数理解析研究所講究録
第 1797 巻 2012 年 187-203

Phase Space Feynman Path Integrals – as Analysis on Path Space via Piecewise Constant Paths

By

Naoto KUMANO-GO*

Abstract

This survey of [20] is based on the introductory talk at RIMS.

§ 1. Introduction

Let $T>0$ and $x\in\mathbf{R}^{d}$. We consider the fundamental solution $U(T,0)$ for the Schrödinger equation

$$\left(i\hbar\partial_{T}-H(T,x,\frac{\hbar}{i}\partial_{x})\right)U(T,0)=0,\ \ U(0,0)=I, \tag{1.1}$$

with the Planck parameter $0<\hbar<1$. By the Fourier transform with respect to $x_{0}\in\mathbf{R}^{d}$ and the inverse Fourier transform with respect to $\xi_{0}\in\mathbf{R}^{d}$, the identity operator I is given by

$$Iv(x)=v(x)=\left(\frac{1}{2\pi\hbar}\right)^{d}\int_{\mathbf{R}^{2d}}e^{\frac{i}{\hbar}(x-x_{0})\cdot\xi_{0}}v(x_{0})dx_{0}d\xi_{0},$$

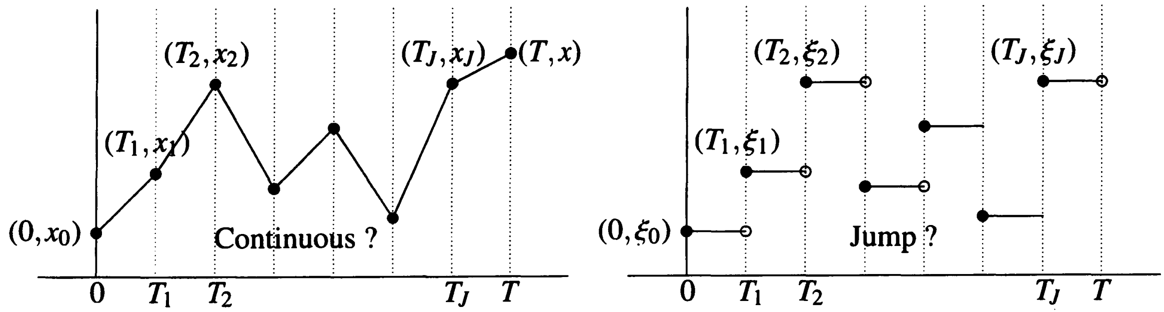
and the Hamiltonian operator $H(T,x,\frac{\hbar}{i}\partial_{x})$ is given by

$$H(T,x,\frac{\hbar}{i}\partial_{x})v(x)=\left(\frac{1}{2\pi\hbar}\right)^{d}\int_{\mathbf{R}^{2d}}e^{\frac{i}{\hbar}(x-x_{0})\cdot\xi_{0}}H(T,x,\xi_{0})v(x_{0})dx_{0}d\xi_{0}.$$

As an approximation of $U(T,0)$, we use the operator $I(T,0)$ given by

$$I(T,0)v(x)=\left(\frac{1}{2\pi\hbar}\right)^{d}\int_{\mathbf{R}^{2d}}e^{\frac{i}{\hbar}(x-x_{0})\cdot\xi_{0}}e^{-\frac{i}{\hbar}\int_{0}^{T}H(t,x,\xi_{0})dt}v(x_{0})dx_{0}d\xi_{0}.$$

2010 Mathematics Subject Classification(s): 81S40 ; 35S30 ; 81Q20 ; 58D30
Key Words: Path integrals, Fourier integral operators, Semiclassical approximation
Supported by JSPS KAKENHI(C)21540196
*Division of Liberal Arts, Kogakuin University, Tokyo 163-8677, Japan.



What is the position path q ? What is the momentum path p ?

Figure 1.

For any division $\Delta_{T,0} : T = T_{J+1} > T_J > \cdots > T_1 > T_0 = 0$ of $[0,T]$, we have

$$U(T,0)v(x) = U(T,T_J)U(T_J,T_{J-1})\cdots U(T_2,T_1)U(T_1,0)v(x).$$

Set $t_j = T_j - T_{j-1}$ and $|\Delta_{T,0}| = \max_{1\leq j\leq J+1} t_j$. Under some condition, using $I(T_j,T_{j-1})$ as an approximation of $U(T_j,T_{j-1})$ as $|\Delta_{T,0}| \to 0$, we can get

$$\begin{aligned}
(1.2) \qquad U(T,0)v(x) &= \lim_{|\Delta_{T,0}|\to 0} I(T,T_J)I(T_J,T_{J-1})\cdots I(T_2,T_1)I(T_1,0)v(x) \\
&= \lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^{d(J+1)} \int_{\mathbf{R}^{2d(J+1)}} e^{\frac{i}{\hbar}\sum_{j=1}^{J+1}((x_j-x_{j-1})\cdot\xi_{j-1} - \int_{T_{j-1}}^{T_j} H(t,x_j,\xi_{j-1})dt)} \\
&\qquad \times v(x_0)\prod_{j=0}^{J} dx_j d\xi_j,
\end{aligned}$$

with $x = x_{J+1}$. When T is small, we consider the function $U(T,0,x,\xi_0)$ satisfying

$$U(T,0)v(x) = \left(\frac{1}{2\pi\hbar}\right)^{d}\int_{\mathbf{R}^{2d}} e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0} U(T,0,x,\xi_0)v(x_0)dx_0d\xi_0.$$

Then we formally write

$$\begin{aligned}
(1.3) \qquad & e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0} U(T,0,x,\xi_0) \\
&= \lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^{dJ} \int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\sum_{j=1}^{J+1}((x_j-x_{j-1})\cdot\xi_{j-1} - \int_{T_{j-1}}^{T_j} H(t,x_j,\xi_{j-1})dt)} \prod_{j=1}^{J} dx_j d\xi_j.
\end{aligned}$$

According to R. P. Feynman [8, Appendix B], we introduce the position path $q(t)$ and the momentum path $p(t)$ with $q(T_j) = x_j$ and $p(T_j) = \xi_j$ (Figure 1). Let $\phi[q,p]$ be the action given by

$$\phi[q,p] = \int_{[0,T)} p(t)\cdot dq(t) - \int_{[0,T)} H(t,q(t),p(t))dt.$$

for the phase space path (q,p) with $q(0)=x_0$, $q(T)=x$ and $p(0)=\xi_0$. Then we formally write

$$e^{\frac{i}{\hbar}(x-x_0)\cdot\xi_0}U(T,0,x,\xi_0)=\int e^{\frac{i}{\hbar}\phi[q,p]}\mathcal{D}[q,p]. \tag{1.4}$$

Here the path integral $\int\sim\mathcal{D}[q,p]$ is a new sum over all the paths (q,p). The expressions (1.2) and (1.3) are now called the time slicing approximation.

However, in the sense of mathematics, the measure $\mathcal{D}[q,p]$ of the path integral (1.4) does not exist. Why can we say (1.4) is an integral ? In the sense of the uncertain principle, we can not have the position $q(t)$ and the momentum $p(t)$ at the same time t. Furthermore, the convergence (1.2) in the sense of operator does not distinguish the position x_0 and the momentum ξ_0. Why can we say $(q.p)$ is a phase space path ?

In [20], using piecewise constant paths, we proved the existence of the phase space Feynman path integrals

$$\int e^{\frac{i}{\hbar}\phi[q,p]}F[q,p]\mathcal{D}[q,p], \tag{1.5}$$

with general functional $F[q,p]$ as integrand. More precisely, we gave the two general classes $\mathcal{F}_{\mathcal{Q}}, \mathcal{F}_{\mathcal{P}}$ such that for any $F[q,p]\in\mathcal{F}_{\mathcal{Q}}$ or $\mathcal{F}_{\mathcal{P}}$, the time slicing approximation of (1.5) converges uniformly on compact subsets with respect to the endpoint x of position paths and the starting point ξ_0 of momentum paths. In this survey, we explain some properties of the phase space path integrals along the talk at RIMS.

Remark. For the phase space path integral (1.4) via Fourier integral operators, see H. Kumano-go–H. Kitada [17] and N. Kumano-go [19]. We regard (1.4) as the particular case of (1.5) with $F[q,p]=1$. Using broken line paths of position and piecewise constant paths of momentum, W. Ichinose [14] gave some functionals $F[q,p]=\prod_{k=1}^{K}B_k(q(\tau_k),p(\tau_k))$, $0<\tau_1<\tau_2<\cdots<\tau_K<T$ for which the time slicing approximations of (1.5) diverge as an operator. We exclude these functionals from our classes $\mathcal{F}_{\mathcal{Q}}, \mathcal{F}_{\mathcal{P}}$ to avoid the uncertain principle.

Remark. Inspired by the forward and backward approach of K. L. Chung–J.-C. Zambrini [4, §2.4], we use left-continuous paths and right-continuous paths. Furthermore, inspired by L. S. Shulman [25, §31], we pay attention to the operations which are valid in the phase space path integrals.

Since [8, Appendix B], the phase space path integral (1.4) has been rediscovered repeatedly (cf. W. Tobocman [26], H. Davies [6], C. Garrod [10]) and developed in various forms (cf. L. S. Schulman [25, §31], H. Kleinert [22], C. Grosche–F. Steiner [12], P. Cartier–C. DeWitt-Morette [3, §3.4], J. R. Klauder [21, §6.2]). For giving a well-defined mathematical meaning, various approaches have been proposed. C. DeWitt-Morette–A. Maheshwari–B. Nelson [7] and M. M. Mizrahi [24] introduced the formulation without limiting procedure. K. Gawedzki [11] used the technique analogous to that used by K. Itô [15]. I. Daubechies–J. R. Klauder [5] presented the phase space path integral via analytic continuation from Wiener measure. Furthermore, S. Albeverio–G. Guatteri–S. Mazzucchi [2] (cf. [1, §10.5.3], [23, §3.3]) realized the phase space path integral as an infinite dimensional oscillatory integral. O. G. Smolyanov–A. G. Tokarev–A. Truman [27] formulated the phase space path integral via Chernoff formula. For the main part of [8], G. W. Johnson–M. Lapidus [16] and T. L. Gill–W. W. Zachary [13] developed Feynman's operational calculus.

§ 2. Existence of Phase Space Path Integrals

Our assumption for the Hamiltonian function $H(t,x,\xi)$ of (1.1) is the following.

Assumption 1 (Hamiltonian function). *$H(t,x,\xi)$ is a real-valued function of (t,x,ξ) in $\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d$, and for any multi-indices α, β, $\partial_x^\alpha\partial_\xi^\beta H(t,x,\xi)$ is continuous. For any non-negative integer k, there exists a positive constant κ_k such that*

$$|\partial_x^\alpha\partial_\xi^\beta H(t,x,\xi)| \le \kappa_k(1+|x|+|\xi|)^{\max(2-|\alpha+\beta|,0)},$$

for any multi-indices α, β with $|\alpha+\beta|=k$.

A typical example of the Hamiltonian operator $H(t,x,\frac{\hbar}{i}\partial_x)$ of (1.1) is the following.

Example 1 (Hamiltonian operator).

$$\begin{aligned} H(t,x,\frac{\hbar}{i}\partial_x) = &\sum_{j,k=1}^{d}(a_{j,k}(t)\frac{\hbar}{i}\partial_{x_j}\frac{\hbar}{i}\partial_{x_k}+b_{j,k}(t)x_j\frac{\hbar}{i}\partial_{x_k}+c_{j,k}(t)x_jx_k)\\ &+\sum_{j=1}^{d}(a_j(t)\frac{\hbar}{i}\partial_{x_j}+b_j(t)x_j)+c(t,x). \end{aligned}$$

Here $a_{j,k}(t)$, $b_{j,k}(t)$, $c_{j,k}(t)$, $a_j(t)$, $b_j(t)$ and $\partial_x^\alpha c(t,x)$ with any multi-index α are real-valued continuous bounded functions.

Let $\Delta_{T,0}=(T_{J+1},T_J,\dots,T_1,T_0)$ be any division of the interval $[0,T]$ given by

$$\Delta_{T,0}: T=T_{J+1}>T_J>\cdots>T_1>T_0=0.$$

Set $x_{J+1}=x$. Let $x_j\in\mathbf{R}^d$ and $\xi_j\in\mathbf{R}^d$ for $j=1,2,\dots,J$. We define the position path

$$q_{\Delta_{T,0}}=q_{\Delta_{T,0}}(t,x_{J+1},x_J,\dots,x_1,x_0)$$

by $q_{\Delta_{T,0}}(0)=x_0$, $q_{\Delta_{T,0}}(t)=x_j$, $T_{j-1}<t\le T_j$ and the momentum path

$$p_{\Delta_{T,0}}=p_{\Delta_{T,0}}(t,\xi_J,\dots,\xi_1,\xi_0)$$

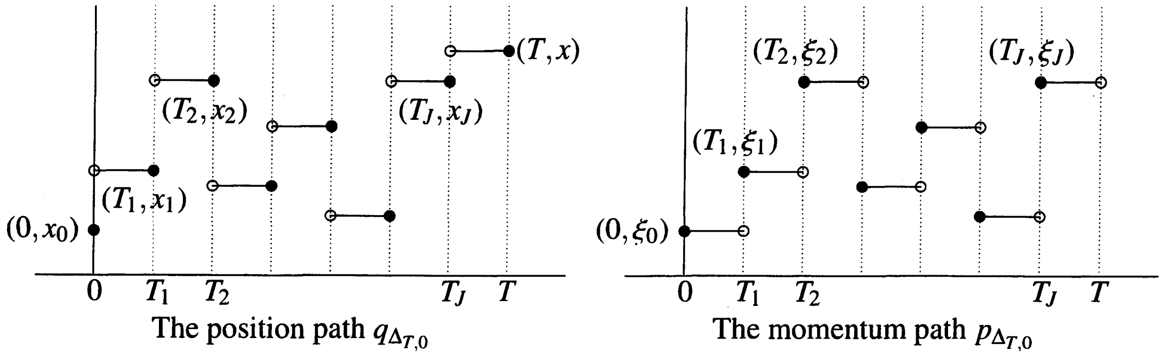


Figure 2.

by $p_{\Delta_{T,0}}(t) = \xi_{j-1}$, $T_{j-1} \leq t < T_j$ for $j = 1, 2, \ldots, J, J+1$ (Figure 2). Let $t_j = T_j - T_{j-1}$ and $|\Delta_{T,0}| = \max_{1 \leq j \leq J+1} t_j$. According to Feynman's first definition of (1.4), we define the phase space path integral (1.5) with the general functional $F[q,p]$ as integrand by

$$
\begin{aligned}
(2.1) \qquad & \int e^{\frac{i}{\hbar}\phi[q,p]} F[q,p] \mathcal{D}[q,p] \\
&= \lim_{|\Delta_{T,0}| \to 0} \left(\frac{1}{2\pi\hbar} \right)^{dJ} \int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]} F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] \prod_{j=1}^{J} dx_j d\xi_j ,
\end{aligned}
$$

if the limit of the right hand side exists.

Theorem 1 (Existence of phase space path integrals). *Let T be sufficiently small. Then, for any $F[q,p] \in \mathcal{F}_{\mathcal{Q}}$ or $\mathcal{F}_{\mathcal{P}}$, the right hand side* (2.1) *converges uniformly on compact sets of* $\mathbf{R}^{3d}$ *with respect to* (x, ξ_0, x_0), *i.e., the phase space path integral* (2.1) *is well-defined.*

For simplicity, we will state the definition of the classes $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ in §5. Because if we apply Theorem 2 to Example 2.1, we can produce many $F[q,p] \in \mathcal{F}_{\mathcal{Q}}$ or $\mathcal{F}_{\mathcal{P}}$.

Remark. Even when $F[q,p] = 1$, each integral of the right hand side of (2.1) does not converge absolutely.

$$
\int_{\mathbf{R}^{2d}} d\xi_j dx_j = \infty .
$$

Furthermore, the number J of integrals (division points) tends to ∞.

$$
\infty \times \infty \times \infty \times \infty \times \cdots\cdots\cdots , \quad J \to \infty .
$$

We treat the multiple integral of (2.1) as an oscillatory integral (cf. H. Kumano-go [18, §1.6]).

Though the functionals $\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$, $F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$ are the functions $\phi_{\Delta_{T,0}}$, $F_{\Delta_{T,0}}$ given by

$$\begin{aligned}
\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] &= \sum_{j=1}^{J+1} \int_{[T_{j-1},T_j)} p_{\Delta_{T,0}} \cdot dq_{\Delta_{T,0}}(t) - \sum_{j=1}^{J+1} \int_{[T_{j-1},T_j)} H(t, q_{\Delta_{T,0}}, p_{\Delta_{T,0}})dt \\
&= \sum_{j=1}^{J+1} (x_j - x_{j-1}) \cdot \xi_{j-1} - \sum_{j=1}^{J+1} \int_{[T_{j-1},T_j)} H(t, x_j, \xi_{j-1})dt \\
&= \phi_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0), \\
F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] &= F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \ldots, \xi_1, x_1, \xi_0, x_0),
\end{aligned}$$

we keep $\phi[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$, $F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}]$ in the multiple integral of (2.1).

Roughly speaking, typical examples of $F[q,p] \in \mathcal{F}_\mathcal{Q}$ or $\mathcal{F}_\mathcal{P}$ are the following.

Example 2.1 ($F[q,p] \in \mathcal{F}_\mathcal{Q}$ or $\mathcal{F}_\mathcal{P}$). For the details, see Theorem 3.

(1) If $|\partial_x^\alpha B(t,x)| \le C_\alpha(1+|x|)^m$, the functionals independent of p or q,

$$F[q] = B(t, q(t)) \in \mathcal{F}_\mathcal{Q},$$
$$F[p] = B(t, p(t)) \in \mathcal{F}_\mathcal{P}.$$

In particular, $F[q,p] = 1 \in \mathcal{F}_\mathcal{Q} \cap \mathcal{F}_\mathcal{P}$.

(2) If $|\partial_x^\alpha \partial_\xi^\beta B(t,x,\xi)| \le C_{\alpha,\beta}(1+|x|+|\xi|)^m$, then

$$F[q,p] = \int_{[T',T'')} B(t, q(t), p(t))dt \in \mathcal{F}_\mathcal{Q} \cap \mathcal{F}_\mathcal{P}.$$

(3) If $|\partial_x^\alpha \partial_\xi^\beta B(t,x,\xi)| \le C_{\alpha,\beta}$, then

$$F[q,p] = e^{\int_{[T',T'')} B(t,q(t),p(t))dt} \in \mathcal{F}_\mathcal{Q} \cap \mathcal{F}_\mathcal{P}.$$

To explain some properties of the classes $\mathcal{F}_\mathcal{Q}$, $\mathcal{F}_\mathcal{P}$, we prepare some notations.

Definition 2.2 (Two spaces $\mathcal{Q}$, $\mathcal{P}$ of piecewise constant paths).

(1) We write $q \in \mathcal{Q}$ if q is left-continuous and piecewise constant, i.e., there exists $q_{\Delta_{T,0}}$ such that $q = q_{\Delta_{T,0}}$.
(2) We write $p \in \mathcal{P}$ if p is right-continuous and piecewise constant, i.e., there exists $p_{\Delta_{T,0}}$ such that $p = p_{\Delta_{T,0}}$.

Definition 2.3 (Fuctional derivatives). For any q, $q' \in \mathcal{Q}$ and any p, $p' \in \mathcal{P}$, we define the functional derivatives $D_{q'}F[q,p]$ and $D_{p'}F[q,p]$ by

$$D_{q'}F[q,p] = \frac{\partial}{\partial\theta} F[q+\theta q', p]\Big|_{\theta=0}, \quad D_{p'}F[q,p] = \frac{\partial}{\partial\theta} F[q, p+\theta p']\Big|_{\theta=0}.$$

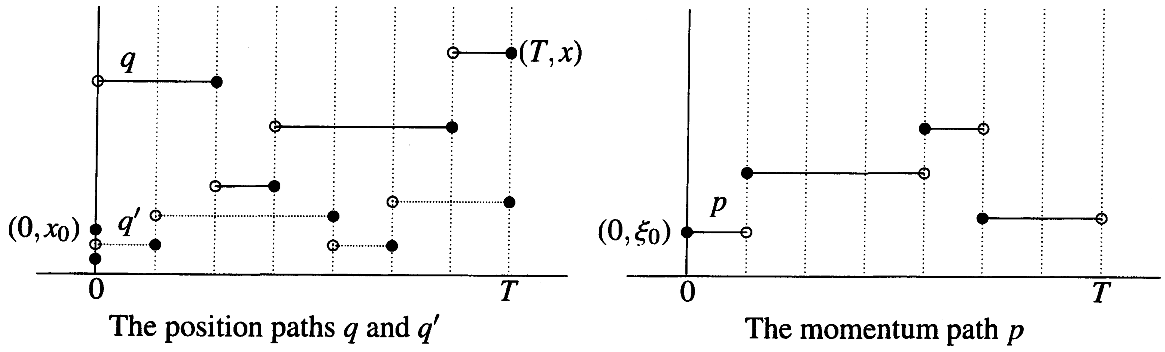


Figure 3.

Remark. For any q, $q' \in \mathcal{Q}$ and $p \in \mathcal{P}$, choose $\Delta_{T,0}$ which contains all times when q, q' or p breaks (Figure 3). Set $q(T_j) = x_j$, $q'(T_j) = x'_j$ for $j = 0, 1, \ldots, J, J+1$ and $p(T_{j-1}) = \xi_{j-1}$ for $j = 1, 2, \ldots, J, J+1$. Since $(q+\theta q')(0) = x_0 + \theta x'_0$, $(q+\theta q')(t) = x_j + \theta x'_j$ on $(T_{j-1}, T_j]$ and $p(t) = \xi_{j-1}$ on $[T_{j-1}, T_j)$ for $j = 1, 2, \ldots, J, J+1$, we have

$$F[q+\theta q', p] = F_{\Delta_{T,0}}(x_{J+1} + \theta x'_{J+1}, \xi_J, x_J + \theta x'_J, \ldots, \xi_0, x_0 + \theta x'_0).$$

Hence we can treat $D_{q'}F[q,p]$ as a finite sum of functions, i.e.,

$$D_{q'}F[q,p] = \frac{\partial}{\partial\theta} F[q+\theta q', p]\bigg|_{\theta=0} = \sum_{j=0}^{J+1} (\partial_{x_j} F_{\Delta_{T,0}})(x_{J+1}, \xi_J, \ldots, \xi_0, x_0) \cdot x'_j .$$

Because we restrict the directions of functional derivatives to piecewise constant paths, the functional derivatives are easy to treat.

Theorem 2 (Smooth algebra).

(1) *For any $F[q,p]$, $G[q,p] \in \mathcal{F}_{\mathcal{Q}}$, any $q' \in \mathcal{Q}$, any $p' \in \mathcal{P}$ and any real $d \times d$ matrices A, B, we have*

$$F[q,p] + G[q,p] \in \mathcal{F}_{\mathcal{Q}}\,,\ F[q,p]G[q,p] \in \mathcal{F}_{\mathcal{Q}}\,,\ F[q+q', p+p'] \in \mathcal{F}_{\mathcal{Q}}$$

$$F[Aq, Bp] \in \mathcal{F}_{\mathcal{Q}}\,,\ D_{q'}F[q,p] \in \mathcal{F}_{\mathcal{Q}}\,,\ D_{p'}F[q,p] \in \mathcal{F}_{\mathcal{Q}}$$

(2) *For any $F[q,p]$, $G[q,p] \in \mathcal{F}_{\mathcal{P}}$, any $q' \in \mathcal{Q}$, any $p' \in \mathcal{P}$ and any real $d \times d$ matrices A, B, we have*

$$F[q,p] + G[q,p] \in \mathcal{F}_{\mathcal{P}}\,,\ F[q,p]G[q,p] \in \mathcal{F}_{\mathcal{P}}\,,\ F[q+q', p+p'] \in \mathcal{F}_{\mathcal{P}}$$

$$F[Aq, Bp] \in \mathcal{F}_{\mathcal{P}}\,,\ D_{q'}F[q,p] \in \mathcal{F}_{\mathcal{P}}\,,\ D_{p'}F[q,p] \in \mathcal{F}_{\mathcal{P}}$$

Remark. The two classes $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ are closed under addition, multiplication, translation, real linear transformation and functional differentiation. However, as we will see in Theorems 4 and

6, because q', p' are piecewise constant, the part $\int_{[0,T)} p(t) \cdot dq(t)$ of the action $\phi[q,p]$ does not always have good properties under these operations. Therefore, we must pay attention to which operations are valid in the phase space path integrals.

§ 3. Properties of Phase Space Path Integrals

Assuming Theorems 1, 2, we explain the properties of the phase space path integrals.

Theorem 3 (Fubini type). *Let m be a non-negative integer.*

(a) *Assume that for any multi-index α, $\partial_x^\alpha B(t,x)$ is continuous on $\mathbf{R} \times \mathbf{R}^d$ and there exists a positive constant C_α such that $|\partial_x^\alpha B(t,x)| \leq C_\alpha (1+|x|)^m$. Then the values at the fixed time t, $0 \leq t \leq T$*

$$F[q] = B(t,q(t)) \in \mathcal{F}_Q,$$
$$F[p] = B(t,p(t)) \in \mathcal{F}_\mathcal{P}.$$

In particular, $F[q,p] = 1 \in \mathcal{F}_Q \cap \mathcal{F}_\mathcal{P}$.

(b) *Let $0 \leq T' \leq T'' \leq T$. Assume that for any multi-indices α, β, $\partial_x^\alpha \partial_\xi^\beta B(t,x,\xi)$ is continuous on $\mathbf{R} \times \mathbf{R}^d \times \mathbf{R}^d$ and there exists a positive constant $C_{\alpha,\beta}$ such that $|\partial_x^\alpha \partial_\xi^\beta B(t,x,\xi)| \leq C_{\alpha,\beta}(1+|x|+|\xi|)^m$. Then the integral*

$$F[q,p] = \int_{[T',T'')} B(t,q(t),p(t))dt \in \mathcal{F}_Q \cap \mathcal{F}_\mathcal{P}.$$

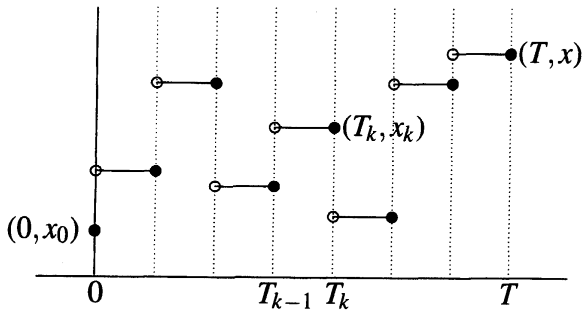
Furthermore let T be sufficiently small. Then we have the following:

(1) *For any $F[q,p] \in \mathcal{F}_Q$ including $F[q,p] = 1$, we have*

$$\int_{q(T)=x,\, p(0)=\xi_0, q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]} \Big(\int_{[T',T'')} B(t,q(t))dt \Big) F[q,p] \mathcal{D}[q,p]$$
$$= \int_{[T',T'')} \Big(\int_{q(T)=x,\, p(0)=\xi_0, q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]} B(t,q(t)) F[q,p] \mathcal{D}[q,p] \Big) dt.$$

(2) *For any $F[q,p] \in \mathcal{F}_\mathcal{P}$ including $F[q,p] = 1$, we have*

$$\int_{q(T)=x,\, p(0)=\xi_0, q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]} \Big(\int_{[T',T'')} B(t,p(t))dt \Big) F[q,p] \mathcal{D}[q,p]$$
$$= \int_{[T',T'')} \Big(\int_{q(T)=x,\, p(0)=\xi_0, q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]} B(t,p(t)) F[q,p] \mathcal{D}[q,p] \Big) dt.$$



The position path $q_{\Delta_{T,0}}$

Figure 4.

Remark (Perturbative expansion). If $|\partial_x^\alpha B(t,x)| \le C_\alpha$, we have

$$
\begin{aligned}
&\int e^{\frac{i}{\hbar}\phi[q,p]+\frac{i}{\hbar}\int_{[0,T)}B(\tau,q(\tau))d\tau}\mathcal{D}[q,p] \\
&= \sum_{n=0}^{\infty}\left(\frac{i}{\hbar}\right)^n \int_{[0,T)} d\tau_n \int_{[0,\tau_n)} d\tau_{n-1}\cdots\int_{[0,\tau_2)} d\tau_1 \\
&\times \int e^{\frac{i}{\hbar}\phi[q,p]}B(\tau_n,q(\tau_n))B(\tau_{n-1},q(\tau_{n-1}))\cdots B(\tau_1,q(\tau_1))\mathcal{D}[q,p]\,.
\end{aligned}
$$

Proof of Theorem 3 (1). For simplicity, set $F[q,p]=1$ and $0=T'<T''=T$. Using $q_{\Delta_{T,0}}(t)=x_k$ on $(T_{k-1},T_k]$ (Figure 4) and $dt(\{T_k\})=0$, we have

$$
\begin{aligned}
&\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]}\int_{[0,T)}B(t,q(t))dt\mathcal{D}[q,p] \\
&= \lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]}\int_{[0,T)}B(t,q_{\Delta_{T,0}}(t))dt\prod_{j=1}^{J}d\xi_j dx_j \\
&= \lim_{|\Delta_{T,0}|\to 0}\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]}\sum_{k=1}^{J+1}\int_{[T_{k-1},T_k)}B(t,x_k)dt\prod_{j=1}^{J}d\xi_j dx_j\,. \\
&= \lim_{|\Delta_{T,0}|\to 0}\sum_{k=1}^{J+1}\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]}\int_{[T_{k-1},T_k)}B(t,x_k)dt\prod_{j=1}^{J}d\xi_j dx_j\,.
\end{aligned}
$$

Interchanging the order of the integration on $[T_{k-1},T_k)$ and the oscillatory integration on $\mathbf{R}^{2dJ}$, we have

$$
\begin{aligned}
&= \lim_{|\Delta_{T,0}|\to 0}\sum_{k=1}^{J+1}\int_{[T_{k-1},T_k)}\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]}B(t,x_k)\prod_{j=1}^{J}d\xi_j dx_j dt \\
&= \lim_{|\Delta_{T,0}|\to 0}\int_{[0,T)}\left(\frac{1}{2\pi\hbar}\right)^{dJ}\int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]}B(t,q_{\Delta_{T,0}}(t))\prod_{j=1}^{J}d\xi_j dx_j dt\,.
\end{aligned}
$$

Interchanging the order of the integration on $[0,T)$ and the limit, we have

$$= \int_{[0,T)} \lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi\hbar}\right)^{dJ} \int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]} B(t, q_{\Delta_{T,0}}(t)) \prod_{j=1}^{J} d\xi_j dx_j dt$$

$$= \int_{[0,T)} \int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]} B(t,q(t)) \mathcal{D}[q,p] dt\,. \ \square$$

Theorem 4 (Translation).

(1) *For any $p' \in \mathcal{P}$, we have*

$$e^{\frac{i}{\hbar}(\phi[q,p+p']-\phi[q,p])} \in \mathcal{F}_Q\,.$$

Let T be sufficiently small. Then for any $F[q,p] \in \mathcal{F}_Q$, we have

$$\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p+p']} F[q,p+p'] \mathcal{D}[q,p]$$

$$= \int_{q(T)=x,p(0)=\xi_0+p'(0),q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]} F[q,p] \mathcal{D}[q,p]\,.$$

(2) *For any $q' \in \mathcal{Q}$, we have*

$$e^{\frac{i}{\hbar}(\phi[q+q',p]-\phi[q,p])} \in \mathcal{F}_{\mathcal{P}}\,.$$

Let T be sufficiently small. Then for any $F[q,p] \in \mathcal{F}_{\mathcal{P}}$, we have

$$\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q+q',p]} F[q+q',p] \mathcal{D}[q,p]$$

$$= \int_{q(T)=x+q'(T),p(0)=\xi_0,q(0)=x_0+q'(0)} e^{\frac{i}{\hbar}\phi[q,p]} F[q,p] \mathcal{D}[q,p]\,.$$

Proof of Theorem 4 (1). By Theorem 1 and 2 (1), we have

$$\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p+p']} F[q,p+p'] \mathcal{D}[q,p]$$

$$= \int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]} e^{\frac{i}{\hbar}(\phi[q,p+p']-\phi[q,p])} F[q,p+p'] \mathcal{D}[q,p]$$

$$= \lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi\hbar}\right)^{dJ} \int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}+p']} F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}+p'] \prod_{j=1}^{J} d\xi_j dx_j\,,$$

with $q_{\Delta_{T,0}}(T_j) = x_j$ and $p_{\Delta_{T,0}}(T_j) = \xi_j$, $j = 1,2,\ldots,J$. Choose $\Delta_{T,0}$ which contains all times when the path p' breaks (Figure 5). Set $p'(t) = \xi'_{j-1}$ on $[T_{j-1},T_j)$ for $j = 1,2,\ldots,J+1$. Since $(p_{\Delta_{T,0}}+p')(t) = \xi_{j-1}+\xi'_{j-1}$ on $[T_{j-1},T_j)$, we can write

$$= \lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi\hbar}\right)^{dJ} \int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}(x_{J+1},\xi_J+\xi'_J,x_J,\ldots,\xi_1+\xi'_1,x_1,\xi_0+\xi'_0,x_0)}$$

$$\times F_{\Delta_{T,0}}(x_{J+1},\xi_J+\xi'_J,x_J,\ldots,\xi_1+\xi'_1,x_1,\xi_0+\xi'_0,x_0) \prod_{j=1}^{J} d\xi_j dx_j\,,$$

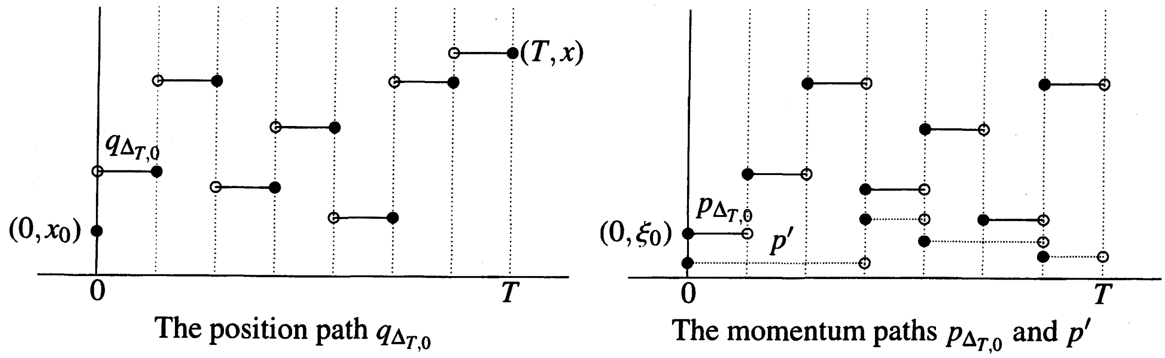


Figure 5.

By the change of variables: $\xi_j + \xi'_j \to \xi_j$, $j = 1,2,\ldots,J$, we have

$$= \lim_{|\Delta_{T,0}|\to 0} \left(\frac{1}{2\pi\hbar}\right)^{dJ} \int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0+\xi'_0,x_0)}$$

$$\times F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,\xi_1,x_1,\xi_0+\xi'_0,x_0)\prod_{j=1}^{J} d\xi_j dx_j .$$

Noting that $p'(0) = \xi'_0$, we can rewrite

$$= \int_{q(T)=x,p(0)=\xi_0+p'(0),q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]} F[q,p]\mathcal{D}[q,p] . \ \square$$

Theorem 5 (Orthogonal transformation). *Let T be sufficiently small. Then for any $F[q,p] \in \mathcal{F}_Q$ or $\mathcal{F}_P$ and any $d\times d$ orthogonal matrix Q,*

$$\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[Qq,Qp]} F[Qq,Qp]\mathcal{D}[q,p]$$
$$= \int_{q(T)=Qx,p(0)=Q\xi_0,q(0)=Qx_0} e^{\frac{i}{\hbar}\phi[q,p]} F[q,p]\mathcal{D}[q,p] .$$

Theorem 6 (Integration by parts).

(1) *For any $p' \in \mathcal{P}$, we have*

$$D_{p'}\phi[q,p] \in \mathcal{F}_Q .$$

Furthermore, let T be sufficiently small. Then for any $F[q,p] \in \mathcal{F}_Q$ and any $p' \in \mathcal{P}$ with $p'(0) = 0$,

$$\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]} (D_{p'}F)[q,p]\mathcal{D}[q,p]$$
$$= -\frac{i}{\hbar} \int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]} (D_{p'}\phi)[q,p]F[q,p]\mathcal{D}[q,p] .$$

(2) *For any $q' \in \mathcal{Q}$, we have*

$$D_{q'}\phi[q,p] \in \mathcal{F}_{\mathcal{P}}.$$

Furthermore, let T be sufficiently small. Then for any $F[q,p] \in \mathcal{F}_{\mathcal{P}}$ and any $q' \in \mathcal{Q}$ with $q'(T) = q'(0) = 0$,

$$\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]}(D_{q'}F)[q,p]\mathcal{D}[q,p]$$
$$= -\frac{i}{\hbar}\int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]}(D_{q'}\phi)[q,p]F[q,p]\mathcal{D}[q,p].$$

Remark (Analogues of canonical equations). Set $F[q,p] = 1$.

(1) For any $p' \in \mathcal{P}$ with $p'(0) = 0$, we have

$$0 = \int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]}\left(\int_{[0,T)} p'dq - (\partial_\xi H)(t,q,p)p'dt\right)\mathcal{D}[q,p].$$

(2) For any $q' \in \mathcal{Q}$ with $q'(T) = q'(0) = 0$, we have

$$0 = \int_{q(T)=x,p(0)=\xi_0,q(0)=x_0} e^{\frac{i}{\hbar}\phi[q,p]}\left(\int_{[0,T)} pdq' - (\partial_x H)(t,q,p)q'dt\right)\mathcal{D}[q,p].$$

Let T be small. For any $(x_{J+1},\xi_0) \in \mathbf{R}^d \times \mathbf{R}^d$, there exists the stationary point $(x_J^*,\xi_J^*,\ldots,x_1^*,\xi_1^*)$ of the phase function $\phi_{\Delta_{T,0}} = \phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]$ given by

$$(\partial_{(\xi_J,x_J,\ldots,\xi_1,x_1)}\phi_{\Delta_{T,0}})(x_{J+1},\xi_J^*,x_J^*,\ldots,\xi_1^*,x_1^*,\xi_0) = 0.$$

Pushing $(x_J^*,\xi_J^*,\ldots,x_1^*,\xi_1^*)$ into the Hessian of $\phi_{\Delta_{T,0}}$, we define $D(T,x_{J+1},\xi_0)$ by

$$D(T,x_{J+1},\xi_0)$$
$$= \lim_{|\Delta_{T,0}|\to 0}(-1)^{dJ}\det(\partial^2_{(\xi_J,x_J,\ldots,\xi_1,x_1)}\phi_{\Delta_{T,0}})(x_{J+1},x_J^*,\xi_J^*,\ldots,x_1^*,\xi_1^*,\xi_0).$$

Let $\bar{q}(t) = \bar{q}(t,x,\xi_0)$ and $\bar{p}(t) = \bar{p}(t,x,\xi_0)$ be the solution of the canonical equations

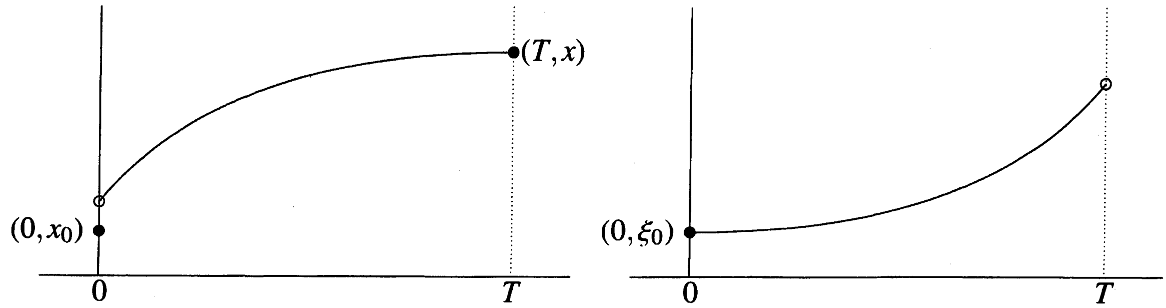
$$\partial_t\bar{q}(t) = (\partial_\xi H)(t,\bar{q}(t),\bar{p}(t)),\quad \partial_t\bar{p}(t) = -(\partial_x H)(t,\bar{q}(t),\bar{p}(t)),\quad 0 \le t \le T,$$

with $\bar{q}(T) = x$ and $\bar{p}(0) = \xi_0$. We define the bicharacteristic paths $q^\flat = q^\flat(t,x,\xi_0,x_0)$ and $p^\flat = p^\flat(t,x,\xi_0)$ by $q^\flat(t) = \bar{q}(t,x,\xi_0)$, $0 < t \le T$, $q^\flat(0) = x_0$ and $p^\flat(t) = \bar{p}(t,x,\xi_0)$, $0 \le t < T$ (Figure 6).

Then the remainder estimate for the semiclassical approximation of Hamiltonian type as $\hbar \to 0$ is the following.

Theorem 7 (Semiclassical approximation of Hamiltonian type as $\hbar \to 0$). *Let T be sufficiently small. Then, for any $F[q,p] \in \mathcal{F}_{\mathcal{Q}}$ or $\mathcal{F}_{\mathcal{P}}$, we have*

$$\int e^{\frac{i}{\hbar}\phi[q,p]}F[q,p]\mathcal{D}[q,p] = e^{\frac{i}{\hbar}\phi[q^\flat,p^\flat]}\left(D(T,x,\xi_0)^{-1/2}F[q^\flat,p^\flat] + \hbar\Upsilon(\hbar,T,x,\xi_0,x_0)\right).$$



The bicharacteristic path $q^\flat$ The bicharacteristic path $p^\flat$

Figure 6.

Here for any multi-indices α and β, the remainder term $\Upsilon(\hbar,T,x,\xi_0,x_0)$ satisfies

$$|\partial_x^\alpha \partial_{\xi_0}^\beta \Upsilon(\hbar,T,x,\xi_0,x_0)| \leq C_{\alpha,\beta}(1+|x|+|\xi_0|+|x_0|)^m,$$

with a positive constant $C_{\alpha,\beta}$.

§ 4. Proof for Theorems 1, 2 and 7

We explain the process of the proof for Theorems 1, 2 and 7. In order to prove the convergence of the multiple integral

$$\left(\frac{1}{2\pi\hbar}\right)^{dJ} \int_{\mathbf{R}^{2dJ}} e^{\frac{i}{\hbar}\phi[q_{\Delta_{T,0}},p_{\Delta_{T,0}}]} F[q_{\Delta_{T,0}},p_{\Delta_{T,0}}] \prod_{j=1}^{J} d\xi_j dx_j, \tag{4.1}$$

as $|\Delta_{T,0}| \to 0$, we have only to add many assumptions for

$$F_{\Delta_{T,0}}(x_{J+1},\xi_J,x_J,\ldots,x_1,\xi_0,x_0) = F[q_{\Delta_{T,0}},p_{\Delta_{T,0}}].$$

The assumptions should be closed under addition and multiplication. Then $\mathcal{F}_Q$, $\mathcal{F}_P$ will be closed under addition and multiplication. Do not consider other things. Then $\mathcal{F}_Q$, $\mathcal{F}_P$ will be larger as a set. If lucky, $\mathcal{F}_Q$, $\mathcal{F}_P$ will contain at least one example $F[q,p]=1$ as the fundamental solution for the Schrödinger equation. Our proof consists of 3 steps. As the first step, by an estimate of H. Kumano-go-Taniguchi's type [18, p.360, (6.94)], we control (4.1) by C^J with a positive constant C as $J \to \infty$. As the second step, by a stationary phase method of Fujiwara's type [9], we control (4.1) by C with a positive constant C independent of $J \to \infty$. As the last step, we add assumptions so that (4.1) converges as $|\Delta_{T,0}| \to 0$.

§ 5. Two classes $\mathcal{F}_Q$, $\mathcal{F}_P$ of functionals $F[q,p]$

In order to state the definition of the classes $\mathcal{F}_Q$, $\mathcal{F}_P$, we introduce the functional derivatives of higher order.

Definition 5.1 (Functional derivatives of higher order). For any division $\Delta_{T,0}$, we assume that

$$F[q_{\Delta_{T,0}}, p_{\Delta_{T,0}}] = F_{\Delta_{T,0}}(x_{J+1}, \xi_J, x_J, \dots, \xi_0, x_0) \in C^\infty(\mathbf{R}^{d(2J+3)}).$$

Let $L_{\mathcal{Q}}$, $L_{\mathcal{P}}$ be non-negative integers. For any q, $q_l \in \mathcal{Q}$, $l = 1, 2, \dots, L_{\mathcal{Q}}$ and any p, $p_l \in \mathcal{P}$, $l = 1, 2, \dots, L_{\mathcal{P}}$, we define the functional derivative $(\prod_{l=1}^{L_{\mathcal{Q}}} D_{q_l})(\prod_{l=1}^{L_{\mathcal{P}}} D_{p_l})F[q,p]$ of higher order by

$$\begin{aligned}
&(\prod_{l=1}^{L_{\mathcal{Q}}} D_{q_l})(\prod_{l=1}^{L_{\mathcal{P}}} D_{p_l})F[q,p] \\
&= (\prod_{l=1}^{L_{\mathcal{Q}}} \frac{\partial}{\partial \theta_l})(\prod_{l=1}^{L_{\mathcal{P}}} \frac{\partial}{\partial \vartheta_l})F[q + \sum_{l=1}^{L_{\mathcal{Q}}} \theta_l q_l, p + \sum_{l=1}^{L_{\mathcal{P}}} \vartheta_l p_l]\Bigg|_{\theta_1 = \dots = \theta_{L_{\mathcal{Q}}} = \vartheta_1 = \dots = \vartheta_{L_{\mathcal{P}}} = 0}.
\end{aligned}$$

The definition of the classes $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ of functionals $F[q,p]$ are the following.

Definition 5.2 (Two classes $\mathcal{F}_{\mathcal{Q}}$, $\mathcal{F}_{\mathcal{P}}$ of functionals $F[q,p]$). Let $F[q,p]$ be a functional of $q \in \mathcal{Q}$ and $p \in \mathcal{P}$.

(1) We write $F[q,p] \in \mathcal{F}_{\mathcal{Q}}$ if $F[q,p]$ satisfies Assumption 2 (1).

(2) We write $F[q,p] \in \mathcal{F}_{\mathcal{P}}$ if $F[q,p]$ satisfies Assumption 2 (2).

Assumption 2. *Let m be a non-negative integer. Let u_j, $j = 1, 2, \dots, J, J+1$ and U be non-negative parameters depending on $\Delta_{T,0}$ such that $\sum_{j=1}^{J+1} u_j = U < \infty$. Set $||q|| = \sup_{0 \le t \le T} |q(t)|$ and $||p|| = \sup_{0 \le t < T} |p(t)|$. For simplicity, we set $[0,0] = (T_{-1}, T_0]$.*

(1) *For any non-negative integer M, there exist positive constants A_M, X_M such that*

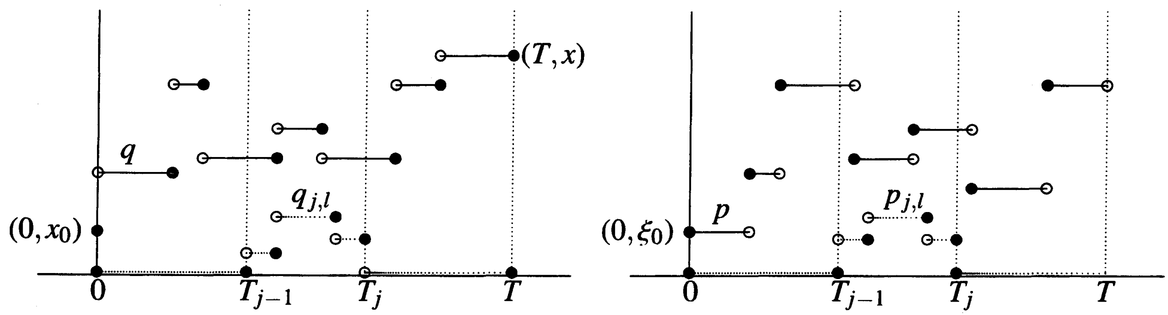
$$\begin{aligned}
&\left|(\prod_{j=0}^{J+1}\prod_{l=1}^{L_{\mathcal{Q},j}} D_{q_{j,l}})(\prod_{j=1}^{J+1}\prod_{l=1}^{L_{\mathcal{P},j}} D_{p_{j,l}})F[q,p]\right| \\
&\le A_M (X_M)^{J+1} (1 + ||q|| + ||p||)^m \\
&\times (\prod_{j=1}^{J+1} (t_j)^{\min(L_{\mathcal{P},j},1)}) \prod_{j=0}^{J+1}\prod_{l=1}^{L_{\mathcal{Q},j}} ||q_{j,l}|| \prod_{j=1}^{J+1}\prod_{l=1}^{L_{\mathcal{P},j}} ||p_{j,l}||, \\
&\left|(\prod_{j=0}^{J+1}\prod_{l=1}^{L_{\mathcal{Q},j}} D_{q_{j,l}})(\prod_{j=1}^{J+1}\prod_{l=1}^{L_{\mathcal{P},j}} D_{p_{j,l}}) D_{q_k} F[q,p]\right| \\
&\le A_M (X_M)^{J+1} (1 + ||q|| + ||p||)^m \\
&\times u_k ||q_k|| (\prod_{j=1, j \ne k}^{J+1} (t_j)^{\min(L_{\mathcal{P},j},1)}) \prod_{j=0}^{J+1}\prod_{l=1}^{L_{\mathcal{Q},j}} ||q_{j,l}|| \prod_{j=1}^{J+1}\prod_{l=1}^{L_{\mathcal{P},j}} ||p_{j,l}||,
\end{aligned}$$

for any division $\Delta_{T,0}$, *any* $L_{\mathcal{Q},j} = 0,1,\ldots,M$, *any* $L_{\mathcal{P},j} = 0,1,\ldots,M$, *any* $q_{j,l} \in \mathcal{Q}$ *with* $q_{j,l}(t) = 0$ *outside* $(T_{j-1}, T_j]$, *any* $q_k \in \mathcal{Q}$ *with* $q_k(t) = 0$ *outside* $(T_{k-1}, T_k]$, *and any* $p_{j,l} \in \mathcal{P}$ *with* $p_{j,l}(t) = 0$ *outside* $[T_{j-1}, T_j)$ (Figure 7).

(2) *For any non-negative integer* M, *there exist positive constants* A_M, X_M *such that*

$$\Big|(\prod_{j=0}^{J+1}\prod_{l=1}^{L_{\mathcal{Q},j}} D_{q_{j,l}})(\prod_{j=1}^{J+1}\prod_{l=1}^{L_{\mathcal{P},j}} D_{p_{j,l}})F[q,p]\Big|$$
$$\leq A_M(X_M)^{J+1}(1+\|q\|+\|p\|)^m$$
$$\times(\prod_{j=1}^{J+1}(t_j)^{\min(L_{\mathcal{Q},j},1)})\prod_{j=0}^{J+1}\prod_{l=1}^{L_{\mathcal{Q},j}}\|q_{j,l}\|\prod_{j=1}^{J+1}\prod_{l=1}^{L_{\mathcal{P},j}}\|p_{j,l}\|,$$
$$\Big|(\prod_{j=0}^{J+1}\prod_{l=1}^{L_{\mathcal{Q},j}} D_{q_{j,l}})(\prod_{j=1}^{J+1}\prod_{l=1}^{L_{\mathcal{P},j}} D_{p_{j,l}})D_{p_k}F[q,p]\Big|$$
$$\leq A_M(X_M)^{J+1}(1+\|q\|+\|p\|)^m$$
$$\times u_k\|p_k\|(\prod_{j=1,j\neq k}^{J+1}(t_j)^{\min(L_{\mathcal{Q},j},1)})\prod_{j=0}^{J+1}\prod_{l=1}^{L_{\mathcal{Q},j}}\|q_{j,l}\|\prod_{j=1}^{J+1}\prod_{l=1}^{L_{\mathcal{P},j}}\|p_{j,l}\|,$$

for any division $\Delta_{T,0}$, *any* $L_{\mathcal{Q},j} = 0,1,\ldots,M$, *any* $L_{\mathcal{P},j} = 0,1,\ldots,M$, *any* $q_{j,l} \in \mathcal{Q}$ *with* $q_{j,l}(t) = 0$ *outside* $(T_{j-1}, T_j]$, *any* $p_k \in \mathcal{P}$ *with* $p_k(t) = 0$ *outside* $[T_{k-1}, T_k)$, *and any* $p_{j,l} \in \mathcal{P}$ *with* $p_{j,l}(t) = 0$ *outside* $[T_{j-1}, T_j)$ (Figure 7).



The position paths q, $q_{j,l}$ The momentum paths p, $p_{j,l}$

Figure 7.

References

[1] Albeverio, S., Høegh-Krohn, and Mazzucchi, S., *Mathematical theory of Feynman path integrals*, Lecture notes of Math. **523**, Springer, Berlin, 1976 (The 2nd edition, 2008).

[2] Albeverio, S., Guatteri, G. and Mazzucchi, S., Phase space Feynman path integrals, *J. Math. Phys.* **43** (2002), 2847–2857.

[3] Cartier, P. and DeWitt-Morette, C., *Functional integration*, Cambridge university press (2006)

[4] Chung, K. L. and Zambrini, J. C., *Introduction to Random Time and Quantum Randomness*, World Scientific Pub Co Inc, 2003.

[5] Daubechies, I. and Klauder, J. R., Quantum mechanical path integrals with Wiener measure for all polynomial Hamiltonians. *J. Math. Phys.* **26** (1985), 2239-2256.

[6] Davies, H., Hamiltonian approach to the method of summation over Feynman histories, Proc. Camb. Phil. Soc. **59** (1963), 147–155.

[7] DeWitt-Morette, C., Maheshwari, A., and Nelson, B., Path integration in phase space, *Gen. Rel. and Grav.* 8 (1977), 581–593.

[8] Feynman, R. P., An operator calculus having applications in quantum electrodynamics, *Appendix B, Phys. Rev.* **84**, (1951), 108–236.

[9] Fujiwara, D., The stationary phase method with an estimate of the remainder term on a space of large dimension, *Nagoya Math. J.* **124** (1991), 61–97.

[10] Garrod, C., Hamiltonian path-integral methods, *Rev. Mod. Phys.* **38** (1966), 483–494.

[11] Gawedzki, K., Construction of quantum-mechanical dynamics by means of path integrals in phase space, *Rep. Math. Phys.* **6** (1974), 327–342.

[12] Grosche, C. and Steiner, F., Handbook of Feynman Path Integrals. Berlin-Heidelberg: Springer-Verlag, 1998.

[13] Gill, T. L. and Zachary, W. W., Foundations for relatives quantum theory. I. Feynman's operational calculus and the Dyson conjecture, *J. Math. Phys.* **43** (2002), 69–93.

[14] Ichinose, W., A mathematical theory of the phase space Feynman path integral of the functional, *Comm. Math. Phys.* **265** (2006), 739–779.

[15] Itô, K., Generalized uniform complex measure in Hilbert metric spaces and their applications to the Feynman path integrals, *Proc. 5th Berkeley Sympos. on Math. Statist. and Prob.* **2**, part 1, 1967, pp.145–161.

[16] Johnson, G. W. and Lapidus, M., *The Feynman Integral and Feynman's Operational Calculus*, Oxford University Press, New York, 2000.

[17] Kitada, H. and Kumano-go, H., A family of Fourier integral operators and the fundamental solution for a Schrödinger equation, *Osaka J. Math.* **18** (1981), 291-360.

[18] Kumano-go, H., *Pseudo-Differential Operators*, The MIT press, Cambridge, MA, 1981.

[19] Kumano-go, N., A construction of the fundamental solution for Schrödinger equations, *J. Math. Sci. Univ. Tokyo* **2** (1995), 441–498.

[20] Kumano-go, N., Phase space Feynman path integrals with smooth functional derivatives by time slicing approximation, Bull. Sci. math. **135** (2011), 936–987.

[21] Klauder, J. R., A Modern Approach to Functional Integration, Birkhauser, 2010.

[22] Kleinert, H., Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets, World Scientific Pub Co Inc, 1990 (5th edition 2009).

[23] Mazzucchi, S., Mathematical Feynman Path Integrals and Their Applications, World Scientific Pub Co Inc, 2009.

[24] Mizrahi, M. M., Phase space path integrals, without limiting procedure, *J. Math. Phys.* **19** (1978), 298-307; Erratum, ibid. **21** (1980), 1965.

[25] Schulman, L. S., *Techniques and Applications of Path Integration*, Monographs and Texts in Physics and Astronomy, Wiley-Interscience, New York, 1981 (with new supplementary section, Dover Publications, Inc, Mineola, New York 2005).

[26] Tobocman, W., Transition amplitudes as sums over histories, *Nuovo Cim.* **3** (1956) 1213–1229.

[27] Smolyanov, O. G., Tokarev, A. G. and Truman, A., Hamiltonian Feynman path integrals via Chernoff formula, *J. Math. Phys.* **43** (2002), 5161-5171.