Residual vanishing of concentration arising in the mean field equations

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Abstract

In this short report, we study the Sawada-Suzuki equation. In the positive case, we prove the property called *Residual vanishing* which means that a blow-up solution sequence (more precisely, its subsequence) converges to a finite sum of Dirac's measures in the sense of measure.

1 Introduction

In this report, we consider the Sawada-Suzuki equation ([6]):

\[
\begin{cases}
-\Delta v_n = \lambda_n \int_I \alpha \left( \frac{e^{\alpha v_n}}{\int_{\Omega} e^{\alpha v_n}} - \frac{1}{|\Omega|} \right) \mathcal{P}(d\alpha) & \text{in } \Omega \\
\int_{\Omega} v_n = 0,
\end{cases}
\]

(1.1)

where \((\lambda_n, v_n)\) is a solution sequence to (1.1), \(\lambda_n\) a non-negative number sequence tending to some non-negative number \(\lambda_0\), \(I = [-1,1]\), \(\Omega = (\Omega, g)\) a two dimensional orientable compact Riemannian manifold, and \(\mathcal{P}(d\alpha)\) a Borel probability measure on \(I\). According to the result of [4], the following alternative holds:

(i) *(Compactness)* \(\limsup_{narrow \infty} \|v_n\|_\infty < +\infty\), namely, there exist \(v \in \mathcal{E}\) and a subsequence \(\{v_{n_k}\} \subset \{v_n\}\) such that \(v_{n_k} \rightarrow v\) in \(\mathcal{E}\) as \(k \rightarrow \infty\), where

\[
\mathcal{E} = \left\{ v \in H^1(\Omega) \mid \int_{\Omega} v = 0 \right\}.
\]

(ii) *(Concentration)* \(\limsup_{narrow \infty} \|v_n\|_\infty = +\infty\), namely, the set \(S = S_+ \cup S_-\) is a non-empty and finite set, and there exists \(0 \leq s_+ \in L^1(\Omega)\) such that

\[
\nu_{\pm,n} := \lambda_n \int_{I_{\pm}} \frac{\alpha e^{\alpha v_n}}{\int_{I} e^{\alpha v_n}} \mathcal{P}(d\alpha) dx \rightarrow \nu_{\pm} = s_\pm dx + \sum_{x_0 \in S_\pm} m(x_0) \delta_{x_0}(dx) \quad (1.2)
\]

in \(\mathcal{M}(\Omega)\) with \(m(x_0) \geq 4\pi\) for all \(x_0 \in S_\pm\), where \(I_+ = (0,1]\), \(I_- = [-1,0)\), \(\delta_x\) is the Dirac measure supported at \(x\), \(\mathcal{M}(\Omega) = C(\Omega)^*\) and

\[
S_\pm = \{ x_0 \in \Omega \mid \text{there exists } \{x_n\} \subset \Omega \text{ such that } x_n \rightarrow x_0 \text{ and } v_n(x_n) \rightarrow \pm \infty \}. \quad (1.3)
\]
It is natural to ask whether $s \pm$ is zero or not in (1.2). If this is the case, we call this property residual vanishing in this report. In the positive case, we obtain

**Proposition 1.** If (ii) above holds and $I = I_+$, then $s = s_+ = 0$.

**Remark 1.** We note that $S = S_+$ in the case $I = I_+$, see [4] for details. The proof of this fact is based on the boundedness from below of the Green function associated to $-\Delta$ on $\Omega$, i.e.,

$$
\begin{cases}
-\Delta z G(x, y) = \delta_y - \frac{1}{|\Omega|} & \text{in } \Omega \\
\int_{\Omega} G(x, y) dx = 0, & \forall y \in \Omega,
\end{cases}
$$

see [1].

**Remark 2.** Residual vanishing also holds in the case $I = I_-$.

**Remark 3.** It is open whether residual vanishing is true or not in the general case. On the contrary, the problem is not solved even in the simple case $\mathcal{P}(d\alpha) = \frac{1}{2}(\delta_1 + \delta_{-1})$ treated in [5].

It is not difficult to show residual vanishing in the case $\mathcal{P}(d\alpha) = \delta_p$ for $p \in I$ by a direct application of the result (Theorem 3) of [2]. Just to be safe, we show it here, assuming $p = 1$ for simplicity, i.e.,

$$
-\Delta v_n = \lambda_n \left( \frac{e^{v_n}}{\int_{\Omega} e^{v_n}} - \frac{1}{|\Omega|} \right).
$$

Fix $x_0 \in S$. If it fails then it holds that

$$
\liminf_{n \to \infty} \int_{\Omega} e^{v_n} < +\infty.
$$

We introduce

$$
z_n = v_n - \log \int_{\Omega} e^{v_n}
$$

and obtain

$$
-\Delta z_n = \lambda_n e^{z_n} - \frac{\lambda_n}{|\Omega|} \quad \text{in } \Omega.
$$

It follows from the assumption of contradiction that $z_n \to +\infty$ (for some subsequence still denoted by the same notation). Since $\lambda_n$ is uniformly bounded and $-\lambda_n/|\Omega|$ can be regarded as a simple perturbed term, we can safely apply the result of [2] to the equation of $z_n$ to find that $z_n \to -\infty$ in $B(x_0, r_0) \setminus \{x_0\}$ for $0 < r_0 < 1$, where $B(x, r)$ denotes a disk centered at $x$ with radius $r$ for $x \in \mathbb{R}^2$ and $r > 0$, in particular, $B_r$ in the case $x = 0$. On the other hand, $z_n$ is bounded below in $B(x_0, r_0) \setminus \{x_0\}$ since $S = S_+ \neq \emptyset$, a contradiction.
Still, it seems to be difficult to directly apply the result of [2] to the general positive case. To overcome this difficulty, we introduce the key transformation, see (2.3) below, and then develop a blowup analysis.

This report consists of three sections. We prove Proposition 1 in Section 2, and several lemmas stated there are shown in Section 3.

2 Proof of Proposition 1

In this section, we write $I$ and $S$ by $I_+$ and $S_+$, respectively, in order to stress that we treat the positive case.

To prove the proposition, we have only to show

$$\mathcal{P}(\{\alpha \in I_+ \mid \liminf_{n \to \infty} \int_{\Omega} e^{\alpha v_n} = +\infty\}) = \mathcal{P}(I_+). \quad (2.1)$$

To confirm this, we fix $\omega \subset \subset \Omega \setminus S_+$. Then, it holds that

$$0 \leq \int_{\omega} s_+ dx = \lim_{n \to \infty} \int_{\omega} \nu_{+,n} = \lim_{n \to \infty} \int_{\omega} \int_{I_+} \left( \frac{\alpha e^{\alpha v_n}}{\int_{\Omega} e^{\alpha v_n}} - \frac{1}{|\Omega|} \mathcal{P}(d\alpha) \right)$$

$$\leq (\lambda_0 + 1) C(\omega) \lim_{n \to \infty} \int_{I_+} \frac{\mathcal{P}(d\alpha)}{\int_{\Omega} e^{\alpha v_n}} = 0$$

because $\lambda_n \to \lambda_0$ and $v_n$ is uniformly bounded in $\omega$. Hence, we obtain $s = 0$ in $\omega$ by $0 \leq s_{+,n} \in L^1(\Omega)$. Since $\omega \subset \subset \Omega \setminus S_+$ is arbitrary, the proposition holds if (2.1) is true.

Now, we suppose that (2.1) is false. Then, there exists a number $\alpha_*$ such that

$$0 < \alpha_* := \sup\{\alpha \in I_+ \mid \liminf_{n \to \infty} \int_{\Omega} e^{\alpha v_n} < +\infty\} \quad \text{and} \quad \mathcal{P}((0, \alpha_*]) > 0. \quad (2.2)$$

Fix $x_0 \in S_+$ and take $r_0 > 0$ satisfying $\overline{B(x_0, r_0)} \cap S_+ = \{x_0\}$. It is possible to take such an $r_0$ because $S$ is a finite set. We may assume $x_0 = 0$ by a translation. Then, there exist $x_n \in B_{r_0}$ and $\alpha_n \in \mathbb{R}$ such that

$$x_n \to 0 \quad v_n(x_n) = \max_{B_{3r_0}} v_n \to +\infty,$$

$$e^{\alpha_n v_n(x_n)} = \int_{I_+} \frac{\alpha e^{\alpha v_n(x_n)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha). \quad (2.3)$$

For this $\alpha_n$, we obtain the following lemmas shown in next section.

Lemma 1. There exists $C_1 > 0$, independent of $n$, such that

$$\int_{I_+} \frac{\alpha e^{(\alpha - \alpha_n)v_n(x)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha) \leq C_1$$

for all $x \in \overline{B_{2r_0}}$. 
Lemma 2. We have
\[ \alpha_n \to \alpha_0 \in [\alpha_*, 1], \]
passing to a subsequence.

Here, we develop a blow-up argument. Set
\[
\begin{cases}
    w_n(x) = \alpha_n v_n(x_n) - L, \\
    \tilde{w}_n(x) = w_n(\sigma_n x + x_n) + 2 \log \sigma_n, \\
    \sigma_n = e^{-w_n(x_n)/2} (\to 0 \text{ by Lemma 2}),
\end{cases}
\]
where \( L \gg 1 \) will be determined later on. The function \( \tilde{w}_n = \tilde{w}_n(x) \) is a solution to
\[
\begin{cases}
    -\Delta \tilde{w}_n = \alpha_n \tilde{V}_n(x) e^{\tilde{w}_n} - \sigma_n^2 \frac{\alpha_n \lambda_n}{|\alpha_n|} \int_{I^+} \alpha P(d\alpha) \quad \text{in } B_{r_0/\sigma_n}, \\
    \tilde{w}_n \leq \tilde{w}_n(0) = 0 \quad \text{in } B_{r_0/\sigma_n}, \\
    \int_{B_{r_0/\sigma_n}} \tilde{V}_n e^{\tilde{w}_n} \leq m(0),
\end{cases}
\]
(2.4)
where
\[
\tilde{V}_n(x) = e^L \cdot \lambda_n \int_{I^+} \frac{\alpha e^{(\alpha - \alpha_n) v_n(\sigma_n x + x_n)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha).
\]

Lemma 3. There exist \( \tilde{w} \in C^2(\mathbb{R}^2) \) and \( 0 < \tilde{V} \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \) such that
\[ \tilde{w}_n \to \tilde{w}, \quad \tilde{V}_n \to \tilde{V} \quad \text{in } \mathbb{R}^2 \]
and
\[
\begin{cases}
    -\Delta \tilde{w} = \alpha_0 \tilde{V}(x) e^{\tilde{w}} \quad \text{in } \mathbb{R}^2, \\
    \tilde{w} \leq \tilde{w}(0) = 0 \quad \text{in } \mathbb{R}^2, \\
    \int_{\mathbb{R}^2} \tilde{V} e^{\tilde{w}} \leq m(0).
\end{cases}
\]
(2.5)

Lemma 3 is also shown in next section.

For a solution \( \tilde{w} \) to (2.5), we set
\[
\tilde{\phi}(x) = \frac{\alpha_0}{2\pi} \int_{\mathbb{R}^2} \tilde{V}(y) e^{\tilde{w}(y)} \log \frac{|x - y|}{1 + |y|} dy,
\]
(2.6)
complying [3]. Noting that
\[ \tilde{V} e^{\tilde{w}} \in L^1 \cap L^\infty(\mathbb{R}^2), \]
(2.7)
we find that the function \( \tilde{\phi} \) set by (2.6) is well-defined in \( \mathbb{R}^2 \), and can show the following lemma because the proof of Lemma 1.1 of [3] is applicable to our case, see also Remark below.
Lemma 4. There exists $C_2 > 0$, independent of $L$, such that
\[ \tilde{w}(x) \geq -\beta \log(1 + |x|) - C_2 \] (2.8)
for $x \in \mathbb{R}^2$, where
\[ \beta = \frac{\alpha_0}{2\pi} \int_{\mathbb{R}^2} \tilde{V} e^{\tilde{w}}. \] (2.9)

Remark 4. In Lemma 1.1 of [3], the integrability condition $\int_{\mathbb{R}^2} e^{\tilde{w}} dx < +\infty$ is assumed to show the estimates from above and below for solutions and the estimate from below for $\beta$. However, it is not required if one only needs the estimate from below (2.8).

Proof of Proposition 1: Fix $R \gg 1$. It follows from Lemmas 3-4 that
\[ v_n(x) \geq v_n(x_n) - \frac{\beta}{\alpha_n} \log \left( 1 + \frac{|x - x_n|}{\sigma_n} \right) - \frac{C_2}{\alpha_n} + \epsilon_n \]
for all $x \in B(x_n, \sigma_n R)$, where $\epsilon_n$ is a quantity converging to 0 as $n \to \infty$. This $\epsilon_n$ may be changed in the following but keeps the property that $\epsilon_n \to 0$
We obtain
\[ \int_{B(x_n, \sigma_n)} e^{\alpha v_n} \geq e^{\alpha v_n(x_n) - \alpha C_2 / \alpha_n - 1} \int_{B(x_n, \sigma_n R)} \left( 1 + \frac{|x - x_n|}{\sigma_n} \right)^{-\alpha \beta / \alpha_n} \]
\[ = e^{(\alpha_0 - \alpha) v_n(x_n)} \cdot e^L \alpha C_2 / \alpha_n - 1 \int_{B_R} (1 + |x|)^{-\alpha \beta / \alpha_n} dx \] (2.10)
for all $\alpha \in I_+$. Thus, (2.3) and (2.10) yield
\[ 1 = \int_{I_+} \frac{\alpha e^{(\alpha_0 - \alpha) v_n(x_n)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha) \]
\[ \leq \epsilon_n + \int_{[\alpha_0, 1]} \frac{\int_{B(x_n, \sigma_n)} e^{\alpha v_n}}{\int_{\Omega} e^{\alpha v_n}} \cdot e^{L - \alpha C_2 / \alpha_n - 1} \int_{B_R} (1 + |x|)^{-\alpha \beta / \alpha_n} \mathcal{P}(d\alpha) \]
\[ \leq \epsilon_n + \frac{1}{e^{L - C_2 / \alpha_n - 1} \int_{B_R} (1 + |x|)^{-\beta / \alpha_n} dx}. \] (2.11)
Since $\beta / \alpha_n \leq (\alpha_0 / \alpha_n) \cdot (m(0) / 2\pi)$ by (2.9) and Lemma 3, inequality (2.11) implies
\[ 1 \leq \epsilon_n + \frac{1}{e^{L - C_2 / \alpha_n - 1} \int_{B_R} (1 + |x|)^{-\alpha_0 / \alpha_n} \frac{m(0)}{2\pi} dx}, \]
or
\[ 1 \leq \frac{1 + C_2 / \alpha_0 - L}{\int_{B_R} (1 + |x|)^{-\frac{m(0)}{2\pi} dx}}, \]
which is a contradiction if $L$ is sufficiently large. The proof is complete. \qed
3 Proof of Lemmas 1-3

As having announced in the previous sections, we show Lemmas 1-3 in this section. We again consider the positive case (i.e., $S = S_+$ and $I = I_+$) in what follows.

Proof of Lemma 1: Since $S = S_+$, there exists $C_3 > 0$, independent of $n$, such that $v_n > -C_3$ in $\Omega$. We use (2.3) and Jensen’s inequality to calculate

\[
\int_{I_+} \frac{\alpha e^{(\alpha - \alpha_n)v_n(x)}}{\int_{\Omega} e^{\alpha v_n}} P(d\alpha) \\
\leq \int_{I_{+,n}'} \frac{\alpha e^{-(\alpha_n - \alpha)v_n(x)}}{\int_{\Omega} e^{\alpha v_n}} + \int_{I_+} \frac{\alpha e^{(\alpha - \alpha_n)v_n(x)}}{\int_{\Omega} e^{\alpha v_n}} P(d\alpha) \\
\leq \frac{\alpha_n P(I_{+,n}')e^{\alpha_n C_3}}{|\Omega|} + 1 \leq \frac{e^{C_3}}{|\Omega|} + 1
\]

for all $x \in \overline{B_{2r_0}}$ and $n$, where

\[
I_{+,n}' = \begin{cases} (0, \alpha_n) & \text{if } \alpha_n > 0 \\
\emptyset & \text{if } \alpha_n \leq 0.
\end{cases}
\]

The lemma is completely shown. \(\square\)

Proof of Lemma 2: Put $\alpha_0 = \lim_{n \to} \alpha_n$.
Assume that $\alpha_0 > 1$. Then, there exists $\delta > 0$ such that

\[
e^{(1+\delta)v_n(x_n)} \leq e^{\alpha_n v_n(x_n)},
\]

that is, by Jensen’s inequality,

\[
e^{\frac{\delta}{2}v_n(x_n)} \leq \int_{I_+} \frac{\alpha e^{(\alpha - 1 - \delta/2)v_n(x_n)}}{\int_{\Omega} e^{\alpha v_n}} P(d\alpha) \leq e^{-\frac{\delta}{2}v_n(x_n)|\Omega|^{-1}}
\]

for $n \gg 1$, which is a contradiction because $v_n(x_n) \to +\infty$.

Next, assume that $\alpha_0 \leq 0$. In the case that $P((0, \alpha_*)) > 0$, there exists $0 < \epsilon \ll 1$ such that $P([\epsilon, \alpha_* - \epsilon]) > 0$, and therefore

\[
1 = \int_{I_+} \frac{\alpha e^{(\alpha - \alpha_n)v_n(x_n)}}{\int_{\Omega} e^{\alpha v_n}} P(d\alpha) \\
\geq \int_{[\epsilon, \alpha_* - \epsilon]} \frac{\alpha e^{(\alpha - \epsilon/2)v_n(x_n)}}{\int_{\Omega} e^{\alpha v_n}} P(d\alpha) \\
\geq c(\epsilon)e^{\frac{\epsilon}{2}v_n(x_n)}P([\epsilon, \alpha_* - \epsilon]) \to +\infty
\]
as $n \to \infty$, a contradiction. In the case that $\mathcal{P}(\{\alpha_*\}) = \mathcal{P}((0, \alpha_*]) > 0$, it holds that $\lim\inf_{n \to \infty} \int_{\Omega} e^{\alpha_n v_n} < +\infty$, and hence

$$1 = \int \frac{\alpha e^{(\alpha-\alpha_n)v_n(x_n)}}{\int_{\Omega} e^{\alpha_n v_n}} \mathcal{P}(d\alpha)$$

$$\geq \alpha_* e^{(\alpha_*-\alpha_n)v_n(x_n)} \left( \int_{\Omega} e^{\alpha_* v_n} \right)^{-1} \mathcal{P}(\{\alpha_*\}) \to +\infty$$

as $n \to \infty$, a contradiction.

We have shown that $\alpha_0 \in (0,1]$. It is left to show that $\alpha_0 \geq \alpha_*$. To prove this, we finally assume that $\alpha_0 \in (0, \alpha_*)$. Consider

$$\varphi_n = \alpha_n v_n - \log \int_{\Omega} e^{\alpha_n v_n}.$$ 

Passing to a subsequence, we have

$$\varphi_n(x_n) \to +\infty. \quad (3.1)$$

The function $\varphi_n = \varphi_n(x)$ satisfies

$$\begin{aligned}
-\Delta \varphi_n &= K_n(x)e^{\varphi_n} - \frac{\alpha_n \lambda_n}{|\Omega|} \int_{I_+} \alpha \mathcal{P}(d\alpha) \quad \text{in } B_{2r_0} \\
\int_{\Omega} e^{\varphi_n} &= 1,
\end{aligned} \quad (3.2)$$

where

$$K_n(x) = \alpha_n \lambda_n \left( \int_{\Omega} e^{\alpha_n v_n} \right) \int_{I_+} \frac{\alpha e^{(\alpha-\alpha_n)v_n(x)}}{\int_{\Omega} e^{\alpha v_n}} \mathcal{P}(d\alpha).$$

Lemma 1 and the boundedness $\lim\inf_{n \to \infty} \int_{\Omega} e^{\alpha_n v_n} < +\infty$ show that there exists $C_4 > 0$, independent of $n$, such that

$$0 \leq K_n \leq C_4 \quad \text{in } B_{2r_0}. \quad (3.3)$$

Consequently, (3.1)-(3.3) assure that

$$\varphi_n \to -\infty \quad \text{locally uniformly in } B_{2r_0\setminus\{0\}} \quad (3.4)$$

by virtue of the result of [2]. However, (3.4) is false since $S = S_+$ and $\lim\inf_{n \to \infty} \int_{\Omega} e^{\alpha_n v_n} < +\infty$. \qed

**Proof of Lemma 3**: It follows from Lemma 2 that

$$0 \leq \tilde{V}_n \leq e^{L(\lambda_0+1)} C_1 \quad \text{in } B_{r_0/\sigma_n}$$

for $n \gg 1$. We also have

$$0 \leq e^{\tilde{w}_n} \leq 1 \quad \text{in } B_{r_0/\sigma_n}$$
for all $n$, and
\[ \sigma_{n}^{2} \lambda_{n} \int_{I^{+}} \alpha \mathcal{P}(d\alpha) \to 0 \]
as $n \to \infty$. Combining these properties with $\tilde{w}_{n}(0) = 0$, we can safely apply the result of [2] to find that, for every $R > 0$, there exists $C_{5}(R) > 0$ such that
\[ \tilde{w}_{n} \geq -C_{5}(R) \quad \text{in } B_{R} \quad (3.5) \]
for $n \gg 1$. Thus, the elliptic regularity and a diagonal argument show that there exists $\tilde{w} \in C^{1+\alpha}(\mathbb{R}^{2})$, $\alpha \in (0, 1)$, such that
\[ \tilde{w}_{n} \to \tilde{w} \quad \text{in } C^{1+\alpha}_{loc}(\mathbb{R}^{2}). \quad (3.6) \]
Noting the definitions of $\tilde{V}_{n}$ and $\tilde{w}_{n}$, we see that there exists $\tilde{V} \in C^{1+\alpha}(\mathbb{R}^{2})$, $\alpha \in (0, 1)$, such that
\[ \tilde{V}_{n} \to \tilde{V} \quad \text{in } C^{1+\alpha}_{loc}(\mathbb{R}^{2}). \quad (3.7) \]
We again use the elliptic regularity, together with (3.6)-(3.7), and conclude the relation (2.5) and $\tilde{w}, \tilde{V} \in C^{2}(\mathbb{R}^{2})$.

It is clear that $\tilde{V} \in L^{\infty}(\mathbb{R}^{2})$ by Lemma 1, and therefore, we must show that $\int_{\mathbb{R}^{2}} \tilde{V} e^{\tilde{w}} \leq m(0)$ and that $\tilde{V} > 0$ in $\mathbb{R}^{2}$.

For every $R > 0$ and $0 < r \ll 1$,
\[ \int_{B_{R}} \tilde{V} e^{\tilde{w}} \leq \liminf_{n \to \infty} \int_{B_{R}} \tilde{V}_{n} e^{\tilde{w}_{n}} \leq \liminf_{n \to \infty} \int_{B_{r/\sigma_{n}}} \tilde{V}_{n} e^{\tilde{w}_{n}} \]
\[ = \liminf_{n \to \infty} \int_{B(x_{n}, r)} \nu_{+, n} \leq m(0) + \int_{B_{2r}} \nu_{+} \]
by the Fatou lemma, the definitions of $w_{n}$, $\tilde{w}_{n}$, $\sigma_{n}$ and $\tilde{V}_{n}$, and (1.2). Letting $R \uparrow +\infty$ and $r \downarrow 0$, we obtain $\int_{\mathbb{R}^{2}} \tilde{V} e^{\tilde{w}} \leq m(0)$.

Finally, we use the definitions of $w_{n}$, $\tilde{w}_{n}$, $\sigma_{n}$ and $\tilde{V}_{n}$, (3.5), $\tilde{w}_{n} \leq 0$ and (1.2) to obtain $C_{6}(R) > 0$, independent of $n \gg 1$, such that
\[ \tilde{V}_{n}(x) = e^{L} \lambda_{n} \int_{I^{+}} \alpha e^{\frac{\alpha_{\sigma_{n}}(\tilde{w}_{n}(x)+\alpha_{n}v_{n}(x_{n}))}{\int_{\Omega} e^{\alpha v_{n}}}} \mathcal{P}(d\alpha) \]
\[ \geq e^{L-C_{6}(R)} \lambda_{n} \int_{I^{+}} \alpha e^{(\alpha-\alpha_{n})v_{n}(x_{n})} \mathcal{P}(d\alpha) = e^{L-C_{6}(R)} \lambda_{n} \]
for all $x \in B_{R}$ and $n \gg 1$, and for every $R > 0$, which means $\tilde{V} > 0$ in $\mathbb{R}^{2}$ because $\lambda_{n} \to \lambda_{0} > 0$ by $S = S_{+} \neq \emptyset$.

\textbf{References}


