数理解析研究所講究録
第 1798 巻 2012 年 101-106

Analytical derivation of diffusion coefficient of two-dimensional point vortex system with Klimontovich formalism

Yuichi YATSUYANAGI
Faculty of Education, Shizuoka University

Tadatsugu HATORI
Faculty of Science, Kanagawa University

1 Introduction

A diffusion coefficient of a two-dimensional (2D) point vortex system is analytically derived with Klimontovich formalism.

The main motif of this international seminar is to provide an opportunity for collaboration between mathematician and physicist. In such case, jargons in a group prevent the other group from active discussion. So, I will try to explain meanings of the words which may be potentially jargons.

The 2D inviscid Euler equation has a formal solution of singular point vortices. However, the Euler equation is the macroscopic fluid equation and should have macroscopic smooth solutions. We regard the Euler equation that has the singular point vortex solution as a kinetic equation. The kinetic equation is formally identical with the macroscopic Euler equation. It happens that the macroscopic Euler equation and the kinetic equation have the same form.

Similar case can be found in plasma physics. The Klomontovich-Dupree equation is a kinetic equation that has a discretized exact solution by the Dirac delta function in a phase space. By coarse-graining (averaging) the equation, the Fokker-Planck type collision term is obtained. A kinetic equation with the Fokker-Planck type collision term is called the Fokker-Planck equation, which is the version of the Boltzmann equation applicable to the case of long-range interparticle forces. The above procedure is called the Klimontovich formalism. This time, we apply the Klimontovich formalism to the point-vortex system, and collision term is analytically obtained.

The organization of this paper is as follows: In Sec. 2, the point vortex system is introduced. In Sec. 3, outline of the Klimontovich formalism is given. In Sec. 4 our result will be given. In Sec. 5 we give our conclusion.

2 Point vortex system

The 2D Euler equation is a partial differential equation which describes incompressible flow in 2D plane.

$$\frac{\partial \boldsymbol{u}(\boldsymbol{r},t)}{\partial t} + \boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla \boldsymbol{u}(\boldsymbol{r},t) = 0. \tag{1}$$

The vorticity equation is obtained by taking the rotation differential:

$$\frac{\partial \omega_z(\boldsymbol{r},t)}{\partial t} + \boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla \omega_z(\boldsymbol{r},t) = 0, \tag{2}$$

and it has a point vortex solution:

$$\omega_z(\boldsymbol{r},t) = \sum_i \Omega_i \delta(\boldsymbol{r} - \boldsymbol{r}_i(t)), \tag{3}$$

where $\omega_z(\boldsymbol{r},t)$, $\boldsymbol{u}(\boldsymbol{r},t)$ are the nonzero component of the vorticity and the flow field in 2D plane pointed by $\boldsymbol{r} = (x,y)$. The circulation (strength) of the i-th point vortex at position $\boldsymbol{r}_i$ is denoted by Ω_i whose value is either Ω_0 or $-\Omega_0$ where Ω_0 is a positive constant. This solution (3) is discretized by the Dirac delta function. In general, macroscopic fluid equation should have a smooth solution. Thus, we regard the point vortex solution is a solution for a kinetic equation that is formally identical with the 2D Euler equation. We call this equation the microscopic Euler equation. To distinguish the microscopic Euler equation from the macroscopic one, we indicate the microscopic variable with a hat.

$$\frac{\partial \hat{\omega}_z(\boldsymbol{r},t)}{\partial t} + \hat{\boldsymbol{u}}(\boldsymbol{r},t)\cdot\nabla \hat{\omega}_z(\boldsymbol{r},t) = 0 \tag{4}$$

The microscopic and macroscopic variables are related by (ensemble) average operator $\langle\cdot\rangle$

$$\omega_z(\boldsymbol{r},t) = \langle \hat{\omega}_z(\boldsymbol{r},t)\rangle. \tag{5}$$

The microscopic vorticity $\hat{\omega}_z(\boldsymbol{r},t)$, velocity $\hat{\boldsymbol{u}}(\boldsymbol{r},t)$ and stream function $\hat{\psi}(\boldsymbol{r},t)$ satisfy the following relations,

$$\hat{\boldsymbol{u}}(\boldsymbol{r},t) = \nabla\times(\hat{\psi}(\boldsymbol{r},t)\hat{\boldsymbol{z}}) = -\hat{\boldsymbol{z}}\times\nabla\hat{\psi}(\boldsymbol{r},t), \tag{6}$$

$$\hat{\omega}_z(\boldsymbol{r},t) = \nabla\times\hat{\boldsymbol{u}}(\boldsymbol{r},t) = -\nabla^2\hat{\psi}(\boldsymbol{r},t) \tag{7}$$

where $\hat{\boldsymbol{z}}$ is a unit vector in z direction. Positions of the point vortices are governed by the microscopic Euler equation and by coarse-graining the distribution of the point vortices, macroscopic vorticity distribution $\omega_z(\boldsymbol{r},t)$ is obtained. Here, a question arises. The microscopic solution governed by the microscopic Euler equation and the macroscopic solution governed by the macroscopic Euler equation is the same as is shown in Fig. 1? To answer this question, we present a similar case in plasma physics in the next section.

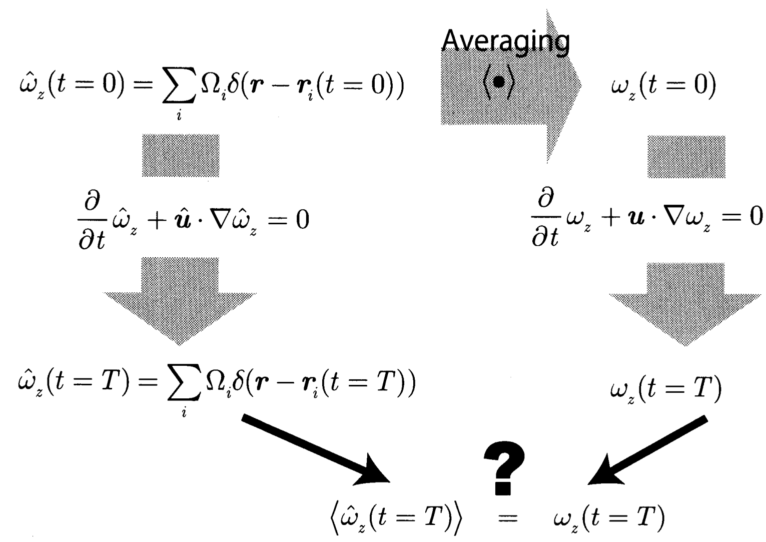

Figure 1: The microscopic solution governed by the microscopic Euler equation and the macroscopic solution governed by the macroscopic Euler equation is the same?

3 Klimontovich formalism

Klimontovich-Dupree equation

$$\frac{\partial \hat{f}(\boldsymbol{r},\boldsymbol{v},t)}{\partial t}+\boldsymbol{v}\cdot\nabla\hat{f}(\boldsymbol{r},\boldsymbol{v},t)+\frac{q}{m}(\hat{\boldsymbol{E}}+\boldsymbol{v}\times\hat{\boldsymbol{B}})\cdot\frac{\partial}{\partial\boldsymbol{v}}\hat{f}(\boldsymbol{r},\boldsymbol{v},t)=0 \tag{8}$$

is an equation for 6-dimensional phase space with $(\boldsymbol{r},\boldsymbol{v})$ and has an exact solution for a particle density function $\hat{f}(\boldsymbol{r},\boldsymbol{v},t)$ [1].

$$\hat{f}(\boldsymbol{r},\boldsymbol{v},t)=\sum_i\delta(\boldsymbol{r}-\boldsymbol{r}_i(t))\delta(\boldsymbol{v}-\boldsymbol{v}_i(t)) \tag{9}$$

The hat on the particle density function f means this function microscopic. The microscopic electric and magnetic fields $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{B}}$ obey the microscopic Maxwell equation that is identical with the macroscopic (usual) Maxwell equation as the Maxwell equation does not have nonlinear term. It is assumed that the microscopic quantity consists of a macroscopic part and a fluctuation.

$$\hat{f}(\boldsymbol{r},\boldsymbol{v},t) = \langle\hat{f}(\boldsymbol{r},\boldsymbol{v},t)\rangle+\delta\hat{f}(\boldsymbol{r},\boldsymbol{v},t)=f(\boldsymbol{r},\boldsymbol{v},t)+\delta\hat{f}(\boldsymbol{r},\boldsymbol{v},t) \tag{10}$$

$$\hat{\boldsymbol{E}}(\boldsymbol{r},\boldsymbol{v},t) = \boldsymbol{E}(\boldsymbol{r},\boldsymbol{v},t)\rangle+\delta\hat{\boldsymbol{E}}(\boldsymbol{r},\boldsymbol{v},t) \tag{11}$$

$$\hat{\boldsymbol{B}}(\boldsymbol{r},\boldsymbol{v},t) = \boldsymbol{B}(\boldsymbol{r},\boldsymbol{v},t)+\delta\hat{\boldsymbol{B}}(\boldsymbol{r},\boldsymbol{v},t) \tag{12}$$

Note that a macroscopic quantity is obtained by averaging a microscopic quantity with a hat as shown in Eq. (5). Substituting the above microscopic variables into the

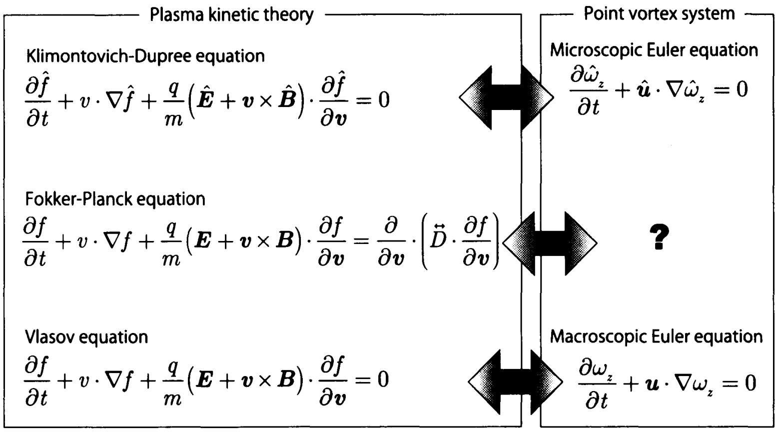

Figure 2: Klimontovich equation is related to the microscopic Euler equation that has the point vortex solution. By averaging Klimontovich equation, macroscopic Fokker-Planck equation is obtained. Ignoring the Fokker-Planck collision term, Vlasov equation is obtained. We regard that the macroscopic Euler equation is related to the macroscopic Vlasov equation. There is no related equation for the Fokker-Planck equation.

Klimontovich-Dupree equation and averaging the equation, the following macroscopic equation is obtained.

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\nabla f+\frac{q}{m}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f}{\partial\boldsymbol{v}}=\frac{q}{m}\left\langle\left(\delta\hat{\boldsymbol{E}}+\boldsymbol{v}\times\delta\hat{\boldsymbol{B}}\cdot\frac{\partial}{\partial\delta\boldsymbol{v}}\delta\hat{f}\right)\right\rangle \tag{13}$$

Further calculation yields the Fokker-Planck collision term from the right hand side of Eq. (13) and a kinetic equation with Fokker-Planck collision term is called Fokker-Planck equation:

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\nabla f+\frac{q}{m}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f}{\partial\boldsymbol{v}}=\frac{\partial}{\partial\boldsymbol{v}}\cdot\left(\vec{D}\cdot\frac{\partial f}{\partial\boldsymbol{v}}\right). \tag{14}$$

If the collision term is dropped completely, it is called Vlasov equation:

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\nabla f+\frac{q}{m}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f}{\partial\boldsymbol{v}}=0 \tag{15}$$

We summarize the relation between the Klimontovich equation and the point vortex equation in Fig. 2. Klimontovich equation is related to the microscopic Euler equation that has the point vortex solution. By averaging Klimontovich equation, macroscopic Fokker-Planck equation is obtained. Ignoring the Fokker-Planck collision term, Vlasov equation is obtained. We regard that the macroscopic Euler equation is related to the macroscopic Vlasov equation. There is no related equation for the Fokker-Planck equation. This time, we focus ourselves to obtain the corresponding macroscopic equation that may include collision term.

4 Kinetic theory for 2D point vortex system

To obtain the Fokker-Planck type collision term for the 2D point vortex system, the same procedure as the Klimontovich formalism is applied. Microscopic vorticity is assumed to consist of the macroscopic vorticity and the fluctuation.

$$\hat{\omega}_z(\boldsymbol{r},t) = \langle\hat{\omega}_z(\boldsymbol{r},t)\rangle + \delta\hat{\omega}_z(\boldsymbol{r},t) = \omega_z(\boldsymbol{r},t) + \delta\hat{\omega}_z(\boldsymbol{r},t) \tag{16}$$

$$\hat{\boldsymbol{u}}(\boldsymbol{r},t) = \boldsymbol{u}(\boldsymbol{r},t) + \delta\hat{\boldsymbol{u}}(\boldsymbol{r},t) \tag{17}$$

Same as before, substituting the microscopic variables into the microscopic Euler equation and averaging it, the following macroscopic equation is obtained.

$$\frac{\partial\omega_z(\boldsymbol{r},t)}{\partial t} + \boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla\omega_z(\boldsymbol{r},t) = -\nabla\cdot\langle\delta\hat{\boldsymbol{u}}(\boldsymbol{r},t)\delta\omega_z(\boldsymbol{r},t)\rangle \tag{18}$$

To evaluate the diffusion term on the right hand side, we need an evolution equation for $\delta\omega_z(\boldsymbol{r},t)$. For this purpose, linearlized equation is introduced, which is obtained by substituting the microscopic variables (16) and (17) into the microscopic Euler equation (4) and dropping the zero-th order macroscopic terms. The obtained linearlized equation is given by

$$\frac{\partial}{\partial t}\delta\hat{\omega}_z(\boldsymbol{r},t) + \boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla\delta\hat{\omega}_z(\boldsymbol{r},t) = -\delta\hat{\boldsymbol{u}}(\boldsymbol{r},t)\cdot\nabla\omega_z(\boldsymbol{r},t). \tag{19}$$

This linearlized equation can be integrated as $\boldsymbol{u}(\boldsymbol{r},t)$ in the second term on the left hand side and $\omega_z(\boldsymbol{r},t)$ are assumed to be constant in the microscopic scale:

$$\delta\hat{\omega}_z(\boldsymbol{r},t) = -\int_{-\infty}^{t} d\tau\,\delta\hat{\boldsymbol{u}}(\boldsymbol{r}-(t-\tau)\boldsymbol{u},\tau)\cdot\nabla\omega_z(\boldsymbol{r},t) \tag{20}$$

Finally, the corresponding macroscopic equation is obtained:

$$\frac{\partial}{\partial t}\omega_z(\boldsymbol{r},t) + \boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla\omega_z(\boldsymbol{r},t) = \nabla\cdot(\vec{\vec{\eta}}\cdot\nabla\omega_z(\boldsymbol{r},t)) \tag{21}$$

$$\vec{\vec{\eta}} = \int_{-\infty}^{t}\langle\delta\hat{\boldsymbol{u}}(\boldsymbol{r},t)\delta\hat{\boldsymbol{u}}(\boldsymbol{r}-(t-\tau)\boldsymbol{u},\tau)\rangle d\tau \tag{22}$$

The right hand side is the diffusion term due to the discreteness of the vorticity. It may be an extension of the well-known Green-Kubo formula. This result includes the position and time correlations, while Green-Kubo formula includes the time correlation only.

5 Conclusion

We have derived the diffusion term implicitly included in 2D Euler equation. On the analogy with plasma physics, the obtained equation is similar to Fokker-Planck equation.

In general, collision term in Fokker-Planck equation consists of two parts: the diffusion term and the friction term.

$$\nabla \cdot (\overleftrightarrow{\eta} \cdot \nabla \omega_z + \boldsymbol{A} \omega_z) \tag{23}$$

In our result, friction term is not included. The friction term may be derived by rewriting Eq. (20) as

$$\begin{aligned} \delta \hat{\omega}_z(\boldsymbol{r}, t) &= -\int_{t_0}^{t} d\tau \delta \hat{\boldsymbol{u}}(\boldsymbol{r} - (t - \tau)\boldsymbol{u}, \tau) \cdot \nabla \omega_z(\boldsymbol{r}, t) \\ &\quad + \delta \omega_z(\boldsymbol{r} - (t - t_0)\boldsymbol{u}, t_0). \end{aligned} \tag{24}$$

Further calculation reveals that the second term is proportional to ω_z. As this effect is, however, evaluated negligible as compared with the diffusion term, we ignore the term.

References

[1] Y. L. Klimontovich: *The statistical theory of non-equilibrium processes in a plasma* (MIT Press, Cambridge, Massachusetts, 1967).