Fast computation of Goursat's infinite integral with very high accuracy

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Abstract We propose an efficient computation method for the infinite integral $\int_0^\infty x dx/(1+x^6 \sin^2 x)$, whose integrand contains a series of spikes, approximately π apart, growing taller and narrower as x increases. Computing the value of this integral has been a problem since 1984. We herein demonstrate a method using the Hilbert transform for changing this type of singular function into a smooth function and computing the value of the integral to more than one million significant digits using a superconvergent double exponential quadrature method.

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Keywords: Numerical quadrature; Multiple-precision computation; Double exponential quadrature method; Computational complexity

1 Introduction

The convergence of the infinite integral

$$I = \int_0^\infty \frac{x}{1+x^6 \sin^2 x} dx \tag{1}$$

was demonstrated by Goursat [1] and Hardy [2] early in the 20th century. In the present paper, we refer to this integral as Goursat's integral. In the RIMS conference at Kyoto University in 1984, Toda of Chiba University proposed the problem of calculating not only the convergence of the integral but also the value of the integral [3]. In 1986, Ninomiya of Nagoya University obtained a value approximated to about 20 significant digits by using an automatic quadrature routine and an acceleration method in quadruple-precision computation [3]. In 2009, Hatano, Ninomiya, Sugiura, and Hasegawa obtained a value approximated to about 73 significant digits by using a contour integral in octuple-precision computation [4] (see Figure 1).

In the present paper, we propose a new method to evaluate Goursat's integral that allows high-speed and high-accuracy computation.

Specifically, we change the integral using a type of Hilbert transform and obtain an easily computable integral using the double exponential quadrature method (DE quadrature method) [5], [9], [10].

Moreover, we propose a superconvergent DE quadrature method constructed specifically for this integral, in order to enable higher-speed and higher-accuracy computation. This quadrature method achieves the desired high performance, and the number of function evaluations necessary to obtain a value of the integral to N significant digits is $O(N)$. Finally, numerical integration to one million or more significant digits is performed, and the relationship between the number of digits and the computational complexity is considered.

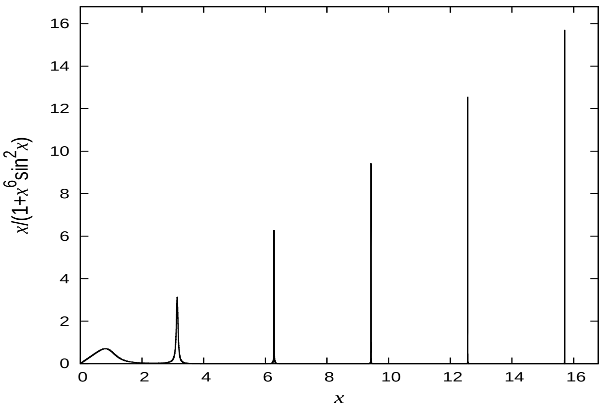


Figure 1: Integrand of Goursat's integral

2 Change by Hilbert transform

First, we analyze the integrand of (1) as a complex function and find that the function has poles $p \approx 0.3 + 0.9i$, $q \approx 0.9 + 0.4i$, $\bar{p}$, $\bar{q}$, and $r_k \approx \pi k + i/(\pi k)^3$, $\bar{r}_k$ $(k = 1, 2, 3, \cdots)$ [4]. Based on the above considerations, the integral of (1) is rewritten as a complex integral:

$$I = \frac{1}{2\pi i} \int_C \varphi(z) dz,$$

$$\varphi(z) = \begin{cases} \dfrac{-\pi i z}{1+z^6\sin^2 z}, & \Im z > 0 \\ \dfrac{+\pi i z}{1+z^6\sin^2 z}, & \Im z < 0 \end{cases},$$

where the contour C passes through the origin and around $(0, +\infty)$, and no pole exists inside C. The contour C is shown in Figure 2.

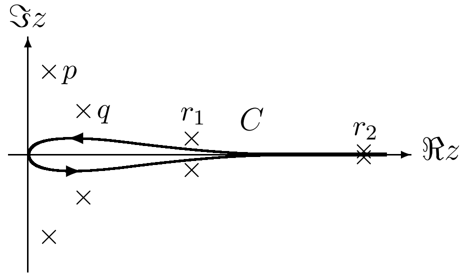


Figure 2: Contour C

Next, in order to eliminate these poles, we consider the Hilbert transform of a modified integrand of (1)

$$\psi(x) = \mathrm{pv}\int_{-\infty}^{\infty} \frac{x}{1+x^6\sin^2 y} \cdot \frac{dy}{x-y},$$

where pv indicates the principal value integral. This transform is analytically calculable (see A.1), and we obtain

$$\psi(z) = \frac{\pi z^6 \sin z \cos z}{\sqrt{1+z^6}} \frac{z}{1+z^6\sin^2 z}. \tag{2}$$

After analyzing the poles, we find that the poles of $\psi(z)$ and $\varphi(z)$ are located at the same positions, the signs of the residues of $\psi(z)$ and $\varphi(z)$ are opposite, $\psi(z)$ is analytic in the neighborhood of the real axis, and

$$0 = \frac{1}{2\pi i}\int_C \psi(z) dz.$$

The poles are then canceled using this character:

$$I = \frac{1}{2\pi i}\int_C (\varphi(z) + \psi(z)) dz$$

and the path C can be moved from the real axis. By changing the integral into the integral on the imaginary axis and the branch of $\sqrt{1+z^6}$, we obtain

$$
\begin{aligned}
I &= I_1 + I_2 \\
&= \int_0^\infty \left[\frac{t}{1+t^6\sinh^2 t} + \Re \frac{2(1+\sqrt{3}i)t}{2-t^6+t^6\cos((\sqrt{3}+i)t)} \right] dt \\
&+ \int_0^1 \frac{t^7}{\sqrt{1-t^6}} \left[\frac{\sinh t \cosh t}{1+t^6\sinh^2 t} + \Im \frac{(1+\sqrt{3}i)\sin((\sqrt{3}+i)t)}{2-t^6+t^6\cos((\sqrt{3}+i)t)} \right] dt. \quad (3)
\end{aligned}
$$

The detailed derivation of (3) is shown in A.2, and the integrands of (3) are shown in Figure 3. The first term of (3) is an infinite integral of function decay $O(t^{-5}e^{-t})$ as $t \to \infty$, and the second term is the integral of a function of order $O((1-t)^{-1/2})$ as $t \to 1$. Integrals of these types are efficiently computed by the DE quadrature method. In this case, the number of function evaluations to compute N significant digits is $O(N \log N)$. Moreover, since the second term of (3) is transformed into the integral of a smooth periodic function by the change of variable $t = \sin\theta$, high-accuracy computation is possible by applying the trapezoidal rule to this periodic integral. The second term of (3) is then calculated as follows:

$$
I_2 = \frac{\pi}{2M}\left(\frac{1}{2}G(1) + \sum_{n=1}^{M-1} G\left(\sin\frac{\pi n}{2M}\right)\right) + E_{2,M} + \Delta I_{2,M},
$$

where $G(t)$ is the integrand such that

$$
G(t) = \frac{t^7}{\sqrt{1+t^2+t^4}} \left[\frac{\sinh t \cosh t}{1+t^6\sinh^2 t} + \Im \frac{(1+\sqrt{3}i)\sin((\sqrt{3}+i)t)}{2-t^6+t^6\cos((\sqrt{3}+i)t)} \right],
$$

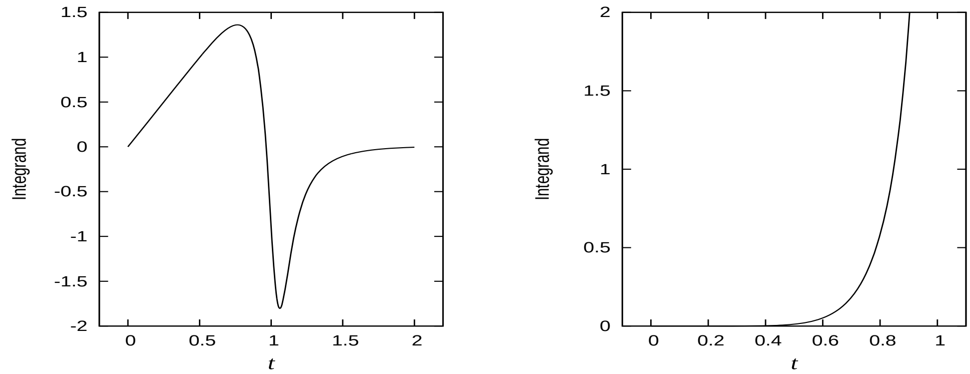
and $E_{2,M}$ is a correction term based on the residues at the poles $t = ip, i\bar{p}, iq, i\bar{q}$ using the characteristic function $\Psi_h(z)$ of the error [8], [7], [5], [6],

$$
E_{2,M} = \Im\left(\frac{-2\pi p^2 P_M}{3+ip^4\cos p} + \frac{2\pi q^2 Q_M}{3-iq^4\cos q} \right),
$$

$$
P_M = \frac{1}{1-(p+\sqrt{1+p^2})^{4M}} + \frac{1}{1-(p'+\sqrt{1+p'^2})^{4M}}, \quad p' = -pe^{\pi i/3},
$$

$$
Q_M = \frac{1}{1-(q'+\sqrt{1+q'^2})^{4M}}, \quad q' = qe^{\pi i/3},
$$

and $|\Delta I_{2,M}|$ is an error term bounded by $O(e^{-3.3M})$ using the saddle point method. In this case, the number of function evaluations needed in order to compute N significant digits is $O(N)$.



First term Second term

Figure 3: Integrands of (3)

3 Superconvergent DE quadrature method

Here, we propose a DE quadrature method for integrals (3). For simplicity, we consider the first term of (3). By moving the integral path to the imaginary axis, the first term is

$$I_1 = -\int_0^\infty \frac{t}{1+t^6\sinh^2 t}dt + \Im \frac{2\pi p^2}{3+ip^4\cos p},$$

where p is the pole of (1). The above equation is then rewritten as

$$I_1 = -\frac{1}{\pi i}\int_{-\infty}^\infty \frac{t\log(ict)}{1+t^6\sinh^2 t}dt + \Im \frac{2\pi p^2}{3+ip^4\cos p},$$

where $c > 0$ is a constant, and

$$I_1 = \frac{1}{\pi}\int_{-\infty}^\infty \frac{(it+T)\log(ict+cT)}{1+(it+T)^6\cosh^2 t}dt + R_K$$

is obtained by moving the integral path in the direction of $-iT$, $T = \pi(K + 1/2)$, where K is a positive integer, and R_K is the sum of the residues at poles $-ip \approx 0.9-0.3i$, $-i\bar{p}$, $-iq \approx 0.4-0.9i$, $-i\bar{q}$, and $-ir_k, -i\bar{r}_k \approx \pm 1/(\pi k)^3 - \pi k i$ ($k = 1, 2, 3, \ldots, K$) (see Figure 4). Setting $c = 1/T$ and applying the change of variables $t = T'\sinh u$ ($T' = \pi(K-1/2)$) and the trapezoidal rule with mesh size h, we obtain

$$I_1 \;=\; \frac{2h}{\pi}\sum_{n=1}^{M'} \Re \frac{(T+iT'\sinh nh)\log(1+i(T'/T)\sinh nh)T'\cosh nh}{1+(T+iT'\sinh nh)^6\cosh^2(T'\sinh nh)}$$

$$+R_K + E_{h,2K} + \Delta I_{1,h,K,M'},$$

where $E_{h,2K}$ is a correction term based on the residues at the poles $T'\sinh u = iT - ip$, $iT - i\bar{p}$, $iT - iq$, $iT - i\bar{q}$, and $T'\sinh u = iT - ir_k$, $iT - i\bar{r}_k$ ($k = 1, 2, 3, \ldots, 2K$) using the characteristic function of the error [5], [6]. R_K and $R_K + E_{h,2K}$ are calculated by

$$R_K = \Im\frac{2\pi p^2}{3 + ip^4\cos p} + \sum_{k=-1}^{K}\Re\frac{2r_k^2\log(r_k/T)}{3 - i(-1)^k r_k^4\cos r_k},$$

$$R_K + E_{h,2K} = \Im\frac{2\pi p^2}{3 + ip^4\cos p} + \sum_{k=-1}^{2K}\Re\frac{2r_k^2\log(r_k/T)}{3 - i(-1)^k r_k^4\cos r_k}\frac{1}{1 - \exp(2\pi i u_k/h)},$$

$$u_k = \log\left(-i(r_k - T)/T' + \sqrt{1 - (r_k - T)^2/T'^2}\right), \quad r_0 = q, r_{-1} = p.$$

Taking K, $1/h$, and M' in proportion to the number of significant digits N, such that

$$K = \lceil \alpha N \rceil, \quad h = \frac{\pi^2 - 2(\Im\sqrt{p+\pi})\sqrt{2\pi/K}}{N\log 10}, \quad M' = \left\lfloor \frac{1}{h}\sinh^{-1}\frac{N\log 10}{2T'} \right\rfloor,$$

the approximation error $|\Delta I_{1,h,K,M'}|$ is bounded by

$$O(10^{-N}) = O(\exp(-C'M')) = O(\exp(-C''(M' + 2K))),$$

where $0 < C' < \pi^2/\sinh^{-1}(\log 10/(2\pi\alpha))$ and $C'' = C'/(1 + 2\alpha)$, and we are ready to apply the superconvergent DE quadrature method and find that the number of function evaluations needed in order to compute N significant digits is $O(N)$ (including the evaluation of the residues).

4 Computation Example

We use GMP [11] (ver. 5.0.1) & MPFR [12] (ver. 2.4.2) as the multiple-precision computation library of four arithmetic operations, and we construct a computation routine of elementary functions that combines the Taylor series expansion with the CORDIC method of the multi-bit unit. This algorithm is shown in A.3. In order to compute N significant digits using this algorithm, the number of multiplications must be approximately $O(\sqrt{N})$.

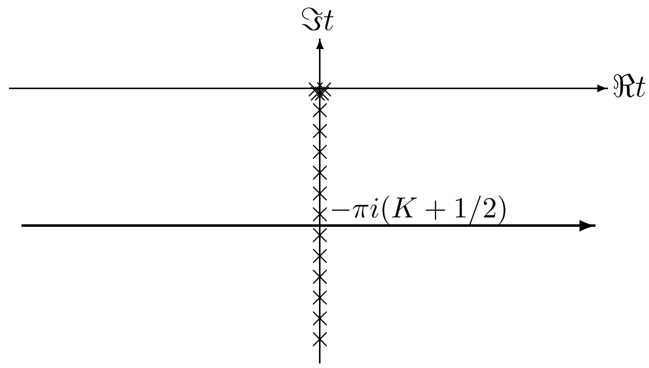


Figure 4: Integration Path

Although the algorithm requires a far larger number of multiplications than the arithmetic–geometric mean algorithm, this algorithm has the advantage that a proportionality coefficient of the computational complexity can be made small according to the size of the memory.

As another computational complexity reduction method, we use the DE quadrature method, which, if possible, does not include exponential function operations. The above described algorithm has two characteristics. Namely, the mesh size is selected as $h = \log 2/K'$ (where K' is a positive integer), and the step size $K'h$ with respect to the inner loop is computed. Since the $n+1$th value of the double exponential function is $a_{n+1} = \exp(\exp((n+1)K'h)) = \exp(2\exp(nK'h)) = a_n^2$ at this time, high-speed computation is enabled by squaring the nth value. However, computation of $\log 2$, a small number of computations of exponential functions, and the memory for the array are needed for initial computation. When applying this algorithm to Goursat's integral, as compared with the usual DE quadrature method, the computational complexity is approximately half.

Moreover, an efficient computation method for the poles r_k (which are the zeros of $i - (-1)^k z^3 \sin z$) is also required. We use the third-order Newton's method, the accuracy of which is controlled to be optimal in each iteration step. Note that a higher-order Newton's method (Halley's method, etc.) is effective in multiple-precision computing, since the precision of each iteration can be controlled to the necessary minimum accuracy.

The results in the computation environment of four 2.8-GHz Opteron

(K10 six-core) CPUs are shown in Tables 1 and 2. The value to 100 digits is

$$
\begin{aligned}
I = 1.\,&16965\ 25542\ 24486\ 47772\ 59225\ 81661\ 19775\ 95884\ 81416\ 66271 \\
&46180\ 73171\ 51391\ 33835\ 19905\ 81627\ 12111\ 09181\ 62126\ 67625 \\
&\cdots.
\end{aligned}
$$

Table 2 shows that approximately 1.54 function evaluations per digit are needed and the number of function evaluations is proportional to the number of significant digits. On the other hand, the computation time is proportional to approximately $N^{2.6}$. The reason for this is that approximately twice the number of computations of elementary functions, e.g., log, exp, atan2, and sincos, and approximately 10 multiplication computations are needed for each computation of a function.

In order to analyze the computation time in greater detail, the rate of the computation time of an elementary function and the computation time of multiplication is shown in Table 3. The computation time of built-in elementary functions of the MPFR library is also shown for comparison. This table shows that the computation time of the elementary functions is proportional to $N^{1.6}$, and the proposed algorithm is approximately 10 times faster than the MPFR library. Moreover, the multiplication time of the GMP library is proportional to $N^{1.3}$.

Next, we predict the computation time for the case in which the fastest currently known algorithm is used. The computational complexity of the Schönhage-Strassen multiplication algorithm is $O(N \log N \log\log N)$ for N digits. The computational complexity of algorithms of elementary functions using the arithmetic–geometric mean is $O(N(\log N)^2 \log\log N)$. The computational complexity of Goursat's integral in this case is $O((N \log N)^2 \log\log N)$.

Table 1: Example of the Computation of Goursat's Integral

Number of Digits	Execution Time	Error
10,000 (33300 bits)	2.73 s	$3.0 \cdot 10^{-10024}$
100,000 (333,000 bits)	815.44 s (13.6 min)	$4.1 \cdot 10^{-100243}$
1,000,000 (3,330,000 bits)	464,758 s (5.4 days)	$6.0 \cdot 10^{-1002429}$

Table 2: Details of the Computation

Number of Digits	Time (I_1, I_2)	Function Evaluations (I_1, I_2)
10,000	1.36 s, 1.37 s	$8,485$ ($K = 257$), $6,995$
100,000	417.50 s, 397.94 s	$84,400$ ($K = 2,562$), $69,945$
1,000,000	262,130 s, 202,628 s	$842,626$ ($K = 25,616$), $699,449$

Table 3: Mean Execution Time (CPU Time) of log, exp, atan2, and sincos

Number of Digits	Present Method	MPFR	Multiplication Time μ
10,000	16.8μ	151μ	$\mu = 0.0641$ ms
100,000	25.1μ	243μ	$\mu = 1.71$ ms
1,000,000	65.1μ	543μ	$\mu = 26.8$ ms

5 Conclusion

Goursat's integral is transformed into a smooth integral, various improvements are applied, and the value is evaluated to more than one million significant digits using the superconvergent DE quadrature method. The obtained results are presented below.

1. A number of integrals with computation difficulty owing to the poles are easily computable by finding a conjugate function using the Hilbert transform.

2. A superconvergent DE quadrature method for a specific type of integral is proposed.

3. A DE quadrature method with few exponential function computations is proposed, and the integration speed is increased. This method of modifying the basic DE quadrature method can be applied to many types of DE quadrature methods and is especially effective for multiple-precision computation.

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A.1 Derivation of the transform (2)

$$\begin{aligned}
\psi(x) &= \mathrm{pv}\int_{-\infty}^{\infty}\frac{x}{1+x^6\sin^2 y}\cdot\frac{dy}{x-y}\\
&= \lim_{n\to\infty}\mathrm{pv}\int_{-\pi n}^{\pi(n+1)}\frac{x}{1+x^6\sin^2 y}\cdot\frac{dy}{x-y}\\
&= \lim_{n\to\infty}\mathrm{pv}\int_{0}^{\pi}\frac{x}{1+x^6\sin^2 y}\cdot\sum_{k=-n}^{n}\frac{1}{x-y-\pi k}dy,
\end{aligned}$$

considering

$$\cot z = \lim_{n\to\infty}\sum_{k=-n}^{n}\frac{1}{z-\pi k},$$

we have

$$\begin{aligned}
\psi(x) &= \mathrm{pv}\int_{0}^{\pi}\frac{x}{1+x^6\sin^2 y}\cot(x-y)dy\\
&= \mathrm{pv}\oint_{|z|=1}\frac{x}{1-x^6(z+z^{-1}-2)/4}\cdot\frac{e^{2ix}+z}{e^{2ix}-z}\cdot\frac{dz}{2z}\\
&= \frac{\pi x^6\sin x\cos x}{\sqrt{1+x^6}}\frac{x}{1+x^6\sin^2 x}.
\end{aligned}$$

A.2 Derivation of the integral (3)

$$\begin{aligned}
I &= \frac{1}{2\pi i}\int_C(\varphi(z)+\psi(z))dz\\
&= \frac{1}{2\pi i}\lim_{n\to\infty}\left(-\int_0^{R_n e^{i\theta}}(\varphi(z)+\psi(z))dz+\int_{R_n}^{R_n e^{i\theta}}(\varphi(z)+\psi(z))dz\right)+\mathrm{cc},
\end{aligned}$$

where $0<\theta<\pi/6$, $R_n=(n+1/2)\pi$, and cc is the complex conjugate. Since $|\varphi(z)+\psi(z)|\sim O(1/|z|^2)$ as $|z|=(n+1/2)\pi\to\infty$, the second term vanishes (see Path 1 of Figure 5), and we have

$$\begin{aligned}
I &= -\frac{1}{2\pi i}\int_0^{Re^{\pi i(1-\varepsilon)/6}}(\varphi(z)+\psi(z))dz+\mathrm{cc}\\
&= -\frac{1}{2\pi i}\int_0^{Re^{\pi i(1+\varepsilon)/6}}(\varphi(z)+\psi(z))dz+\frac{1}{\pi i}\int_{e^{\pi i/6}}^{Re^{\pi i(1+\varepsilon)/6}}\psi(z)dz+\mathrm{cc}\\
&= -\frac{1}{2\pi i}\int_0^{R(i+\varepsilon)}(\varphi(z)+\psi(z))dz+\frac{1}{\pi i}\int_{e^{\pi i/6}}^{Re^{\pi i(1+\varepsilon)/6}}\psi(z)dz+\mathrm{cc},
\end{aligned}$$

where the notation $\lim_{R\to+\infty}\lim_{\varepsilon\to+0}$ is omitted and

$$\begin{aligned}\int_{e^{\pi i/6}}^{Re^{\pi i(1+\varepsilon)/6}}\psi(z)dz &= \int_{e^{\pi i/6}}^{Re^{\pi i(1+\varepsilon)/6}}(\varphi(z)+\psi(z))dz - \int_{e^{\pi i/6}}^{Re^{\pi i/6}}\varphi(z)dz \\ &= \left[\int_{e^{\pi i/6}}^{0} + \int_{0}^{R(i+\varepsilon)}\right](\varphi(z)+\psi(z))dz - \int_{e^{\pi i/6}}^{Re^{\pi i/6}}\varphi(z)dz\end{aligned}$$

(see Path 2 of Figure 5). Considering that $\frac{1}{2\pi i}\int_i^{R(i+\varepsilon)}\psi(z)dz$ is a pure imaginary number, we have

$$I = \frac{1}{2\pi i}\left[\int_0^{Ri} - 2\int_0^{Re^{\pi i/6}}\right]\varphi(z)dz + \frac{1}{2\pi i}\left[\int_0^{i} - 2\int_0^{e^{\pi i/6}}\right]\psi(z)dz + \text{cc.}$$

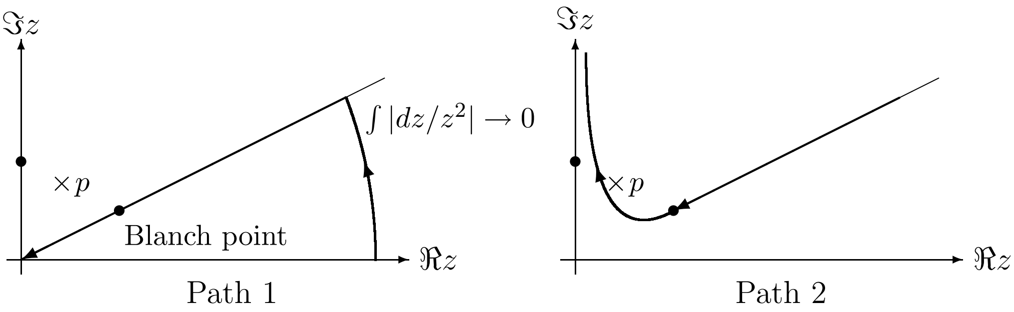

Figure 5: Integration Path

A.3 Fast algorithms for elementary functions in multiple-precision computation

We present an example that combines the CORDIC method of an R-bit unit K step and Taylor series expansion. First, we present the algorithms for the exponential and logarithmic functions.

Initial computation:

$L(j,k) := \log(1 + j/2^{Rk}) \quad (j = 1, 2, \cdots, 2^R - 1, k = 1, 2, \cdots K)$

Algorithms for the exponential and logarithmic functions:

Computation of $y = \exp x$ $(0 \le x < \log 2)$:	**Computation of** $y = \log x$ $(1/2 < x \le 1)$:

for $k = 1, 2, \cdots, K$, do
  $m_k := 0$
  for $r = R-1, R-2, \cdots, 0$, do
    $j := m_k + 2^r$
    if $x \ge L(j,k)$ then $m_k := j$
  end for
  if $m_k > 0$ then $x := x - L(m_k, k)$
end for
$y := \exp x$ (*1)
for $k = 1, 2, \cdots, K$, do
  if $m_k > 0$ then $y := y + y * m_k/2^{Rk}$
end for
	$t := 1 - x$, $y := 0$
	for $k = 1, 2, \cdots, K$, do
	  $x := 1 - t$, $m := 0$
	  for $r = R-1, R-2, \cdots, 0$, do
	    if $x/2^{Rk-r} \le t$ then (*2)
	      $t := t - x/2^{Rk-r}$
	      $m := m + 2^r$
	    end if
	  end for
	  if $m > 0$ then $y := y - L(m, k)$
	end for
	$y := y + \log(1-t)$ (*3)

(*1) and (*3) are computed using the Taylor series considering $0 \le x < 2^{-RK}$. The computation of $y * m_k/2^{Rk}$ is accelerated using bit operations and integer multiplication. In order to accelerate the inner loop (*2), low-accuracy comparison is required, except for boundary values. Furthermore, in order to accelerate Taylor series computation of log, an expansion $\log(1-t) = -2\tanh^{-1}(t/(2-t))$, $\tanh^{-1} x = x + x^3/3 + x^5/5 + x^7/7 + \cdots$ is used.

Next, we present the algorithms for trigonometric and inverse trigonometric functions.

Initial computation:

$T(j,k) := \tan^{-1}(j/2^{Rk}) \quad (j = 1, 2, \cdots, 2^R - 1, k = 1, 2, \cdots K)$

Algorithms for trigonometric and inverse trigonometric functions:

Computation of $y = \sin x$, $z = \cos x$ $(0 \le x < \pi/4)$:	**Computation of** $z = \tan^{-1}(x/y)$ $(0 \le \arg(y + ix) < \pi/4)$:

$x := x/2$
for $k = 1, 2, \cdots, K$, do
  $m_k := 0$
  for $r = R-1, R-2, \cdots, 0$, do
    $j := m_k + 2^r$
    if $x \ge T(j,k)$ then $m_k := j$
  end for
  if $m_k > 0$ then $x := x - T(m_k, k)$
end for
$y := \sin x$, $z := \cos x$
for $k = 1, 2, \cdots, K$, do
  if $m_k > 0$ then
    $t := z * m_k/2^{Rk}$, $u := y * m_k/2^{Rk}$
    $y := y + t$, $z := z - u$
  end if
end for
$t := 2 * y/(y^2 + z^2)$, $u := y$
$y := z * t$, $z := 1 - u * t$
	$z := 0$
	for $k = 1, 2, \cdots, K$, do
	  $t := x$, $m := 0$
	  for $r = R-1, R-2, \cdots, 0$, do
	    if $y/2^{Rk-r} \le t$ then
	      $t := t - y/2^{Rk-r}$, $m := m + 2^r$
	    end if
	  end for
	  if $m > 0$ then
	    $t := y * m/2^{Rk}$, $u := x * m/2^{Rk}$
	    $x := x - t$, $y := y + u$
	    $z := z + T(m,k)$
	  end if
	end for
	$z := z + \tan^{-1}(x/y)$

Finally, we present a technique for high-speed, multiple-precision Taylor series computation. For simplicity, a series by which to compute is $y = \sum_{n=0}^{M} x^n/a_n$, where a_n are integers, is assumed. Then, we choose positive integers M' and P such that $M < M'2^P$.

High-speed, multi-precision Taylor series computation:

Computation of $y = \sum_{n=0}^{M'2^P} x^n/a_n$:

for $k = 0, 1, \cdots, 2^P - 1$, do $s_k := 1/a_k$
$t := 1$, $u := x^{2^P}$, $y := 0$
for $m = 1, 2, \cdots, M' - 1$, do
  $t := t * u$
  for $k = 0, 1, \cdots, 2^P - 1$, do $s_k := s_k + t/a_{k+m2^P}$
end for
for $k = 2^P - 1, 2^P - 2, \cdots, 0$, do $y := y * x + s_k$

This algorithm contains operations of $M' + 2^P + P - 3$ multiplications,

$M'2^P$ integer divisions, and a number of additions. If we choose M' as approximately $\sqrt{M}$, then the number of multiplications is reduced to $O(\sqrt{M})$ operations.

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