n})}$ defined above, one has the following factorization formula for the Grothendieck polynomial corresponding to $w^{(\mathbf{n})}$, [26],

$$\mathfrak{G}^{(\beta)}_{w^{(\mathbf{n})}}=\mathfrak{G}^{(\beta)}_{w_0^{(n_1)}}\times\mathfrak{G}^{(\beta)}_{1^{n_1}\times w_0^{(n_2)}}\times\mathfrak{G}^{(\beta)}_{1^{n_1+n_2}\times w_0^{(n_3)}}\times\cdots\times\mathfrak{G}^{(\beta)}_{1^{n_1+\cdots n_{p-1}}\times w_0^{(n_p)}}.$$

In particular, if

$$w^{(\mathbf{n})}=w_0^{(n_1)}\times w_0^{(n_2)}\times\cdots\times w_0^{(n_p)}\in\mathbb{S}_n, \qquad (3.25)$$

then the principal specialization $\mathfrak{G}^{(\beta)}_{w^{(\mathbf{n})}}$ of the Grothendieck polynomial corresponding to the permutation w, is the product of q-Schröder–Hankel polynomials. Finally, we observe that from discussions in Section 3.4, **Grothendieck and Narayana polynomials**, one can deduce that

$$\mathfrak{G}^{(\beta-1)}_{w^{(\mathbf{n})}}(x_1=1,\ldots,x_n=1)=\prod_{j=1}^{p-1}\mathfrak{N}^{(n^{(j)})}_{n^{(j+1)}}(\beta).$$

In particular, the polynomial $\mathfrak{G}^{(\beta-1)}_{w^{(\mathbf{n})}}(x_1,\ldots,x_n)$ is a symmetric polynomial in β with non-negative integer coefficients.

Example 3.5

(1) Let us take (non vexillary) permutation $w = 2143 = s_1\, s_3$. One can check that $\mathfrak{G}_w^{(\beta)}(1,1,1,1) = 3 + 3\,\beta + \beta^2 = 1 + (\beta+1) + (\beta+1)^2$, and $\mathfrak{N}_4(\beta) = (1,6,6,1)$, $\mathfrak{N}_3(\beta) = (1,3,1)$, $\mathfrak{N}_2(\beta) = (1,1)$. It is easy to see that $\beta\ \mathfrak{G}_w^{(\beta)}(1,1,1,1) = DET \begin{vmatrix} \mathfrak{N}_4(\beta) & \mathfrak{N}_3(\beta) \\ \mathfrak{N}_3(\beta) & \mathfrak{N}_2(\beta) \end{vmatrix}$. On the other hand, $DET \begin{vmatrix} P_4(\beta) & P_3(\beta) \\ P_3(\beta) & P_2(\beta) \end{vmatrix} = (3,6,4,1) = \underline{(3+3\beta+\beta^2)}\ (1+\beta)$. It is more involved to check that

$$q^5(1+\beta)\ \mathfrak{G}_w^{(\beta)}(1,q,q^2,q^3) = DET \begin{vmatrix} S_4(q;\beta) & S_3(q;\beta) \\ S_3(q;q\beta) & S_2(q;q\beta) \end{vmatrix}.$$

(2) Let us illustrate Theorem 3.3 by a few examples. For the sake of simplicity, we consider the case $\beta = 0$, i.e. the case of Schubert polynomials. In this case $P_n(q;\beta = 0) = C_n(q)$ is equal to the Carlitz–Riordan q-analogue of Catalan numbers. We are reminded that the q-Catalan– Hankel polynomials are defined as follows

$$C_n^{(k)}(q) = q^{k(1-k)(4k-1)/6}\ DET|C_{n+k-i-j}(q)|_{1 \le i,j \le n}.$$

In the case $\beta = 0$ the Theorem 3.3 states that if $\mathbf{n} = (n_1,\ldots,n_p) \in \mathbb{N}^p$ and the permutation $w_{(\mathbf{n})} \in \mathbb{S}_n$ is defined by the use of (3.25), then

$$\mathfrak{S}_{w^{(\mathbf{n})}}(1,q,q^2,\ldots) = q^{\sum \binom{n_i}{3}}\ C_{n_1+n_2}^{(n_1)}(q) \times C_{n_1+n_2+n_3}^{(n_1+n_2)}(q) \times C_n^{(n-n_p)}(q).$$

Now let us consider a few examples for $n = 6$.

- $\mathbf{n} = (1,5)$, $\Longrightarrow\ \mathfrak{S}_{w^{(\mathbf{n})}}(1,q,\ldots) = q^{10}\ C_6^{(1)}(q) = C_5(q)$.
- $\mathbf{n} = (2,4)$, $\Longrightarrow\ \mathfrak{S}_{w^{(\mathbf{n})}}(1,q,\ldots) = q^4\ C_6^{(2)}(q) = DET \begin{vmatrix} C_6(q) & C_5(q) \\ C_5(q) & C_4(q) \end{vmatrix}$.

Note that $\mathfrak{S}_{w^{(2,4)}}(1,q,\ldots) = \mathfrak{S}_{w^{(1,1,4)}}(1,q,\ldots)$.

- $\mathbf{n} = (2,2,2)$ $\Longrightarrow\ \mathfrak{S}_{w^{(\mathbf{n})}}(1,q,\ldots) = C_4^{(2)}(q)\ C_6^{(4)}(q)$.
- $\mathbf{n} = (1,1,4)$ $\Longrightarrow\ \mathfrak{S}_{w^{(\mathbf{n})}}(1,q,\ldots) = q^4\ C_2^{(1)}(q)\ C_4^{(2)}(q) = q^4\ C_4^{(2)}(q)$,

the last equality follows from that $C_{k+1}^{(k)}(q) = 1$ for all $k \ge 1$.

- $\mathbf{n} = (1,2,3)$ $\Longrightarrow\ \mathfrak{S}_{w^{(\mathbf{n})}}(1,q,\ldots) = q\ C_3^{(1)}(q)\ C_6^{(3)}(q)$. On the other hand,
- $\mathbf{n} = (3,2,1)$ $\Longrightarrow\ \mathfrak{S}_{w^{(\mathbf{n})}}(1,q,\ldots) = q\ C_5^{(3)}(q)\ C_6^{(5)}(q) = q\ C_5^{(3)}(q) = q(1,1,1,1)$.

Note that $C_{k+2}^{(k)}(q) = \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q$.

Exercise.

Let $1 \le k \le m \le n$ be integers, $n \ge 2k+1$. Consider permutation

$$w = \begin{pmatrix} 1 & 2 & \ldots & k & k+1 & \ldots & n \\ m & m-1 & \ldots & m-k+1 & n & \ldots\ \ldots & 1 \end{pmatrix} \in \mathbb{S}_n.$$

Show that

$$\mathfrak{S}_w(1,q,\ldots) = q^{n(D(w))}\ C_{n-m+k}^{(m)}(q),$$

where for any permutation w, $n(D(w)) = \sum \binom{d_i(w)}{2}$ and $d_i(w)$ denotes the number of boxes in the i-th column of the (*Rothe*) diagram $D(w)$ of the permutation w, see [26]. p.8.

■

(C) A determinantal formula for the Grothendieck polynomials $\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}$

Define polynomials

$$\Phi_n^{(m)}(X_n) = \sum_{a=m}^{n} e_a(X_n)\ \beta^{a-m},$$

$$A_{i,j}(X_{n+k-1}) = \frac{1}{(i-j)!}\left(\frac{\partial}{\partial\beta}\right)^{j-1} \Phi_{k+n-i}^{(n+1-i)}(X_{k+n-i}), \quad if \ \ 1 \le i \le j \le n,$$

and

$$A_{i,j}(X_{k+n-1}) = \sum_{a=0}^{i-j-1} e_{n-i-a}(X_{n+k-i}) \binom{i-j-1}{a}, \quad if \ \ 1 \le j < i \le n.$$

Theorem 3.6

$$DET|A_{i,j}|_{1\le i,j\le n} = \mathfrak{G}^{(\beta)}_{\pi_{k+n}^{(k)}}(X_{k+n-1}).$$

Comments 3.5 One can compute the Grothendieck polynomials for yet another interesting family of permutations. namely, permutations $\sigma_k^{(n)} =$

$$\sigma_k^{(n)} = \begin{pmatrix} 1 & 2 & \dots & k-1 & k & k+1 & k+2 & \dots & n+k \\ 1 & 2 & \dots & k-1 & n+k & k & k+1\dots & n+k-1 & \end{pmatrix} =$$

$$s_k s_{k+1} \dots s_{n+k-1} \in \mathbb{S}_{n+k}.$$

<u>Then</u>

$$\mathfrak{G}^{(\beta)}_{\sigma_k^{(n)}}(x_1,\dots,x_{n+k}) = \sum_{j=0}^{k-1} \binom{n+j-1}{j}\ e_{n+j}(x_1,\dots,x_{n+k})\ (1+\beta)^j.$$

In particular,

$$\mathfrak{G}^{(\beta)}_{\sigma_k^{(n)}}(x_1=1,\dots,x_{n+k}=1) = \sum_{j=0}^{k} \binom{n+j-1}{j}\ \beta^j.$$

■

Problems 3.2

(1) *Give a bijective prove of Theorem 3.3, i.e. construct a bijection between*

• *the set of k-tuple of mutually non-crossing Schröder paths $(\mathfrak{p}_1,\dots,\mathfrak{p}_k)$ of lengths $(n, n-1,\dots,n-k+1)$ correspondingly,* <u>*and*</u>

• *the set of pairs $(\mathfrak{m}, \mathcal{T})$, where $\mathcal{T}$ is a k-dissection of a convex $(n+k+1)$-gon, and $\mathfrak{m}$ is a upper triangle $(0,1)$-matrix of size $(k-1)\times(k-1)$,*

which is compatible with natural statistics on both sets.

(2) *Let $w \in \mathbb{S}_n$ be a permutation, and $CS(w)$ be the set of compatible sequences corresponding to w, see e.g. [4].*

Define statistics $c(\bullet)$ on the set $CS(w)$ such that

$$\mathfrak{G}_w^{(\beta-1)}(x_1=1,x_2=1,\ldots)=\sum_{a\in CS(w)}\beta^{c(a)}.$$

3.2.3 Specialization of Schubert polynomials

Let $n,\ k,\ r$ be positive integers and $p,\ b$ be non-negative integers such that $r \le p+1$. It is well-known [26] that in this case there exists a unique vexillary permutation $\varpi := \varpi_{\lambda,\phi} \in \mathbb{S}_\infty$ which has the shape $\lambda=(\lambda_1,\ldots,\lambda_{n+1})$ and the flag $\phi=(\phi_1,\ldots,\phi_{n+1})$, where

$$\lambda_i=(n-i+1)\,p+b,\qquad \phi_i=k+1+r\,(i-1),\quad 1\le i\le n+1.$$

According to a theorem by M.Wachs [43], the Schubert polynomial $\mathfrak{S}_\varpi(X)$ admits the following determinantal representation

$$\mathfrak{S}_\varpi(X)=DET\Big(h_{\lambda_i-i+j}(X_{\phi_i})\Big)_{1\le i,j\le n+1}.$$

Therefore we have $\quad \mathfrak{S}_\varpi(1):=\mathfrak{S}_\varpi(x_1=1,x_2=1,\ldots)=$

$$DET\left(\binom{(n-i+1)p+b-i+j+k+(i-1)r}{k+(i-1)r}\right)_{1\le i,j\le n+1}.$$

We denote the above determinant by $D(n,k,r,b,p)$.

Theorem 3.7 $\qquad D(n,k,r,b,p)=$

$$\prod_{(i,j)\in\mathcal{A}_{n,k,r}}\frac{i+b+jp}{i}\prod_{(i,j)\in\mathcal{B}_{n,k,r}}\frac{(k-i+1)(p+1)+(i+j-1)r+r(b+np)}{k-i+1+(i+j-1)r},$$

where

$$\mathcal{A}_{n,k,r}=\Big\{(i,j)\in\mathbb{Z}_{\ge0}^2\ \ |\ j\le n,\ \ j<i\le k+(r-1)(n-j)\Big\},$$

$$\mathcal{B}_{n,k,r}=\Big\{(i,j)\in\mathbb{Z}_{\ge1}^2\ \ |\ i+j\le n+1,\ \ i\ne k+1+r\,s,\ \ s\in\mathbb{Z}_{\ge0}\Big\}.$$

It is convenient to re-wright the above formula for $D(n,k,r,b,p)$ in the following form

$$D(n,k,r,b,p)=$$

$$\prod_{j=1}^{n+1}\frac{\Big((n-j+1)p+b+k+(j-1)(r-1)\Big)!\,(n-j+1)!}{\Big(k+(j-1)r\Big)!\,\Big((n-j+1)(p+1)+b\Big)!}\times$$

$$\prod_{1\le i\le j\le n}\Big((k-i+1)(p+1)+jr+(np+b)r\Big).$$

($\Longrightarrow$) **The case r=1**

We consider below some special cases of Theorem 3.5 in the case $r=1$. To simplify notation, we set $D(n,k,b,p) := D(n,k,r=1,b,p)$. Then we can rewrite the above formula for $D(n,k,r,b,p)$ as follows $D(n,k,b,p) =$

$$\prod_{j=1}^{n+1} \frac{\Big((n+k-j+1)(p+1)+b\Big)!\ \Big((n-j+1)p+b+k\Big)!\ (j-1)!}{\Big((n-j+1)(p+1)+b\Big)!\ \Big((k+n-j+1)p+b+k\Big)!\ (k+j-1)!}.$$

Corollary 3.2

(1) *If* $k \le n+1$, *then* $D(n,k,b,p) =$

$$\prod_{j=1}^{k} \binom{(n+k+1-j)(p+1)+b}{n-j+1} \binom{(k-j)p+b+k}{j} \frac{j!\ (k-j)!\ (n-j+1)!}{(n+k-j+1)!}.$$

In particular,

- *If* $k=1$, *then*

$$D(n,1,b,p) = \frac{1+b}{1+b+(n+1)p} \binom{(p+1)(n+1)+b}{n+1} := F_{n+1}^{(p+1)}(b),$$

where $F_n^p(b) := \frac{1+b}{1+b+(p-1)n}\binom{pn+b}{n}$ *denotes the generalized Fuss-Catalan number.*

- *if* $k=2$, *then*

$$D(n,2,b,p) = \frac{(2+b)(2+b+p)}{(1+b)(2+b+(n+1)p)(2+b+(n+2)p)}\ F_{n+1}^{(p+1)}(b)\ F_{n+2}^{(p+1)}(b).$$

(2) *(R.A. Proctor [35]) Consider the Young diagram*

$$\lambda := \lambda_{n,p,b} = \{(i,j) \in \mathbb{Z}_{\ge 1} \times \mathbb{Z}_{\ge 1} \mid 1 \le i \le n+1, 1 \le j \le (n+1-i)p+b\}.$$

For each box $(i,j) \in \lambda$ *define the numbers* $c(i,j) := n+1-i+j$, *and*

$$l_{(i,j)}(k) = \begin{cases} \frac{k+c(p,j)}{c(i,j)}, & if \quad j \le (n+1-i)(p-1)+b, \\ \frac{(p+1)k+c(i,j)}{c(i,j)}, & if \quad (n+1-i)(p-1) < j-b \le (n+1-i)p. \end{cases}$$

Then

$$D(n,k,b,p) = \prod_{(i,j)\in\lambda} l_{(i,j)}(k). \qquad (3.26)$$

Therefore, $D(n,k,b,p)$ *is a polynomial in* k *with rational coefficients.*

(3) *If* $p=0$, *then*

$$D(n,k,b,0) = dim\ V_{(n+1)^k}^{\mathfrak{gl}(b+k)} = \prod_{j=1}^{n+k} (\frac{j+b}{j})^{min(j,n+k+1-j)},$$

where for any partition μ, $\ell(\mu) \le m$, $V_\mu^{\mathfrak{gl}(m)}$ *denotes the irreducible* $\mathfrak{gl}(m)$*-module with the highest weight* μ. In particular,

- $D(n,2,b,0) = \frac{1}{n+2+b}\binom{n+2+b}{b}\binom{n+2+b}{b+1}$ is equal to the Narayana number $N(n+b+2,b)$;

- $$D(1,k,b,0) = \frac{(b+k)!\,(b+k+1)!}{k!b!(k+1)!(b+1)!} := N(b+k+1,k),$$

and therefore the number $D(1,k,b,0)$ counts the number of pairs of non-crossing lattice paths inside a rectangular of size $(b+1)\times(k+1)$, which go from the point $(1,0)$ (resp. from that $(0,1)$) to the point $(b+1,k)$ (resp. to that $(b,k+1)$), consisting of steps $U=(1,0)$ and $R=(0,1)$, see [37], A001263, for some list of combinatorial interpretations of the Narayana numbers.

(4) *If $p=b=1$, then*

$$D(n,k,1,1) = \prod_{1\le l<j\le n+2} \frac{2k+i+j-1}{i+j-1}.$$

(5) *(R.A. Proctor [33],[34]) If $p=1$ and b is odd integer, then $D(n,k,b,1)$ is equal to the dimension of the irreducible representation of the symplectic Lie algebra $Sp(b+2n+1)$ with the highest wright $k\omega_{n+1}$.*

(6) *(Cf [13]) Let ϖ_λ be a unique dominant permutation of shape $\lambda := \lambda_{n,p,b}$ and $\ell := \ell_{n,p,b} = \frac{1}{2}(n+1)(np+2b)$ be its length. Then*

$$\sum_{\mathbf{a}\in R(\varpi_\lambda)} \prod_{i=1}^{\ell}(x+a_i) = \ell!\; B(n,x,p,b).$$

Here for any permutation w of length l, we denote by $R(w)$ the set $\{\mathbf{a}=(a_1,\ldots,a_l)\}$ of all reduced decompositions of w.

Remark 3.2

It is well-known, see e.g. [35], that the number $D(n,k,b,p)$ is equal to the total number $pp^{\lambda_{n,p,b}}(k)$ of reverse (weak) plane partitions [4] bounded by k and contained in the shape $\lambda_{n,b,p}$. Finally we recall that the generalized Fuss-Catalan number $F_{n+1}^{(p+1)}(b)$ counts the number of lattice paths from $(0,0)$ to $(b+np,n)$ that do not go above the line $x=py$, see e.g. [24].

■

Theorem 3.8 *Let $\varpi_{n,k,p}$ be a unique vexillary permutation of the shape $\lambda_{n,p} := (n,n-1,\ldots,2,1)p$ and flag $\phi_{n,k} := (k+1,k+2,\ldots,k+n-1,k+n)$. Then*

- $$\mathfrak{G}^{(\beta-1)}_{\varpi_{n,1,p}}(1) = \sum_{j=1}^{n+1} \frac{1}{n+1}\binom{n+1}{j}\binom{(n+1)p}{j-1}\beta^{j-1}.$$

- *If $k\ge 2$, then $G_{n,k,p}(\beta) := \mathfrak{G}^{(\beta-1)}_{\varpi_{n,k,p}}(1)$ is a polynomial of degree nk in β, and $Coeff_{[\beta^{nk}]}(G_{n,k,p}(\beta)) = D(n,k,1,p-1,0)$.*

---

[4] Let λ be a partition. A reverse plane partition bounded by d and shape λ is a filling of the shape λ by the numbers from the set $\{0,1,\ldots,d\}$ in such a way that the numbers along columns and rows are weakly increasing.

The polynomial $\sum_{j=1}^{n} \frac{1}{n} \binom{n}{j} \binom{pn}{j-1} t^{j-1} := \mathfrak{FN}_n(t)$ is known as the Fuss-Narayana polynomial and can be considered as a t-deformation of the Fuss-Catalan number $FC_n^p(0)$.

Recall that the number $\frac{1}{n} \binom{n}{j} \binom{pn}{j-1}$ counts paths from $(0,0)$ to $(np,0)$ in the first quadrant, consisting of steps $U=(1,1)$ and $D=(1,-p)$ and have j peaks (i.e. UD's), cf. [37], A108767.

For example, take $n=3, k=2, p=3, r=1, b=0$. Then $\varpi_{3,2,3} = [1,2,12,9,6,3,4,5,7,8,10,11] \in \mathbb{S}_{12}$, and $G_{3,2,3}(\beta) = (1,18,171,747,1767,1995,1001)$. Therefore, $G_{3,2,3}(1) = 5700 = D(3,2,3,0)$ and $Coeff_{[\beta^6]}(G_{3,2,3}(\beta)) = 1001 = D(3,2,2,0)$.

Comments 3.6

It follows from Theorem 3.5 that in the case $r=0$ and $k \geq n$, one has

$$D(n,k,0,p,b) = dim\ V_{\lambda_{n,p,b}}^{\mathfrak{gl}(k+1)} = (1+p)^{\binom{n+1}{2}} \prod_{j=1}^{n+1} \frac{\binom{(n-j+1)p+b+k-j+1}{k-j+1}}{\binom{(n-j+1)(p+1)+b}{n-j+1}}.$$

Now consider the conjugate $\nu := \nu_{n,p,b} := ((n+1)^b, n^p, (n-1)^p, \ldots, 1^p)$ of the partition $\lambda_{n,p,b}$, and a rectangular shape partition $\psi = (\underbrace{k,\ldots,k}_{np+b})$. If $k \geq np+b$, then there exists a unique grassmannian permutation $\sigma := \sigma_{n,k,p,b}$ of the shape ν and the flag ψ, [26]. It is easy to see from the above formula for $D(n,k,0,p,b)$, that

$$\mathfrak{S}_{\sigma_{n,k,p,b}}(1) = dim\ V_{\nu_{n,p,b}}^{\mathfrak{gl}(k-1)} =$$

$$(1+p)^{\binom{n}{2}} \binom{k+n-1}{b} \prod_{j=1}^{n} \frac{(p+1)(n-j+1)}{(n-j+1)(p+1)+b} \prod_{j=1}^{n} \frac{\binom{k+j-2}{(n-j+1)p+b}}{\binom{(n-j+1)(p+1)+b-1}{n-j}}.$$

After the substitution $k := np+b+1$ in the above formula we will have

$$\mathfrak{S}_{\sigma_{n,np+b+1,p,b}}(1) = (1+p)^{\binom{n}{2}} \prod_{j=1}^{n} \frac{\binom{np+b+j-1}{(n-j+1)p}}{\binom{j(p+1)-1}{j-1}}.$$

In the case $b=0$ some simplifications are happened, namely

$$\mathfrak{S}_{\sigma_{n,k,p,0}}(1) = (1+p)^{\binom{n}{2}} \prod_{j=1}^{n} \frac{\binom{k+j-2}{(n-j+1)p}}{\binom{(n-j+1)p+n-j}{n-j}}.$$

Finally we observe that if $k = np+1$, then

$$\prod_{j=1}^{n} \frac{\binom{np+j-1}{(n-j+1)p}}{\binom{(n-j+1)p+n-j}{n-j}} = \prod_{j=2}^{n} \frac{\binom{np+j-1}{(p+1)(j-1)}}{\binom{j(p+1)-1}{j-1}} = \frac{\prod_{i=1}^{p} \prod_{j=0}^{n-1} ((p+1)j+i)}{\prod_{j=0}^{n(p-1)-1} (n+j)} := A_n^{(p)},$$

where the numbers $A_n^{(p)}$ are integers that generalize the numbers of **alternating sign matrices** (ASM) of size $n \times n$, recovered in the case $p=2$, see [31], [6] for details.

Examples 3.1

(1) Let us consider polynomials $\mathfrak{G}_n(\beta) := \mathfrak{G}^{(\beta-1)}_{\sigma_{n,2n,2,0}}(1)$.

- If $n = 2$, then $\sigma_{2,4,2,0} = 235614 \in \mathbb{S}_6$, and $\mathfrak{G}_2(\beta) = (1, 2, \mathbf{3}) := 1 + 2\beta + \mathbf{3}\beta^2$.

Moreover, $\mathfrak{R}_{\sigma_{2,4,2,0}}(q; \beta) = (1, 2)_\beta + \mathbf{3}\ q\beta^2$.

- If $n = 3$, then $\sigma_{3,6,2,0} = 235689147 \in \mathbb{S}_9$, and $\mathfrak{G}_3(\beta) = (1, 6, 21, 36, 51, 48, \mathbf{26})$.

Moreover, $\mathfrak{R}_{\sigma_{3,6,2,0}}(q; \beta) = (1, 6, 11, 16, \mathbf{11})_\beta + q\ \beta^2(10, 20, 35, 34)_\beta + q^2\beta^4(5, 14, \mathbf{26})_\beta$;
$\mathfrak{R}_{\sigma_{3,6,2,0}}(q; 1) = (45, 99, 45)_q$.

- If $n = 4$, then $\sigma_{4,8,2,0} = [2, 3, 5, 6, 8, 9, 11, 12, 1, 4, 7, 10] \in \mathbb{S}_{12}$, and $\mathfrak{G}_4(\beta) = (1, 12, 78, 308, 903, 2016, 3528, 4944, 5886, 5696, 4320, 2280, \mathbf{646})$.

Moreover, $\mathfrak{R}_{\sigma_{4,8,2,0}}(q; \beta) = (1, 12, 57, 182, 392, 602, 763, 730, 493, \mathbf{170})_\beta +$
$q\beta^2(21, 126, 476, 1190, 1925, 2626, 2713, 2026, 804)_\beta +$
$q^2\beta^4(35, 224, 833, 1534, 2446, 2974, 2607, 1254)_\beta + q^3\beta^6(7, 54, 234, 526, 909, 1026, \mathbf{646})_\beta$;
$\mathfrak{R}_{\sigma_{4,8,2,0}}(q; 1) = (3402, 11907, 11907, 3402)_q = 1701\ (2, 7, 7, 2)_q$.

[It will be recalled that here we have used the notation $(a_0, a_1, \ldots, a_r)_\beta := \sum_{j=0}^{r} a_j\ \beta^j$, *etc.*]

One can show that $deg_{[\beta]}\mathfrak{G}_n(\beta) = n(n-1)$, and looking on the numbers 3, 26, 646 we made

Conjecture 2 Let $a(n) := Coeff[\beta^{n(n-1)}]\left(\mathfrak{G}_n(\beta)\right)$. Then

$$a(n) = VSAM(n) = OSASM(n) = \prod_{j=1}^{n-1} \frac{(3j+2)(6j+3)!\ (2j+1)!}{(4j+2)!\ (4j+3)!},$$

where

$VSASM(n)$ is the number of alternating sign $2n+1 \times 2n+1$ matrices symmetric about the vertical axis;

$OSASM(n)$ is the number of $2n \times 2n$ off-diagonal symmetric alternating sign matrices.

See [37], A005156, [31] and references therein, for details.

(2) Let us consider polynomials $\mathfrak{F}_n(\beta) := \mathfrak{G}^{(\beta-1)}_{\sigma_{n,2n+1,2,0}}(1)$.

- If $n = 1$, then $\sigma_{1,3,2,0} = 1342 \in \mathbb{S}_4$, and $\mathfrak{F}_2(\beta) = (1, \mathbf{2}) := 1 + \mathbf{2}\beta$.
- If $n = 2$, then $\sigma_{2,5,2,0} = 1346725 \in \mathbb{S}_7$, and $\mathfrak{F}_3(\beta) = (1, 6, 11, 16, \mathbf{11})$.

Moreover, $\mathfrak{R}_{\sigma_{2,5,2,0}}(q; \beta) = (1, 2, \mathbf{3})_\beta + q\beta(4, 8, 12)_\beta + q^2\beta^3(4, \mathbf{11})_\beta$.

- If $n = 3$, then $\sigma_{3,7,2,0} = [1, 3, 4, 6, 7, 9, 10, 2, 5, 8] \in \mathbb{S}_{10}$, and $\mathfrak{F}_4(\beta) = (1, 12, 57, 182, 392, 602, 763, 730, 493, \mathbf{170})$.

Moreover, $\mathfrak{R}_{\sigma_{3,7,2,0}}(q; \beta) = (1, 6, 21, 36, 51, 48, \mathbf{26})_\beta + q\ \beta\ (6, 36, 126, 216, 306, 288, 156)_\beta$
$+\ q^2\beta^3(20, 125, 242, 403, 460, 289)_\beta + q^3\beta^5(6, 46, 114, 204, \mathbf{170})_\beta$;
$\mathfrak{R}_{\sigma_{3,7,2,0}}(q; 1) = (189, 1134, 1539, 540)_q = 27\ (7, 42, 57, 20)_q$.

- If $n = 4$, then $\sigma_{4,9,2,0} = [1, 3, 4, 6, 7, 9, 10, 12, 13, 2, 5, 8, 11] \in \mathbb{S}_{13}$, and $\mathfrak{F}_5(\beta) = (1, 20, 174, 988, 4025, 12516, 31402, 64760, 111510, 162170, 202957, 220200, 202493, 153106, 89355, 35972, \mathbf{7429})$.

Moreover,
$\mathfrak{R}_{\sigma_{4,9,2,0}}(q; \beta) = (1, 12, 78, 308, 903, 2016, 3528, 4944, 5886, 5696, 4320, 2280, \mathbf{646})_\beta +$
$q\beta\ (8, 96, 624, 2464, 7224, 16128, 28224, 39552, 47088, 45568, 34560, 18240, 5168)_\beta +$
$q^2\beta^3(56, 658, 3220, 11018, 27848, 53135, 78902, 100109, 103436, 84201, 47830, 14467)_\beta +$

$q^3\beta^5(56, 728, 3736, 12820, 29788, 50236, 72652, 85444, 78868, 50876, 17204)_{\beta}+$
$q^4\beta^7(8, 117, 696, 2724, 7272, 13962, 21240, 24012, 18768, \mathbf{7429})_{\beta}$;
$\mathfrak{R}_{\sigma_{4,9,2,0}}(q;1) = (30618, 244944, 524880, 402408, 96228)_q = 4374\ (7, 56, 120, 92, 22)_q$.

One can show that $\mathfrak{F}_n(\beta)$ is a polynomial in β of degree n^2, and looking on the numbers $2, 11, 170, 7429$ we made

Conjecture 3 Let $b(n) := Coeff[\beta^{(n-1)^2}]\left(\mathfrak{F}_n(\beta)\right)$. Then

$b(n) = CSTCPP(n)$. In other words, $b(n)$ is equal to the number of cyclically symmetric transpose complement plane partitions in an $2n \times 2n \times 2n$ box. This number is known to be

$$\prod_{j}^{n-1} \frac{(3j+1)(6j)!\ (2j)!}{(4j+1)!\ (4j)!},$$

see [37], A051255, [2], p.199.

Proposition 3.5 *One has*

$$\mathfrak{R}_{\sigma_{4,2n+1,2,0}}(0;\beta) = \mathfrak{S}_n(\beta) = \mathfrak{S}^{(\beta-1)}_{\sigma_{n,2n,2,0}}(1), \quad \mathfrak{R}_{\sigma_{n,2n,2,0}}(0,\beta) = \mathfrak{F}_n(\beta) = \mathfrak{S}^{(\beta-1)}_{\sigma_{n,2n+1,2,0}}(1).$$

Remarks 1 One can compute the principal specialization of the Schubert polynomial corresponding to the transposition $t_{k,n} := (k, n-k) \in \mathbb{S}_n$ that interchanges k and $n-k$, and fixes all other elements of $[1, n]$.

Proposition 3.6

$$q^{(n-1)(k-1)}\ \mathfrak{S}_{t_{k,n-k}}(1, q^{-1}, q^{-2}, q^{-3}, \ldots) = \sum_{j=1}^{k}(-1)^{j-1}\ q^{\binom{j}{2}} \begin{bmatrix} n-1 \\ k-j \end{bmatrix}_q \begin{bmatrix} n-2+j \\ n-k-1 \end{bmatrix}_q.$$

■

Exercises.

(1) Let $n \geq 1$ be a positive integer, consider "zig-zag" permutation

$$w = \begin{pmatrix} 2 & 4 & \ldots & 2k & 2k+2 & \ldots & & 2n \\ 1 & 3 & \ldots & 2k-1 & 2k+1 & \ldots & \ldots & 2n-1 \end{pmatrix} \in \mathbb{S}_{2n}.$$

Show that

$$\mathfrak{R}_w(q,\beta) = \prod_{k=0}^{n-1}\left(\frac{1-\beta^{2k}}{1-\beta} + q\beta^{2k}\right).$$

(2) Let $\sigma_{k,n,m}$ be grassmannian permutation with shape $\lambda = (n^m)$ and flag $\phi = (k+1)^m$, i.e.

$$\sigma_{k,n,m} = \begin{pmatrix} 1 & 2 & \ldots & k & k+1 & \ldots & k+n & k+n+1 & \ldots & k+n+m \\ 1 & 2 & \ldots & k & k+m+1 & \ldots & k+m+n & k+1 & \ldots & k+m \end{pmatrix}.$$

Clearly $\sigma_{k+1,n,m} = 1 \times \sigma_{k,n,m}$. Show that the coefficient

$Coeff_{\beta^m}\left(\mathfrak{R}_{\sigma_{k,n,m}}(1,\beta)\right)$ is equal to the Narayana number $N(k+n+m,k)$.

(3) Show that

$$\sum_{\substack{(a,b,c)\in\mathbb{Z}^3\\ a>0}} \begin{bmatrix} a+b-1 \\ b \end{bmatrix}_q \begin{bmatrix} a+c-1 \\ c \end{bmatrix}_q \begin{bmatrix} b+c \\ b \end{bmatrix}_q = 1+\frac{1}{(q;q)^3}\left(\sum_{k\geq 2}(-1)^k\binom{k}{2}q^{\binom{k}{2}}\right).$$

3.2.4 Specialization of Grothendieck polynomials

Let p,b,n and i, $2i<n$ be positive integers. Denote by $\mathcal{T}^{(i)}_{p,b,n}$ the *trapezoid*, i.e. a convex quadrangle having vertices at the points

$$(ip,i),\ \ (ip,n-i),\ \ (b+ip,i)\ \ and\ \ (b+(n-i)p,n-i).$$

Definition 3.5 *Denote by $FC^{(i)}_{b,p,n}$ the set of lattice path from the point (ip,i) to that $(b+(n-i)p,n-i)$ with east steps $E=(0,1)$ and north steps $N=(1,0)$, which are located inside of the trapezoid $\mathcal{T}^{(i)}_{p,b,n}$.*

If $\mathfrak{p}\in FC^{(i)}_{b,p,n}$ is a path, we denote by $p(\mathfrak{p})$ the number of peaks, i.e.

$$p(\mathfrak{p})=NE(\mathfrak{p})+E_{in}(\mathfrak{p})+N_{end}(\mathfrak{p}),$$

where $NE(\mathfrak{p})$ is equal to the number of steps NE resting on path $\mathfrak{p}$; $E_{in}(\mathfrak{p})$ is equal to 1, if the path $\mathfrak{p}$ starts with step E and 0 otherwise; $N_{end}(\mathfrak{p})$ is equal to 1, if the path $\mathfrak{p}$ ends by the step N and 0 otherwise.

Note that the equality $N_{end}(\mathfrak{p})=1$ may happened only in the case $b=0$.

Definition 3.6 *Denote by $FC^{(k)}_{b,p,n}$ the set of k-tuples $\mathfrak{P}=(\mathfrak{p}_1,\ldots,\mathfrak{p}_k)$ of non-crossing lattice paths, where for each $i=1,\ldots,k$, $\mathfrak{p}_i\in FC^{(i)}_{b,p,n}$.*

Let

$$FC^{(k)}_{b,p,n}(\beta):=\sum_{\mathfrak{P}\in FC^{(k)}_{b,p,n}}\beta^{p(\mathfrak{P})}$$

denotes the generating function of the statistics $p(\mathfrak{P}):=\sum_{i=1}^k p(\mathfrak{p}_i)-k$.

Theorem 3.9 *The following equality holds*

$$\mathfrak{G}^{(\beta)}_{\varrho_{n,k,p,b}}(x_1=1,x_2=1,\ldots)=FC^{(k)}_{p,b,n+k}(\beta+1).$$

3.3 The "longest element" and Chan–Robbins polytope

Assume additionally, cf [40], **6.C8**, (d), that the condition (a) in Definition 3.1 is replaced by that

(a') : x_{ij} and x_{kl} <u>commute</u> for all i, j, k and l.

Consider the element $w_0 := \prod_{1\le i<j\le n} x_{ij}$. Let us bring the element w_0 to the reduced form, that is, let us consecutively apply the defining relations (a') and (b) to the element w_0 in any order until unable to do so. Denote the resulting polynomial by $Q_n(x_{ij};\beta)$. Note that the polynomial itself <u>depends</u> on the order in which the relations (a') and (b) are applied.

We denote by $Q_n(\beta)$ the specialization $x_{ij} = 1$ for all i and j, of the polynomial $Q_n(x_{ij};\beta)$; y $Q_n(\beta,t)$ the specialization $x_{ij} = 1$, if $(i,j) \ne (1,n)$, and $x_{1,n} = t$, and by $Q_n(z_1,\dots,z_{i-1})$ the specialization $x_{ij} = z_i$.

Example 3.6

$Q_3(\beta) = (2,1) = 1 + (\beta+1)$, $\quad Q_4(\beta) = (10,13,4) = 1 + 5(\beta+1) + 4(\beta+1)^2$,
$Q_4(\beta,t) = t^4 + t\ (1 + 2\,t^2 + 2\,t^3)\ (\beta+1) + (t+t^2)^2\ (\beta+1)^2$.
$Q_4(z_1,z_2,z_3) = z_1^7 z_2^3 z_3\ \mathfrak{G}^{(\beta)}_{[1,5,3,4,2]}(z_1^{-1}, z_2^{-1}, z_3^{-1}, z_1^{-1})$,
$Q_5(\beta) = (140,336,280,92,9) = 1 + 16(\beta+1) + 58(\beta+1)^2 + 56(\beta+1)^3 + 9(\beta+1)^4$,
$Q_6(\beta) = 1+42(\beta+1)+448(\beta+1)^2+1674(\beta+1)^3+2364(\beta+1)^4+1182(\beta+1)^5+169(\beta+1)^6$.

What one can say about the polynomial $Q_n(\beta) := Q_n(x_{ij};\beta)|_{x_{ij}=1,\forall i,j}$?

It is known, [40], **6.C8**, (d), that the constant term of the polynomial $Q_n(\beta)$ is equal to the product of Catalan numbers $\prod_{j=1}^{n-1} C_j$. It is not difficult to see that if $n \ge 3$, then $deg_\beta(Q_n(\beta)) = 2(n-3)$ and $Coeff_{[\beta+1]}(Q_n(\beta)) = 2^n - 1 - \binom{n+1}{2}$.

Theorem 3.10 One has

$$Q_n(\beta-1) = \Big(\sum_{m\ge 0} \iota(CR_{n+1},m)\ \beta^m\Big)\ (1-\beta)^{\binom{n+2}{2}+1},$$

where CR_m denotes the Chan–Robbins polytope [3], i.e. the convex polytope given by the following conditions :

$CR_m = \{(a_{ij}) \in Mat_{m\times m}(\mathbb{Z}_{\ge 0})\}$ such that
(1) $\sum_i a_{ij} = 1$, $\quad \sum_j a_{ij} = 1$,
(2) $a_{ij} = 0$, $\ if\ \ j > i+1$.

Here for any integral convex polytope $\mathcal{P} \subset \mathbb{Z}^d$, $\iota(\mathcal{P},n)$ denotes the number of integer points in the set $n\mathcal{P} \cap \mathbb{Z}^d$.

Conjecture 4 **(A)** *Let $n \ge 4$ and write*

$$Q_n(\beta,t) := \sum_{k=0}^{2n-6} (1+\beta)^k\ c_{k,n}(t), \quad \underline{then} \quad c_{k,n}(t) \in \mathbb{Z}_{\ge 0}[t].$$

(B) *All roots of the polynomial* $Q_n(\beta)$ *belong to the set* $\mathbb{R}_{<0}$.

Comments 3.7

(1) We **expect** that for each integer $n \ge 2$ the set

$$\Psi_{n+1} := \{w \in \mathbb{S}_{2n-1} \mid \mathfrak{S}_w(1) = \prod_{j=1}^{n} Cat_j\}$$

contains either one or two elements, whereas the set $\{w \in \mathbb{S}_{2n-2} \mid \mathfrak{S}_w(1) = \prod_{j=1}^{n} Cat_j\}$ is empty. For example. $\Psi_4 = \{\ [1,5,3,4,2]\ \}$, $\Psi_5 = \{\ [1,5,7,3,2,6,4],\ [1,5,4,7,2,6,3]\ \}$, $\Psi_6 = \{\ w := [1,3,2,8,6,9,4,5,7],\ w^{-1}\ \}$, $\Psi_7 = \{???\}$.

Question Does there exist a vexillary (grassmannian ?) permutation $w \in \mathbb{S}_\infty$ such that $\mathfrak{S}_w(1) = \prod_{j=1}^{n} Cat_j$?
For example, $w = [1,4,5,6,8,3,5,7] \in \mathbb{S}_8$ is a grassmannian permutation such that $\mathfrak{S}_w(1) = 140$, and $\mathfrak{R}_w(1,\beta) = (1,9,27,43,38,18,4)$.

Remark 3.3 *We expect that for $n \ge 5$ there are no permutations $w \in \mathbb{S}_\infty$ such that $Q_n(\beta) = \mathfrak{S}_w^{(\beta)}(1)$.*

(**2**) The numbers $\mathfrak{C}_n := \prod_{j=1}^{n} Cat_j$ appear also as the values of the *Kostant partition function* of the type A_{n-1} on some special vectors. Namely,

$$\mathfrak{C}_n = K_{\Phi(1^n)}(\gamma_n), \quad where \quad \gamma_n = (1,2,3,\ldots,n-1,-\binom{n}{2}),$$

see e.g. [40], **6.C10**, and [17], 173–178. More generally [17], (7,18), (7.25),one has

$$K_{\Phi(1^n)}(\gamma_{n,d}) = pp^{\delta_n}(d)\ \mathfrak{C}_{n-1} = \prod_{j=d}^{n+d-2} \frac{1}{2j+1}\binom{n+d+j}{2j},$$

where $\gamma_{n,d} = (d+1, d+2, \ldots, d+n-1, -n(2d+n-1)/2)$, and $pp^{\delta_n}(d)$ denotes the set of reverse (weak) plane partitions bounded by d and contained in the shape $\delta_n := (n-1, n-2, \ldots, 1)$. Clearly, $pp^{\delta_n}(1) = \prod_{1 \le i < j \le n} \frac{i+j+1}{i+j-1} = C_n$, where C_n is the n-th Catalan number [5].

Conjecture 5
For any permutation $w \in \mathbb{S}_n$ there exists a graph $\Gamma_w = (V,E)$, possibly with multiple edges, such that the reduced volume $\widetilde{vol}(\mathcal{F}_{\Gamma_w})$ of the *flow polytope* $\mathcal{F}_{\Gamma_w}$, see e.g. [39] for a definition of the former, is equal to $\mathfrak{S}_w(1)$. ■

For a family of vexillary permutations $w_{n,p}$ of the shape $\lambda = p\delta_{n+1}$ and flag $\phi = (1,2,\ldots,n-1,n)$ the corresponding graphs $\Gamma_{n,p}$ have been constructed in [29], Section 6. In this case the reduced volume of the flow polytope $\mathcal{F}_{\Gamma_{n,p}}$ is equal to the Fuss-Catalan number $\frac{1}{1+(n+1)p}\binom{(n+1)(p+1)}{n+1} = \mathfrak{S}_{w_{n,p}}(1)$, cf Corollary 3.2

Problems 3.3
(1) *Assume additionally to the conditions (a′) and (b) above that*

$$x_{ij}^2 = \beta\ x_{ij} + 1, \quad if \quad 1 \le i < j \le n.$$

What one can say about a reduced form of the element w_0 in this case ?

[5] For example, if $n = 3$, there exist 5 reverse (weak) plane partitions of shape $\delta_3 = (2,1)$ bounded by 1, namely reverse plane partitions $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix} \right\}$.

(2) *According to a result by S. Matsumoto and J. Novak [27], if $\pi \in \mathbb{S}_n$ is a permutation of the cyclic type $\lambda \vdash n$, then the total number of primitive factorizations (see definition in [27]) of π into product of $n-\ell(\lambda)$ transpositions, denoted by $Prim_{n-\ell(\lambda)}(\lambda)$, is equal to the product of Catalan numbers:*

$$Prim_{n-\ell(\lambda)}(\lambda) = \prod_{i=1}^{\ell(\lambda)} Cat_{\lambda_i-1}.$$

Recall that the Catalan number $Cat_n := C_n = \frac{1}{n}\binom{2n}{n}$. Now take $\lambda = (2,3,\ldots,n+1)$. Then

$$Q_n(1) = \prod_{a=1}^{n} Cat_a = Prim_{\binom{n}{2}}(\lambda).$$

Does there exist "a natural" bijection between the primitive factorizations and monomials which appear in the polynomial $Q_n(x_{ij};\beta)$?

Appendix Grothendieck polynomials

Definition A1 Let β be a parameter. The Id-Coxeter algebra $IdC_n(\beta)$ is an associative algebra over the ring of polynomials $\mathbb{Z}[\beta]$ generated by elements $\left\langle e_1,\ldots,e_{n-1}\right\rangle$ subject to the set of relations

- $e_ie_j = e_je_i$, if $\left|i-j\right| \geq 2$,
- $e_ie_je_i = e_je_ie_j$, if $\left|i-j\right| = 1$,
- $e_i^2 = \beta\ e_i$, $1 \leq i \leq n-1$.

It is well-known that the elements $\{e_w,\ w \in \mathbb{S}_n\}$ form a $\mathbb{Z}[\beta]$-linear basis of the algebra $IdC_n(\beta)$. Here for a permutation $w \in \mathbb{S}_n$ we denoted by e_w the product $e_{i_1}e_{i_2}\cdots e_{i_\ell} \in IdC_n(\beta)$, where $(i_1,i_2,\ldots,i_\ell)$ is any *reduced* word for a permutation w, i.e. $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ and $\ell = \ell(w)$ is the length of w.

Let $x_1,x_2,\ldots,x_{n-1},x_n = y, x_{n+1} = z,\ldots$ be a set of mutually commuting variables. We assume that x_i and e_j commute for all values of i and j. Let us define

$$h_i(x) = 1 + xe_i, \quad and \quad A_i(x) = \prod_{a=n-1}^{i} h_a(x), \quad i = 1,\ldots,n-1.$$

Lemma A1 One has

(1) (Addition formula)

$$h_i(x)\ h_i(y) = h_i(x \oplus y),$$

where we set $(x \oplus y) := x + y + \beta xy$;

(2) (Yang–Baxter relation)

$$h_i(x)h_{i+1}(x \oplus y)h_i(y) = h_{i+1}(y)h_i(x \oplus y)h_{i+1}(x).$$

Corollary A1

(1) $[h_{i+1}(x)h_i(x), h_{i+1}(y)h_i(y)] = 0.$
(2) $[A_i(x), A_i(y)] = 0, \ \ i = 1, 2, \ldots, n-1.$

The second equality follows from the first one by induction using the Addition formula, whereas the fist equality follows directly from the Yang–Baxter relation.

Definition A2 (Grothendieck expression)

$$\mathfrak{G}_n(x_1, \ldots, x_{n-1}) := A_1(x_1)A_2(x_2)\cdots A_{n-1}(x_{n-1}).$$

Theorem A ([11]) The following identity

$$\mathfrak{G}_n(x_1, \ldots, x_{n-1}) = \sum_{w \in \mathbb{S}_n} \mathfrak{G}_w^{(\beta)}(X_{n-1})\ e_w$$

holds in the algebra $IdC_n \otimes \mathbb{Z}[x_1, \ldots, x_{n-1}]$.

Definition A3 We will call polynomial $\mathfrak{G}_w^{(\beta)}(X_{n-1})$ as the β-Grothendieck polynomial corresponding to a permutation w.

Corollary A2

(1) If $\beta = -1$, the polynomials $\mathfrak{G}_w^{(-1)}(X_{n-1})$ coincide with the Grothendieck polynomials introduced by Lascoux and M.-P. Schützenberger [25].

(2) The β-Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}(X_{n-1})$ is divisible by $x_1^{w(1)-1}$.

(3) For any integer $k \in [1, n-1]$ the polynomial $\mathfrak{G}_w^{(\beta-1)}(x_k = q, x_a = 1, \forall a \neq k)$ is a polynomial in the variables q and β with non-negative integer coefficients.

Proof (Sketch) It is enough to show that the specialized Grothendieck expression $\mathfrak{G}_n(x_k = q, x_a = 1, \forall a \neq k)$ can be written in the algebra $IdC_n(\beta - 1) \otimes \mathbb{Z}[q, \beta]$ as a linear combination of elements $\{e_w\}_{w \in \mathbb{S}_n}$ with coefficients which are polynomials in the variables q and β with non-negative coefficients. Observe that one can rewrite the relation $e_k^2 = (\beta - 1)e_k$ in the following form $e_k(e_k + 1) = \beta\ e_k$. Now, all possible negative contributions to the expression $\mathfrak{G}_n(x_k = q, x_a = 1, \forall a \neq k)$ can appear only from products of a form $c_a(q) := (1 + qe_k)(1 + e_k)^a$. But using the Addition formula one can see that $(1 + qe_k)(1 + e_k) = 1 + (1 + q\beta)e_k$. It follows by induction on a that $c_a(q)$ is a polynomial in the variables q and β with non-negative coefficients.
■

Definition A4

- The double β-Grothendieck expression $\mathfrak{G}_n(X_n, Y_n)$ is defined as follows

$$\mathfrak{G}_n(X_n, Y_n) = \mathfrak{G}_n(X_n)\ \mathfrak{G}_n(-Y_n)^{-1} \in IdC_n(\beta) \otimes \mathbb{Z}[X_n, Y_n].$$

- The double β-Grothendieck polynomials $\{\mathfrak{G}_w(X_n, Y_n)\}_{w \in \mathbb{S}_n}$ are defined from the decomposition

$$\mathfrak{G}_n(X_n, Y_n) = \sum_{w \in \mathbb{S}_n} \mathfrak{G}_w(X_n, Y_n)\ e_w$$

of the double β-Grothendieck expression in the algebra $IdC_n(\beta)$.

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