GLOBAL DYNAMICS BELOW THE GROUND STATE ENERGY FOR THE ZAKHAROV SYSTEM IN THE 3D RADIAL CASE

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Abstract. We consider the global dynamics below the ground state energy for the Zakharov system in the 3D radial case. We obtain dichotomy between the scattering and the growup.

Contents

1. Introduction

In this paper, we continue our study [7] on the global Cauchy problem for the 3D Zakharov system

$$\begin{cases} i\dot{u} - \Delta u = nu, \\ \ddot{n}/\alpha^2 - \Delta n = -\Delta |u|^2, \end{cases} \tag{1.1}$$

with the initial data

$$u(0,x) = u_0,\ n(0,x) = n_0,\ \dot{n}(0,x) = n_1, \tag{1.2}$$

where $(u,n)(t,x) : \mathbb{R}^{1+3} \to \mathbb{C} \times \mathbb{R}$, and $\alpha > 0$ denotes the ion sound speed. It preserves $\|u(t)\|_{L^2_x}$ and the energy

$$E = \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{|D^{-1}\dot{n}|^2/\alpha^2 + |n|^2}{2} - n|u|^2 dx, \tag{1.3}$$

where $D := \sqrt{-\Delta}$, as well as the radial symmetry.

This system (1.1) in d dimensions was introduced by Zakharov [23] as a mathematical model for the Langmuir turbulence in unmagnetized ionized plasma. It has been extensively studied. Local wellposedness (without symmetry) is well known. For example, the well-posedness in the energy space was proved in [4] for $d = 2, 3$ and in [6] for $d = 1$, and in weighted Sobolev space in [13]. It has been improved to the critical regularity in [6, 3] for $d = 1, 2$, and to the full subcritical regularity in [6, 2] for $d \geq 4, d = 3$. The well-posedness for the system on the torus was studied in [22, 14]. These results except for [13] follow from the iteration argument using Bourgain space, where the estimates depend on α, while in [13] the well-posedness is obtained uniformly for α. For more results on the subsonic limit to NLS (as $\alpha \to \infty$), see [20, 18, 15]. Concerning the long-time behavior, Merle [16] obtained blow-up in finite or infinite time for negative energy (which we will call grow-up for brevity), while the scattering theory was studied in [21, 5, 19], dealing with solutions for given asymptotic free profiles. Recently, in [7] the authors obtained scattering for radial initial data[1] with small energy in the 3D case, by using the normal form reduction and radial-improved Strichartz estimates. The purpose of this paper is to consider the global dynamics for larger data under the radial symmetry.

To simplify the presentation, we rewrite the system into the first order as usual. Let $N := n - iD^{-1}\dot{n}/\alpha$. Then (1.1) can be rewritten as

$$(i\partial_t - \Delta)u = (\Re N)u, \quad (i\partial_t + \alpha D)N = \alpha D|u|^2, \tag{1.4}$$

with initial data $(u_0, N_0) \in H^1 \times L^2$. It has the conserved mass

$$M(u) := \int_{\mathbb{R}^3} \frac{|u|^2}{2} dx, \tag{1.5}$$

and the Hamiltonian

$$E_Z(u, N) := \int_{\mathbb{R}^3} \frac{|\nabla u|^2}{2} + \frac{|N|^2}{4} - \frac{\Re N|u|^2}{2} dx = E_S(u) + \|N - |u|^2\|_{L^2}^2/4, \tag{1.6}$$

where $E_S(u)$ denotes the Hamiltonian for the cubic NLS (the limit $\alpha \to \infty$)

$$(i\partial_t - \Delta)u = |u|^2 u, \tag{1.7}$$

namely

$$E_S(u) := \int_{\mathbb{R}^3} \frac{|\nabla u|^2}{2} - \frac{|u|^4}{4} dx. \tag{1.8}$$

Let Q be the ground state for NLS (1.7), that is the unique positive radial solution for the following equation

$$-\Delta Q + Q = Q^3, \tag{1.9}$$

which minimizes the action

$$J(Q) := E_S(Q) + M(Q) \tag{1.10}$$

[1]After submitting this paper, the authors learned the more recent result by Hani, Pusateri and Shatah [9] of small data scattering, imposing no symmetry, but instead fast decay as $|x| \to \infty$.

among all nontrivial solutions of (1.9) (see, e.g., [10] for further properties of Q). For $\lambda > 0$, let

$$Q_\lambda(x) := \lambda Q(\lambda x), \tag{1.11}$$

then we have

$$-\Delta Q_\lambda + \lambda^2 Q_\lambda = Q_\lambda^3, \quad M(Q_\lambda) = \lambda^{-1} M(Q), \quad E_S(Q_\lambda) = \lambda E_S(Q). \tag{1.12}$$

Thus the Zakharov system (1.4) has the following family of radial standing waves

$$(u, N) = (e^{i(\theta - \lambda^2 t)} Q_\lambda, Q_\lambda^2), \tag{1.13}$$

where $\lambda > 0$ and $\theta \in \mathbb{R}$ can be chosen arbitrarily.

The goal of this study is to determine global dynamics of all the radial solutions "below" the above family of special solutions, in the spirit of Kenig-Merle [12], namely the variational dichotomy into the scattering solutions and the blowup solutions. Such a result has been obtained for the limit equation (1.7) by Holmer-Roudenko [10] in the radial case, as well as in the nonradial case [11]. For the dichotomy, we need to introduce another functional (for NLS), which is the scaling derivative of the action J:

$$K(\varphi) := \partial_\lambda|_{\lambda=1} J(\lambda^{d/2}\varphi(\lambda x)) = \int_{\mathbb{R}^d} |\nabla \varphi|^2 - \frac{d|\varphi|^4}{4} dx. \tag{1.14}$$

We would like to get the same result as in [10] for NLS, but by the virial argument as in [16] we can only prove grow-up, due to the poor control of the wave component N. In fact, existence of any blowup in finite time is still an open question for the 3D Zakharov system. The main result of this paper is

Theorem 1.1. *Assume that $(u_0, N_0) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ is radial and satisfies*

$$E_Z(u_0, N_0) M(u_0) < E_S(Q) M(Q). \tag{1.15}$$

Then we have

(a) if $K(u_0) \ge 0$, then (1.4) has a unique global solution (u, N), which scatters both as $t \to \infty$ and as $t \to -\infty$ in the energy space. More precisely, there are $(u_\pm, N_\pm) \in H^1 \times L^2$ such that

$$\|(u(t), N(t)) - (e^{-it\Delta} u_\pm, e^{it\alpha D} N_\pm)\|_{H^1 \times L^2} \to 0 \quad (t \to \pm\infty). \tag{1.16}$$

(b) if $K(u_0) < 0$, then (1.4) blows up in either finite or infinite time, in the sense that $\sup_{0<t<T^*} \|(u, N)\|_{H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)} = \infty = \sup_{T_*<t<0} \|(u, N)\|_{H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)}$, where (T_*, T^*) is the maximal interval of existence.

Remark 1.1. 1) Assuming $K(u_0) = 0$ and (1.15), one can actually get by variational estimates that $u_0 = 0$, so $u \equiv 0$ and $N = e^{it\alpha D} N_0$, see Section 2.

2) The condition (1.15) is sharp in view of the standing wave solutions (1.13).

The difficulty for the scattering even for small data can be observed by comparing the time decay with the NLS of general power nonlinearity

$$i\dot{u} - \Delta u = |u|^p u, \quad u : \mathbb{R}^{1+d} \to \mathbb{C}. \tag{1.17}$$

It is well known that the scattering for NLS requires $p > 2/d$, corresponding to the time integrability of the optimal decay of the potential

$$\||u|^p\|_{L^\infty_x} \sim |t|^{-dp/2}, \tag{1.18}$$

while the scattering in H^s for any s has been proven only for $p \geq 4/d$. The 3D Zakharov system would be on the borderline in the above sense, since the potential n can decay only by

$$\|n\|_{L^\infty_x} \sim |t|^{-1}, \tag{1.19}$$

as it is solving the 3D wave equation. This suggests that the decay estimates are far insufficient for the scattering in H^1, and so it is essential to exploit nonlinear oscillations, e.g. by the normal form. This part for small radial data has been resolved in the previous paper [7]. Hence our main task in this paper is to carry out the Kenig-Merle approach [12] in accordance with the normal form. Since the normal form produces nonlinear terms without time integration, we need to modify Kenig-Merle's formulation, as well as some estimates in [7]. As a crucial ingredient for that approach, we will derive a virial identity, which is slightly different from Merle's one in [16] and more suitable for the scattering.

2. Hamiltonian and variational structures

2.1. **Virial identity.** We derive a virial identity on $\mathbb{R}^d$, which is slightly different from [16]. Recall that the Zakharov system can be rewritten in the Hamiltonian form

$$\partial_t \begin{pmatrix} u \\ N \end{pmatrix} = \mathbf{J} E'_Z(u, N), \tag{2.1}$$

where $\mathbf{J}$ and E'_Z denote the symplectic operator and the Fréchet derivative given by

$$\mathbf{J} = \begin{pmatrix} i & 0 \\ 0 & 2i\alpha D \end{pmatrix}, \quad E'_Z(u, N) = \begin{pmatrix} E'_S(u) - (\Re N - |u|^2)u \\ (N - |u|^2)/2 \end{pmatrix} = \begin{pmatrix} -\Delta u - (\Re N)u \\ (N - |u|^2)/2 \end{pmatrix}.$$

Let A be the generator for the family of scaling transforms[2]

$$\begin{pmatrix} f \\ g \end{pmatrix} \mapsto S_\lambda \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} \lambda^{d/2} f(\lambda x) \\ \lambda^{(d+1)/2} g(\lambda x) \end{pmatrix} \quad (\lambda > 0), \tag{2.2}$$

hence we have

$$\begin{aligned} A &= \begin{pmatrix} x \cdot \nabla + d/2 & 0 \\ 0 & x \cdot \nabla + (d+1)/2 \end{pmatrix}, \\ A^* &= -\begin{pmatrix} x \cdot \nabla + d/2 & 0 \\ 0 & x \cdot \nabla + (d-1)/2 \end{pmatrix}. \end{aligned} \tag{2.3}$$

[2]The order of scaling, i.e. the exponents $\frac{d}{2}$ and $\frac{d+1}{2}$, is the unique choice such that (2.5) holds.

Let $v := (u, N)$, $\underline{\mathbf{J}} := \mathbf{J}^{-1}$ and denote the real part of L^2 inner product by $\langle\cdot|\cdot\rangle$. Then the virial identity for the Zakharov system is given by

$$\begin{aligned}\partial_t\langle\underline{\mathbf{J}}v|Av\rangle &= \langle\underline{\mathbf{J}}\dot{v}|Av\rangle + \langle\underline{\mathbf{J}}v|A\dot{v}\rangle = \langle\dot{v}|(\underline{\mathbf{J}}^*A + A^*\underline{\mathbf{J}})v\rangle \\ &= 2\langle\dot{v}|\underline{\mathbf{J}}^*Av\rangle = 2\langle JE_Z'(v)|\underline{\mathbf{J}}^*Av\rangle = 2\langle E_Z'(v)|Av\rangle \\ &= 2\partial_{\lambda=1}E_Z(S_\lambda v) = 2\partial_{\lambda=1}[E_S(S_\lambda u) + \|\lambda^{1/2}N - \lambda^{d/2}|u|^2\|_2^2/4] \\ &= 2K(u) + \frac{1}{2}\|N - |u|^2\|_2^2 - \frac{d-1}{2}\langle N - |u|^2||u|^2\rangle,\end{aligned} \tag{2.4}$$

where we used for the third equality that

$$\mathbf{J}^*A^* = i\begin{pmatrix} x\cdot\nabla + d/2 & 0 \\ 0 & 2\alpha(x\cdot\nabla + (d+1)/2)D\end{pmatrix} = A\mathbf{J}. \tag{2.5}$$

Therefore, we have proved

Lemma 2.1 (Virial identity)**.** *Assume $v = (u, N)$ is a smooth decaying solution to Zakharov system* (1.1)*. Then*

$$\begin{aligned}\partial_t\langle\underline{\mathbf{J}}v|Av\rangle &= \partial_t\big[\langle u|ir\partial_r u\rangle + \frac{1}{2\alpha}\langle N|ir\partial_r D^{-1}N\rangle\big] \\ &= 2K(u) + \frac{1}{2}\|N - |u|^2\|_2^2 - \frac{d-1}{2}\langle N - |u|^2||u|^2\rangle.\end{aligned} \tag{2.6}$$

The virial identity by Merle [16] is slightly different from the above one. In our notation, it can be written as

$$\begin{aligned}&\partial_t\big[\langle u|ir\partial_r u\rangle - \frac{1}{\alpha}\langle\Re N|r\partial_r D^{-1}\Im N\rangle\big] \\ &\qquad = 2K(u) + \frac{d}{2}\|N - |u|\|_2^2 - (d-1)\|\Im N\|_2^2 \\ &\qquad = 2dE_Z(u, N) - (d-2)\|\nabla u\|_2^2 - (d-1)\|\Im N\|_2^2.\end{aligned} \tag{2.7}$$

The left hand side differs from (2.6) since $ir\partial_r D^{-1}$ is not self-adjoint, but $ir(\partial_r + (d-1)/2)D^{-1}$ is so. Precisely, the difference is

$$\partial_t\frac{d-1}{2\alpha}\langle\Re N|D^{-1}\Im N\rangle = \frac{1-d}{2}\langle N - |u|^2|N\rangle + (d-1)\|\Im N\|_2^2. \tag{2.8}$$

The advantage of our identity is that it is monotone both in the scattering region ($K > 0$) and in the blow-up region ($K < 0$), as we will show in the next section, while (2.7) is not monotone when $u(t)$ and $n(t)$ are very small compared with $\dot{n}(t)$. Although Merle's identity is more convenient in the blow-up region, our identity can also be used there, as we will see in Section 3.

2.2. **Variational estimates.** In the 3D case $d = 3$, the cubic nonlinearity is L^2-supercritical and $\dot{H}^1$ subcritical. Hence Q is obtained by the constrained minimization

$$J(Q) = \inf\{J(\varphi) \mid 0 \neq \varphi,\ K(\varphi) = 0\}. \tag{2.9}$$

Indeed, Q is the unique minimizer modulo the phase $e^{i\theta}$ and spatial translation. By scaling, we also have for any $\lambda > 0$

$$\lambda J(Q) = J_\lambda(Q_\lambda) = \inf\{J_\lambda(\varphi) \mid 0 \neq \varphi,\ K(\varphi) = 0\}, \quad J_\lambda := E_S + \lambda^2 M, \tag{2.10}$$

and Q_λ is the unique minimizer modulo phase and translation.

Lemma 2.2. *Assume that (u, N) is a solution to (1.4) with maximal interval I satisfying*

$$E_Z(u, N)M(u) < E_S(Q)M(Q). \tag{2.11}$$

Then for some $\lambda > 0$ we have $E_Z(u, N) + \lambda^2 M(u) < \lambda J(Q)$. Moreover, either $u \equiv 0$ on I, or $K(u(t)) \neq 0$ for all $t \in I$. In other words, $K(u(t))$ does not change its sign on I.

Proof. From (2.9), we have $J(Q) = \inf_{\lambda>0} J(Q_\lambda)$, and thus $\partial_\lambda|_{\lambda=1} J(Q_\lambda) = 0$. This implies

$$J^2(Q)/4 = E_S(Q)M(Q).$$

Thus we see that there exists $\lambda > 0$ such that

$$E_Z(u, N) + \lambda^2 M(u) < J_\lambda(Q_\lambda) = \lambda J(Q). \tag{2.12}$$

Since $J_\lambda(u) \le E_Z(u, N) + \lambda^2 M(u)$, by the variational characterization of Q_λ, we have at each $t \in I$,

$$K(u(t)) = 0 \iff u(t) = 0. \tag{2.13}$$

If $K(u(t_0)) = 0$ for some $t_0 \in I$, by uniqueness we have $u \equiv 0$. □

Corollary 2.3. *Assume that (u, N) is a solution to (1.4) with maximal interval I satisfying for some $\lambda > 0$*

$$E_Z(u, N) + \lambda^2 M(u) < \lambda J(Q), \ \ K(u_0) \ge 0. \tag{2.14}$$

Then $I = (-\infty, \infty)$, and moreover,

$$E_Z(u, N) + \lambda^2 M(u) \sim \|u\|_{H^1}^2 + \|N\|_{L^2}^2 \sim \|u_0\|_{H^1}^2 + \|N_0\|_{L^2}^2. \tag{2.15}$$

where the implicit constant depends only on λ and $J(Q)$.

Proof. From Lemma 2.2 we get that if $K(u_0) = 0$, then $u \equiv 0$, and hence this case is trivial. Thus we may assume $K(u_0) > 0$, hence $K(u(t)) > 0$ by Lemma 2.2. From the assumption, we get (2.15) immediately from

$$\begin{aligned}\lambda J(Q) \ge& E_Z(u, N) + \lambda^2 M(u) - K(u(t))/3\\ =& \frac{1}{6}\|\nabla u\|_2^2 + \frac{\lambda^2}{2}\|u\|_2^2 + \frac{1}{4}\|N - |u|^2\|_2^2,\end{aligned}$$

and the Sobolev inequality $\|u\|_{L^4} \lesssim \|u\|_{H^1}$. So $(u, N)(t)$ is a priori bounded in $H^1 \times L^2$, and thus by the local wellposedness we have $I = (-\infty, \infty)$. □

So far, the global well-posedness of part (a) of Theorem 1.1 is proved. It remains to prove the scattering and part (b). For both purposes, the virial estimates play crucial roles. Unlike the NLS case, it is not at all obvious that virial for (1.4) is monotone. The following lemma is our key observation

Lemma 2.4. *Let $\varphi \in H^1(\mathbb{R}^3)$, $\lambda > 0$ and $\widetilde{\nu} \ge 0$ satisfy*

$$E_S(\varphi) + \lambda^2 M(\varphi) + \frac{\widetilde{\nu}^2}{4} \le J_\lambda(Q_\lambda). \tag{2.16}$$

Then we have

$$\begin{cases} K(\varphi) \ge 0 \implies 4K(\varphi) + \widetilde{\nu}^2 \ge \sqrt{6}\widetilde{\nu}\|\varphi\|_4^2, \\ K(\varphi) \le 0 \implies 4K(\varphi) + \widetilde{\nu}^2 \le -2\widetilde{\nu}\|\varphi\|_4^2. \end{cases} \tag{2.17}$$

Proof. First, if $K(\varphi) = 0$ then $\widetilde{\nu} = 0$ and the conclusion is trivial. Hence we may assume $K(\varphi) \ne 0$ as well as $\widetilde{\nu} > 0$. Next by the scaling $(\varphi, \widetilde{\nu}) \mapsto (\lambda\varphi(\lambda x), \sqrt{\lambda}\widetilde{\nu})$, we may remove λ or assume $\lambda = 1$. Then the energy constraint becomes $J(\varphi) + \widetilde{\nu}^2/4 \le J(Q)$. Now consider the L^2 scaling of φ, $S_\mu\varphi = \mu^{d/2}\varphi(\mu x)$ and

$$\begin{aligned} J(S_\mu\varphi) &= \frac{\mu^2}{2}\|\nabla\varphi\|_2^2 + \frac{1}{2}\|\varphi\|_2^2 - \frac{\mu^3}{4}\|\varphi\|_4^4, \\ \mu\partial_\mu J(S_\mu\varphi) &= K(S_\mu\varphi) = \mu^2\|\nabla\varphi\|_2^2 - \frac{3\mu^3}{4}\|\varphi\|_4^4. \end{aligned} \tag{2.18}$$

There is a unique $0 < \mu \ne 1$ such that

$$\|\nabla\varphi\|_2^2 = \frac{3\mu}{4}\|\varphi\|_4^4, \tag{2.19}$$

which is equivalent to $K(S_\mu\varphi) = 0$. Then the variational characterization of Q implies $J(S_\mu\varphi) \ge J(Q)$, and so

$$\begin{aligned} \frac{\widetilde{\nu}^2}{4} \le J(S_\mu\varphi) - J(\varphi) &= \frac{\mu^2 - 1}{2}\|\nabla\varphi\|_2^2 - \frac{\mu^3 - 1}{4}\|\varphi\|_4^4, \\ &= \frac{(\mu - 1)^2(\mu + 2)}{8}\|\varphi\|_4^4, \end{aligned} \tag{2.20}$$

where (2.19) is used in the last step. Let $X := \|\varphi\|_4^2/\widetilde{\nu}$. Then the above inequality is rewritten as

$$|\mu - 1|\sqrt{\mu + 2}X \ge \sqrt{2}. \tag{2.21}$$

Hence it suffices to estimate, under the above constraint,

$$\frac{4K(\varphi) + \widetilde{\nu}^2}{\widetilde{\nu}\|\varphi\|_4^2} = 3(\mu - 1)X + 1/X =: f(X, \mu). \tag{2.22}$$

For $K(\varphi) > 0$, or equivalently $\mu > 1$, $f(X, \mu)$ is increasing in X unless

$$\sqrt{\frac{1}{3(\mu - 1)}} < \frac{1}{\mu - 1}\sqrt{\frac{2}{\mu + 2}}, \tag{2.23}$$

which is solved $\mu > (\sqrt{33} - 1)/2$. In the latter case, we have

$$3(\mu - 1)X + 1/X \ge 2\sqrt{3(\mu - 1)X/X} > \sqrt{6}, \tag{2.24}$$

since $\mu > 3/2$. Otherwise, the minimum is attained at the boundary and equal to

$$f(\frac{1}{\mu - 1}\sqrt{\frac{2}{\mu + 2}}, \mu) = 3\sqrt{\frac{2}{\mu + 2}} + (\mu - 1)\sqrt{\frac{\mu + 2}{2}} =: b(\mu), \tag{2.25}$$

which is increasing[3] in $\mu > 0$, hence $b(\mu) > b(1) = \sqrt{6}$.

For $K(\varphi) < 0$, or equivalently $0 < \mu < 1$, $-f(X, \mu)$ is increasing in X, so its minimum is attained at the boundary and equals to

$$-f(\frac{1}{1-\mu}\sqrt{\frac{2}{\mu+2}}, \mu) = b(\mu) > b(0) = 2. \tag{2.26}$$

Therefore, the proof of the lemma is completed. □

Remark 2.1. Applying the lemma above by letting

$$\widetilde{\nu} > \|N - |u|^2\|_2, \tag{2.27}$$

we get from Lemma 2.1 that the virial $\langle \underline{\mathbf{J}}v|Av\rangle$ is monotone in our consideration. This fact will play crucial role in our consequent analysis.

3. Growup at infinity

This section is devoted to prove part (b) of Theorem 1.1. We assume that under the assumption of part (b), the solution exists for all $t > 0$. We will show that $\sup_{t>0} \|(u, N)\|_{H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)} = \infty$.

In the following computation, the algebraic part is valid in general dimensions $\mathbb{R}^d$, but we need to specify $d = 3$ for the estimates. We will keep d in the identities because they may be recycled in other dimensions, but the reader can regard $d = 3$ throughout this section or the entire paper.

3.1. **Localized virial.** Let $X = X^*$ be the operator of smooth truncation to $|x| < R$ by multiplication with $\psi_R(x) = \psi(x/R)$, where $\psi \in C_0^\infty(\mathbb{R}^3)$ is a fixed radial function satisfying $0 \le \psi \le 1$, $\partial_r\psi \le 0$, $\psi(x) = 1$ for $|x| \le 1$ and $\psi(x) = 0$ for $|x| \ge 2$. We consider the localized virial quantity in the form

$$V_R(t) := \langle \underline{J}v|(AX + XA)v\rangle. \tag{3.1}$$

Then similarly to the non-localized virial identity, we can compute

$$\dot{V}_R = \langle E_Z'(v)|(AX + XA + AJX\underline{J} + J^*X\underline{J}^*A)v\rangle. \tag{3.2}$$

Putting $\nu := N - |u|^2$, the right hand side can be written componentwise

$$\begin{aligned}\dot{V}_R =& \langle E_S'(u) - \nu u|2A_0Xu + 2XA_0u\rangle \\ &+ \langle \nu/2|(XA_1 + A_1X + DXD^{-1}A_1 + A_1DXD^{-1})(\nu + |u|^2)\rangle,\end{aligned} \tag{3.3}$$

where $A_j := x\cdot\nabla + (d+j)/2$. The right hand side is decomposed into the NLS part:

$$NS := \langle E_S'(u)|2A_0Xu + 2XA_0u\rangle, \tag{3.4}$$

the quadratic terms in ν:

$$QN := \langle \nu/2|(XA_1 + A_1X + DXD^{-1}A_1 + A_1DXD^{-1})\nu\rangle, \tag{3.5}$$

and the cubic cross terms:

$$\begin{aligned}CC :=& \langle -\nu u|2A_0Xu + 2XA_0u\rangle \\ &+ \langle \nu/2|(XA_1 + A_1X + DXD^{-1}A_1 + A_1DXD^{-1})|u|^2\rangle,\end{aligned} \tag{3.6}$$

[3] This can be checked by computing $\frac{d(b^2)}{d\mu}$.

i.e., $\dot{V}_R = NS+QN+CC$. Since the NLS part has been treated by Ogawa-Tsutsumi [17] and Holmer-Roudenko [10], while the cross terms are higher order, the main problem for us is to control QN. Indeed, our way of the localization is motivated by a better cancellation in QN, while some other multipliers such as AXv in (3.2) could make the other terms simpler.

It is further decomposed $QN = (QN_1 + QN_2 + QN_3)/2$ with

$$QN_1 := \langle \nu|(XA_1 + A_1X)\nu\rangle = \langle \nu|X(A_1 + A_1^*)\nu\rangle = \langle \nu|X\nu\rangle, \tag{3.7}$$

where we used the symmetry of the bilinear form as well as $X = X^*$ and $A_0 = -A_0^*$. Putting $\eta := D^{-1}\nu$, the other two terms are computed as follows.

$$\begin{aligned} QN_2 :&= \langle \nu|DXD^{-1}A_1\nu\rangle = \langle \eta|D^2XA_{-1}\eta\rangle = \langle \nabla\eta|\nabla XA_{-1}\eta\rangle \\ &= \langle \nabla\eta|XA_1\nabla\eta\rangle + \langle \nabla\eta|(\nabla\psi_R)A_{-1}\eta\rangle, \end{aligned} \tag{3.8}$$

where we used $DA_{-1} = A_1D$ and $\nabla A_{-1} = A_1\nabla$,

$$\begin{aligned} QN_3 :&= \langle \nu|A_1DXD^{-1}\nu\rangle = \langle \eta|DA_1DX\eta\rangle = \langle \nabla\eta|A_1\nabla X\eta\rangle \\ &= \langle \nabla\eta|A_1X\nabla\eta\rangle + \langle \nabla\eta|A_1(\nabla\psi_R)\eta\rangle, \end{aligned} \tag{3.9}$$

where we used $DA_1D = -\nabla\cdot A_1\nabla$. Hence

$$\begin{aligned} QN_2 + QN_3 &= \langle \nabla\eta|X(A_1 + A_1^*)\nabla\eta\rangle + \langle \nabla\eta|(\nabla\psi_R)A_{-1}\eta + A_1(\nabla\psi_R)\eta\rangle \\ &= \langle \nabla\eta|X\nabla\eta\rangle + 2\langle \nabla\eta|(\nabla\psi_R)x\cdot\nabla\eta\rangle + \langle \nabla\eta|\eta A_d\nabla\psi_R\rangle \\ &= \langle \nabla\eta|X\nabla\eta\rangle + 2\langle \eta_r|r\psi_R'\eta_r\rangle - \frac{1}{2}\langle |\eta|^2|A_{d+2}\Delta\psi_R\rangle, \end{aligned} \tag{3.10}$$

where we used the radial symmetry of ψ_R but not of η. Thus we obtain

$$QN = \langle \nu|X\nu\rangle/2 + \langle \nabla\eta|X\nabla\eta\rangle/2 + \langle \eta_r|r\psi_R'\eta_r\rangle - \frac{1}{4}\langle |\eta|^2|A_{d+2}\Delta\psi_R\rangle. \tag{3.11}$$

The first two terms are less than $\|\nu\|_2^2 = \|\nabla\eta\|_2^2$ since $\psi_R \le 1$, while the third term is nonpositive since $\psi_R' \le 0$. The last term is bounded from above and below by[4]

$$\rho_R := \int_{|x|\sim R} \frac{|\eta|^2}{R^2}dx \lesssim \|\nabla\eta\|_2^2 = \|\nu\|_2^2. \tag{3.12}$$

In short, we have

$$QN(t) \le \|\nu\|_2^2 + O(\rho_R(t)). \tag{3.13}$$

$\rho_R(t) \to 0$ as $R \to \infty$ for each fixed t, but some uniform decay is needed for the main term $\dot{V}_\infty(t) = 4K + \|\nu\|_2^2 + (1-d)\langle \nu||u|^2\rangle$ to absorb the error. For that we use the equation of η:

$$(i\partial_t + \alpha D)\eta = D^{-1}(i\partial_t + \alpha D)(N - |u|^2) = -iD^{-1}|u|_t^2, \tag{3.14}$$

[4] Such an error term does not occur in Merle's virial identity [16]. This is a disadvantage of our identity. Nevertheless we can dispose of it using the evolution equation.

and the corresponding integral equation

$$\begin{aligned}&\eta = \eta^0 + \eta^1, \quad \eta^0 := e^{i\alpha Dt}\eta(0),\\&\eta^1 := -\int_0^t e^{i\alpha D(t-s)}D^{-1}|u(s)|_s^2 ds = \eta^2 + \eta^3 + \eta^4,\\&\eta^2 := D^{-1}[e^{i\alpha Dt}|u(0)|^2 - |u(t)|^2],\\&\eta^3 := i\alpha\int_0^{(t-1)_+} e^{i\alpha D(t-s)}|u(s)|^2 ds, \quad \eta^4 := i\alpha\int_{(t-1)_+}^t e^{i\alpha D(t-s)}|u(s)|^2 ds.\end{aligned} \tag{3.15}$$

We use the above equation only for very low frequency. More precisely, with a small parameter $0 < \delta < 1$ independent of t, decompose η smoothly in the Fourier space

$$\eta = \eta_{<\delta} + \eta_{>\delta}, \quad \eta_{<\delta} := \mathcal{F}^{-1}\psi_\delta\mathcal{F}\eta, \tag{3.16}$$

then we have $\|\eta_{>\delta}\|_2 \le \delta^{-1}\|\nu\|_2$. For the low frequency part, we have

$$\begin{aligned}&\|\eta^0_{<\delta}\|_{\dot H^1} = \|\nu_{<\delta}(0)\|_{L^2},\\&\|\eta^2_{<\delta}\|_{\dot H^{-1/2+}} \lesssim \||u(0)|^2\|_{L^1} + \||u(t)|^2\|_{L^1} \lesssim \|u(0)\|_2^2,\\&\|\eta^4_{<\delta}\|_{\dot H^{-3/2+}} \lesssim \alpha\||u|^2\|_{L_t^\infty L_x^1} \lesssim \alpha\|u(0)\|_2^2,\\&\|\eta^3_{<\delta}\|_{L^4} \lesssim \delta^{1/2}\|\eta^3_{<\delta}\|_{\dot B^{-1/2}_{4,\infty}} \lesssim \delta^{1/2}\|\eta^3_{<\delta}\|^{1/2}_{\dot B^{-2}_{\infty,\infty}}\|\eta^3_{<\delta}\|^{1/2}_{\dot H^1},\end{aligned} \tag{3.17}$$

and by the L^∞ decay of the wave equation,

$$\|\eta^3_{<\delta}\|_{\dot B^{-2}_{\infty,\infty}} \lesssim \int_0^{(t-1)_+} \frac{1}{|t-s|}\|u(s)\|_2^2 ds \lesssim \|u(0)\|_2^2\log(t+1). \tag{3.18}$$

Thus we obtain

$$\|\eta^1_{<\delta}\|_{L_t^\infty(0,T;L_x^4)} \lesssim \|u(0)\|_2^2\delta\log(T+2) + \|\nu\|^2_{L_t^\infty(0,T;L_x^2)}, \tag{3.19}$$

and so

$$\begin{aligned}\sup_{0<t<T}\rho_R(t) \lesssim&\|\nu_{<\delta}(0)\|_2^2 + R^{-1/2}[\|u(0)\|_2^2\delta\log(T+2) + \|\nu\|^2_{L_t^\infty(0,T;L_x^2)}]\\&+ (\delta R)^{-2}\|\nu\|^2_{L_t^\infty(0,T;L_x^2)}.\end{aligned} \tag{3.20}$$

Next we estimate the cubic cross terms

$$\begin{aligned}&CC = CC_1 + CC_2 + CC_3,\\&CC_1 := -2\langle \nu u|A_0Xu + XA_0u\rangle = -2\langle \nu|(r\psi_R' + XA_d)|u|^2\rangle,\\&CC_2 := \frac{1}{2}\langle \nu|(XA_1 + A_1X)|u|^2\rangle = \langle \nu|(XA_1 + r\psi_R'/2)|u|^2\rangle,\\&CC_3 := \frac{1}{2}\langle \nu|(DXD^{-1}A_1 + A_1DXD^{-1})|u|^2\rangle.\end{aligned} \tag{3.21}$$

For the last term we use the commuting relations:

$$\begin{aligned}A_1DXD^{-1} = DA_{-1}XD^{-1} &= D(XA_{-1} + r\psi_R')D^{-1}\\&= DXD^{-1}A_1 + Dr\psi_R'D^{-1},\end{aligned} \tag{3.22}$$

and so

$$\begin{aligned}&CC_3 = \langle \nu|(XA_1 + r\psi_R'/2)|u|^2\rangle + CC_3',\\&CC_3' := \langle \nu|([D,X]D^{-1}A_1 + [D,r\psi_R']D^{-1}/2)|u|^2\rangle.\end{aligned} \tag{3.23}$$

Hence

$$CC = (1-d)\langle\nu||u|^2\rangle + \langle\nu|[(1-d)(\psi_R-1) - r\psi_R']|u|^2\rangle + CC_3', \tag{3.24}$$

and the second term on the right is bounded by

$$\int_{|x|\gtrsim R} |\nu u^2|dx \lesssim \|\nu\|_2\|u\|_2\|u\|_{L^\infty(|x|\gtrsim R)} \lesssim R^{-1}\|\nu\|_2\|u\|_2^{3/2}\|\nabla u\|_2^{1/2}, \tag{3.25}$$

since the functions in the brackets [] vanish on $|x| \lesssim R$. We used the radial Sobolev inequality

$$\varphi(x) = \varphi(|x|) \in H^1(\mathbb{R}^3) \implies \|r\varphi\|_{L^\infty(\mathbb{R}^3)} \lesssim \|\varphi\|_2^{1/2}\|\nabla\varphi\|_2^{1/2}. \tag{3.26}$$

For the commutator terms CC_3', we use the elementary commutator estimate[5]

$$\|[D,f]g\|_{L^2} \lesssim \|\mathcal{F}(\nabla f)\|_{L^1}\|g\|_{L^2}, \tag{3.27}$$

together with the (radial/nonradial) Sobolev

$$\begin{aligned} &\|xu^2\|_2 \le \|xu\|_\infty\|u\|_2 \lesssim \|u\|_2^{3/2}\|\nabla u\|_2^{1/2}, \\ &\|D^{-1}|u|^2\|_2 \lesssim \||u|^2\|_{6/5} \le \|u\|_2\|u\|_3 \lesssim \|u\|_2^{3/2}\|\nabla u\|_2^{1/2}. \end{aligned} \tag{3.28}$$

Since $\|\mathcal{F}(\nabla\psi_R)\|_1 = CR^{-1}$, we thus obtain

$$|CC_3'| \lesssim \|\nu\|_2 R^{-1}[\|D^{-1}\nabla\cdot x|u|^2\|_2 + \|D^{-1}|u|^2\|_2] \lesssim R^{-1}\|\nu\|_2\|u\|_2^{3/2}\|\nabla u\|_2^{1/2}. \tag{3.29}$$

In short, we have obtained

$$CC = (1-d)\langle\nu||u|^2\rangle + O(R^{-1}\|\nu\|_2\|u\|_2^{3/2}\|\nabla u\|_2^{1/2}). \tag{3.30}$$

Finally we estimate the NLS part

$$\begin{aligned} NS/2 &= \langle -\Delta u - |u|^2 u | A_0 X u + X A_0 u\rangle \\ &= \langle\nabla u|\nabla(A_0X + XA_0)u\rangle - \langle r\psi_R'||u|^4\rangle - \langle\psi_R|(r\partial_r/2 + d)|u|^4\rangle \\ &=: NS_1 + NS_2 + NS_3. \end{aligned} \tag{3.31}$$

For the first term NS_1 we use

$$\begin{aligned} \nabla(A_0X + XA_0) &= A_2\nabla X + X\nabla A_0 + [\nabla\psi_R]A_0 \\ &= A_2X\nabla + A_2[\nabla\psi_R] + XA_2\nabla + [\nabla\psi_R]A_0 \\ &= (A_2X + XA_2)\nabla + 2[\nabla\psi_R]r\partial_r + [A_{2+d}\nabla\psi_R], \end{aligned} \tag{3.32}$$

where the bracket denotes the multiplication with the inside function. Using $A_0^* = -A_0$ as well, we obtain

$$NS_1 = \langle\nabla u|2X\nabla u\rangle + 2\langle u_r|\psi_R' r u_r\rangle + \frac{1}{2}\langle\nabla|u|^2|A_{2+d}\nabla\psi_R\rangle. \tag{3.33}$$

Since $\psi_R \le 1$ and $\psi_R' \le 0$, the first term is less than $2\|\nabla u\|_2^2$ and the second is nonpositive. The last term equals

$$-\frac{1}{2}\langle|u|^2|A_d\Delta\psi_R\rangle \lesssim \|u\|_2^2\|A_d\Delta\psi_R\|_\infty \lesssim R^{-2}\|u\|_2^2. \tag{3.34}$$

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[5]This follows from Plancherel: $\|[D,f]g\|_2 \sim \|[|\xi|,\hat{f}*]\hat{g}\|_2 \le \|(|\xi||\hat{f}|)*|\hat{g}|\|_2 \le \|\xi\hat{f}\|_1\|\hat{g}\|_2$.

The quartic terms equal

$$NS_2 + NS_3 = -\frac{1}{2}\langle (r\partial_r + d)\psi_R | |u|^4\rangle = -\frac{d}{2}\|u\|_4^4 - \frac{1}{2}\langle r\psi_R' + d(\psi_R - 1) | |u|^4\rangle, \quad (3.35)$$

and the last term is bounded by

$$\|u\|_{L^4(|x|\gtrsim R)}^4 \le \|u\|_2^2 \|u\|_{L^\infty(|x|\gtrsim R)}^2 \lesssim R^{-2}\|u\|_2^3\|\nabla u\|_2, \quad (3.36)$$

using the radial Sobolev inequality. In short, we have obtained

$$NS/2 \le 2K(u) + O(R^{-2}\|u\|_2^3\|\nabla u\|_2). \quad (3.37)$$

Gathering the above estimates on QN, CC and NS, we obtain

$$\begin{aligned} \dot V_R \le\ & 4K(u) + \|\nu\|_2^2 + (1-d)\langle \nu | |u|^2\rangle + O(\rho_R) \\ & + O(R^{-1}\|\nu\|_2\|u\|_2^{3/2}\|\nabla u\|_2^{1/2}) + O(R^{-2}\|u\|_2^3\|\nabla u\|_2), \end{aligned} \quad (3.38)$$

and

$$\begin{aligned} \sup_{0<t<T} \rho_R \lesssim & \|\nu_{<\delta}(0)\|_2^2 + R^{-1/2}[\|u(0)\|_2^2 \delta \log(T+2) + \|\nu\|_{L_t^\infty(0,T;L_x^2)}^2] \\ & + (\delta R)^{-2}\|\nu\|_{L_t^\infty(0,T;L_x^2)}^2. \end{aligned} \quad (3.39)$$

Also we have

$$|V_R| \lesssim R[\|u\|_2\|\nabla u\|_2 + \|N\|_2^2]. \quad (3.40)$$

Now suppose for contradiction that

$$\sup_{t>0} \|u(t)\|_{H_x^1} + \|N(t)\|_{L_x^2} \le M \in [1,\infty), \quad (3.41)$$

then $\|\nu\|_{L_x^2} \lesssim M^2$ and $|V_R| \lesssim RM^2$. The variational lemma 2.4 provides us with an upper bound

$$\dot V_\infty = 4K(u) + \|\nu\|_2^2 + (1-d)\langle \nu | |u|^2\rangle \le -\kappa, \quad (3.42)$$

with $d = 3$ and $\kappa := 4[J_\lambda(Q_\lambda) - E_Z(v) - \lambda^2 M(u)] > 0$, choosing $\widetilde{\nu}^2 = \|\nu\|_2^2 + \kappa$. We can first choose $0 < \delta \ll 1$ so small that $\|\nu_{<\delta}(0)\|_2^2 \ll \kappa$. Secondly we can choose $R \gg 1$ so large that

$$R^{-1/2}M^2\delta\log(RM^2/(\delta\kappa)) \ll \kappa, \quad (R^{-1/2} + (\delta R)^{-2})M^4 \ll \kappa, \quad (3.43)$$

where $\log(RM^2/(\delta\kappa))$ may be replaced with $(RM^2/(\delta\kappa))^{1/6}$ for example. Then for $0 < t < RM^2/\delta\kappa =: T$, we have $\dot V_R \le -\kappa/2$, and so $|V_R(T) - V_R(0)| \ge \kappa T/2 = \frac{RM^2}{2\delta}$, which is contradicting the above bound on $|V_R|$.

4. Concentration-compactness procedure

It remains to prove the scattering in part (a) of Theorem 1.1. Thanks to the variational estimates in Section 2, we can proceed as Kenig-Merle. For each $0 \le a \le J(Q)$ and $\lambda > 0$, let

$$\begin{aligned} &\mathscr{E}_\lambda(f,g) := \lambda^{-1}E_Z(f,g) + \lambda M(f), \\ &\mathcal{K}_\lambda^+(a) := \{(f,g) \in H_r^1 \times L_r^2 \mid \mathscr{E}_\lambda(f,g) < a,\ K(f) \ge 0\}, \\ &\mathcal{S}_\lambda(a) := \sup\{\|(u,N)\|_S \mid (u(0),N(0)) \in \mathcal{K}_\lambda^+(a),\ (u,N) \text{ sol.}\}, \end{aligned} \quad (4.1)$$

where S denotes a norm containing almost all the Strichartz norms for radial free solutions, including $L^\infty_t(H^1 \times L^2)$. See (4.25) for the precise definition. For any time interval I, we will denote by $S(I)$ the restriction of S onto I.

From Corollary 2.3 we already know that all solutions starting from $\mathcal{K}^+_\lambda(a)$ stays there globally in time. What we want to prove is the uniform scattering below the ground state energy, i.e. $\mathcal{S}_\lambda(a) < \infty$ for all $a < J(Q)$. Let

$$E^*_\lambda := \sup\{a > 0 \mid \mathcal{S}_\lambda(a) < \infty\}. \tag{4.2}$$

The small data scattering in [7] implies that $E^*_\lambda > 0$, and the existence of the ground state soliton implies that $E^*_\lambda \le J(Q)$. We will prove $E^*_\lambda = J(Q)$ by contradiction, and thus finish the proof of Theorem 1.1 (a). The main result in this section is

Lemma 4.1 (Existence of critical element)**.** *Suppose $E^*_\lambda < J(Q)$, then there is a global solution (u, N) in $\mathcal{K}^+_\lambda(a)$ satisfying*

$$\mathscr{E}_\lambda(u, N) = E^*_\lambda, \quad \|(u, N)\|_{S(-\infty,0)} = \|(u, N)\|_{S(0,\infty)} = \infty. \tag{4.3}$$

Moreover, $\{(u, N)(t) \mid t \in \mathbb{R}\}$ is precompact in $H^1_x \times L^2_x$.

We will prove this lemma by following the concentration-compactness procedure. The main difference from NLS is that we need to work with the solutions after the normal form transform. In particular, we have some nonlinear terms without time integration (or the Duhamel form). Besides that, we have various different interactions, for which we need to use different norms or exponents.

4.1. **Profiles for the radial Zakharov.** First we recall the free profile decomposition of Bahouri-Gérard type [1]. Actually we do not need its full power, as we can freeze scaling and space positions of the profiles thanks to the radial symmetry and the regularity room of our problem. Hence the setting is essentially the same as the NLS case [10].

Lemma 4.2. *For any bounded sequence (f_n, g_n) in $H^1_r \times L^2_r$, there is a subsequence (f'_n, g'_n), $\bar J \in \mathbb{N} \cup \{\infty\}$, a bounded sequence $\{\mathbf{f}^j, \mathbf{g}^j\}_{1\le j<\bar J}$ in $H^1_r \times L^2_r$, and sequences $\{t^j_n\}_{n\in\mathbb{N},1\le j<\bar J} \subset \mathbb{R}$, such that the following holds. For any $0 \le j \le J < \bar J$, let*

$$\begin{aligned} &u_n(t) := e^{-it\Delta} f'_n, \quad N_n(t) := e^{it\alpha D} g'_n, \\ &\mathbf{u}^j_n(t) := e^{-i(t-t^j_n)\Delta} \mathbf{f}^j, \quad \mathbf{N}^j_n(t) := e^{i(t-t^j_n)\alpha D} \mathbf{g}^j, \\ &u^{>\mathrm{J}}_n := u_n - \sum_{j=1}^J \mathbf{u}^j_n, \quad N^{>\mathrm{J}}_n := N_n - \sum_{j=1}^J \mathbf{N}^j_n. \end{aligned} \tag{4.4}$$

Then for any $j, k \in \{1 \dots J\}$, we have $t^j_\infty := \lim_{n\to\infty} t^j_n \in \{0, \pm\infty\}$,

$$j \ne k \implies \lim_{n\to\infty} |t^j_n - t^k_n| = \infty, \tag{4.5}$$

$$\begin{aligned} &(u^{>\mathrm{J}}_n, N^{>\mathrm{J}}_n)(t^j_n) \to 0 \text{ weakly in } H^1 \times L^2 \text{as } n \to \infty, \\ &(u^{>\mathrm{J}}_n, N^{>\mathrm{J}}_n)(0) \to 0 \text{ weakly in } H^1 \times L^2 \text{as } n \to \infty, \end{aligned} \tag{4.6}$$

and for any $\delta > 0$,

$$\lim_{J\to\bar J} \limsup_{n\to\infty} [\|u^{>\mathrm{J}}_n\|_{L^\infty_t B^{-1/2-\delta}_\infty} + \|N^{>\mathrm{J}}_n\|_{L^\infty_t (\dot B^{-3/2-\delta}_\infty + \dot B^{-3/2+\delta}_\infty)}] = 0. \tag{4.7}$$

Remark 4.1. 1) (4.5)–(4.6) implies the linear orthogonality

$$\begin{aligned} &\lim_{n\to\infty} \|u_n(0)\|_{H^1}^2 - \sum_{j=1}^{J} \|\mathbf{u}_n^j(0)\|_{H^1}^2 - \|u_n^{>\mathrm{J}}(0)\|_{H^1}^2 = 0, \\ &\lim_{n\to\infty} M(u_n(0)) - \sum_{j=1}^{J} M(\mathbf{u}_n^j(0)) - M(u_n^{>\mathrm{J}}(0)) = 0, \\ &\lim_{n\to\infty} \|N_n(0)\|_{L^2}^2 - \sum_{j=1}^{J} \|\mathbf{N}_n^j(0)\|_{L^2}^2 - \|N_n^{>\mathrm{J}}(0)\|_{L^2}^2 = 0, \end{aligned} \tag{4.8}$$

as well as the nonlinear orthogonality

$$\begin{aligned} &\lim_{n\to\infty} \|u_n(0)\|_{L^4}^4 - \sum_{j=1}^{J} \|\mathbf{u}_n^j(0)\|_{L^4}^4 - \|u_n^{>\mathrm{J}}(0)\|_{L^4}^4 = 0, \\ &\lim_{n\to\infty} E_S(u_n(0)) - \sum_{j=1}^{J} E_S(\mathbf{u}_n^j(0)) - E_S(u_n^{>\mathrm{J}}(0)) = 0, \\ &\lim_{n\to\infty} K(u_n(0)) - \sum_{j=1}^{J} K(\mathbf{u}_n^j(0)) - K(u_n^{>\mathrm{J}}(0)) = 0, \\ &\lim_{n\to\infty} E_Z(u_n(0), N_n(0)) - \sum_{j=1}^{J} E_Z(\mathbf{u}_n^j(0), \mathbf{N}_n^j(0)) - E_Z(u_n^{>\mathrm{J}}(0), N_n^{>\mathrm{J}}(0)) = 0. \end{aligned} \tag{4.9}$$

The same orthogonality holds also along $t = t_n^j$ instead of $t = 0$.

2) The norms in (4.7) are related to the Sobolev embedding $L^2 \subset \dot{B}_\infty^{-3/2}$. Interpolation with the Strichartz estimate extends the smallness to any Strichartz norms as far as the exponents are not sharp either in L^p or in regularity (including the low frequency of N).

We call such a sequence of free solutions $\{(\mathbf{u}_n^j, \mathbf{N}_n^j)\}_{n\in\mathbb{N}}$ a *free concentrating wave*. Now we introduce the nonlinear profile associated to a free concentrating wave

$$(\mathbf{u}_n(t), \mathbf{N}_n(t)) = U(t - t_n)(\mathbf{f}, \mathbf{g}), \quad t_\infty = \lim_{n\to\infty} t_n \in \{0, \pm\infty\}, \tag{4.10}$$

where $U(t) = e^{-it\Delta} \oplus e^{it\alpha D}$ denotes the free propagator. With it, we associate the *nonlinear profile* $(\mathfrak{u}, \mathfrak{N})$, defined as the solution of the Zakharov system satisfying

$$(u, N) = U(t)(\mathbf{f}, \mathbf{g}) + \int_{-t_\infty}^{t} U(t - s)(nu, \alpha D|u|^2)(s) ds, \tag{4.11}$$

which is obtained by solving the initial data problem (if $t_\infty = 0$) or by solving the final data problem (if $t_\infty = \pm\infty$). When $t_\infty = \pm\infty$, the existence of wave operators will be given at the end of this paper as appendix .

We call $(\mathfrak{u}_n(t), \mathfrak{N}_n(t)) := (\mathfrak{u}(t - t_n), \mathfrak{N}(t - t_n))$ the *nonlinear concentrating wave* associated with $(\mathbf{u}_n(t), \mathbf{N}_n(t))$. By the above construction we have

$$\begin{aligned} &\|(\mathbf{u}_n, \mathbf{N}_n)(0) - (\mathfrak{u}_n, \mathfrak{N}_n)(0)\|_{H^1 \times L^2} \\ &= \|(\mathfrak{u}, \mathfrak{N})(-t_n) - U(-t_n)(\mathbf{f}, \mathbf{g})\|_{H^1 \times L^2} \to 0. \end{aligned} \tag{4.12}$$

Given a sequence of solutions to the Zakharov system with bounded initial data, we can apply the free profile decomposition Lemma 4.2 to the sequence of initial data, and associate a nonlinear profile with each free concentrating wave. If all nonlinear profiles are scattering and the remainder is small enough, then we can conclude that the original sequence of nonlinear solutions is also scattering with a global Strichartz bound. More precisely, we have

Lemma 4.3. *For each free concentrating wave* $(\mathbf{u}_n^j, \mathbf{N}_n^j)$ *in Lemma 4.2, let* $(\mathfrak{u}_n^j, \mathfrak{N}_n^j)$ *be the associated nonlinear concentrating wave. Let* (u_n, N_n) *be the sequence of nonlinear solutions with* $(u_n, N_n)(0) = (f_n, g_n)$*. If* $\|(\mathfrak{u}_n^j, \mathfrak{N}_n^j)\|_{S(0,\infty)} < \infty$ *for all* $j < \bar{J}$*, then*

$$\limsup_{n\to\infty} \|(u_n, N_n)\|_{S(0,\infty)} < \infty. \tag{4.13}$$

To prove Lemma 4.3, we need some global stability. In the next subsection, we will refine the normal form reduction and the nonlinear estimates that was used in [7], and then prove Lemma 4.3 and Lemma 4.1.

4.2. **Nonlinear estimates with small non-sharp norms.** In order to obtain the nonlinear profile decomposition, we need that the non-sharp smallness (4.7) is sufficient to reduce the nonlinear interactions globally. The idea is to use interpolation, thus we need to do some refined estimates than in [7], more precisely, to avoid using the sharp (or endpoint) norms with L_t^2 or L_t^∞.

4.2.1. *Modifying the nonresonant part.* The first problem in following the Strichartz analysis in [7] is the L_t^2-type norms. In fact, one can observe that the use of L_t^2-type Strichartz norm for N is inevitable for the low-high interactions of nu in very low frequencies, since the regularity exponent becomes bigger than that for the dual Schrödinger admissible exponent as we move the Strichartz norm of N to L_t^{2+}.

However, this problem can be avoided by applying the normal form to those interactions. In fact, there is no resonance in very low frequencies because

$$-|\xi|^2 \pm \alpha|\xi-\eta| + |\eta|^2 \sim \alpha|\xi-\eta| \tag{4.14}$$

when all of $|\xi|, |\xi-\eta|, |\eta|$ are small. Hence we include them into the "non-resonant" interactions, which are integrated in time before the Strichartz estimate.

The second problem is that our solution is no longer small, so the nonlinear terms without time integration (i.e. the boundary terms from the partial integration) do not contain any small factor for the perturbation argument. To overcome this difficulty, we shrink the "non-resonant" part to either higher or lower frequencies, for which we gain a small factor, depending on the frequencies, from the regularity room. Hence our decomposition into the "resonant" and "non-resonant" interactions depends on the solution size.

Thus we are lead to divide the bilinear interactions nu and $|u|^2$ as follows. Let $u = \sum_{k\in\mathbb{Z}} P_k u$ be the standard homogeneous Littlewood-Paley decomposition such that $\operatorname{supp} \mathcal{F} P_k u \subset \{2^{k-1} < |\xi| < 2^{k+1}\}$. For a parameter $\beta \geq 5 + |\log_2 \alpha|$, let

$$\begin{aligned} &XL := \{(j,k) \in \mathbb{Z}^2 \mid j \geq \max(k+5, \beta)\}, \\ &RL := \{(j,k) \in \mathbb{Z}^2 \mid |j| < \beta \text{ and } k \leq \max(j-5, -\beta)\}, \\ &LL := \{(j,k) \in \mathbb{Z}^2 \mid \max(j,k) \leq -\beta\}, \\ &LH := \{(j,k) \in \mathbb{Z}^2 \mid k > \max(j-5, -\beta)\}, \\ &HH := \{(j,k) \in \mathbb{Z}^2 \mid |j-k| < 5 \text{ and } \max(j,k) \geq \beta\}, \\ &RR := \{(j,k) \in \mathbb{Z}^2 \mid \max(j,k) < \beta\}, \end{aligned} \tag{4.15}$$

and $LX := \{(k,j) \mid (j,k) \in XL\}$. Then

$$\mathbb{Z}^2 = (XL \cup LL) \cup (RL \cup LH) = (XL \cup LX) \cup (HH \cup RR), \tag{4.16}$$

where all the unions are disjoint. For any set $A \subset \mathbb{Z}^2$, and any functions $f(x), g(x)$, we denote the bilinear frequency cut-off to A by

$$(fg)_A = \mathcal{F}^{-1} \int \mathcal{P}_A \hat{f}(\xi - \eta)\hat{g}(\eta) d\eta := \sum_{(j,k)\in A} (P_j f)(P_k g). \tag{4.17}$$

For the nonlinear term nu, we apply the time integration by parts on $XL \cup LL$, where the phase factor $\omega = -|\xi|^2 \pm \alpha|\xi - \eta| + |\eta|^2$ is estimated

$$|\omega| \sim |\xi - \eta|\langle \xi - \eta\rangle \sim |\xi - \eta|\langle \xi \rangle, \tag{4.18}$$

which is gained in the bilinear operator

$$\begin{aligned} &\Omega_\pm(f,g) := \mathcal{F}^{-1} \int \mathcal{P}_{XL\cup LL} \frac{\hat{f}(\xi-\eta)\hat{g}(\eta)}{-|\xi|^2 \pm \alpha|\xi - \eta| + |\eta|^2} d\eta, \\ &\Omega(f,g) := \frac{1}{2}\{\Omega_+(f,g) + \Omega_-(\overline{f}, g)\} \end{aligned} \tag{4.19}$$

For the nonlinear term $u\bar{u}$, we integrate by parts on $XL \cup LX$. Then we get a bilinear operator of the form

$$\tilde{\Omega}(f,g) := \mathcal{F}^{-1} \int \mathcal{P}_{XL\cup LX} \frac{\hat{f}(\xi-\eta)\hat{\bar{g}}(\eta)}{|\xi-\eta|^2 - |\eta|^2 - \alpha|\xi|} d\eta. \tag{4.20}$$

After this modification of the normal form, we can rewrite the integral equation for (1.4) as follows. Let

$$\vec{u} := (u, N), \quad \vec{u}^0 := U(t)\vec{u}(0) = (e^{-it\Delta}u(0), e^{it\alpha D}N(0)). \tag{4.21}$$

For the fixed free solution $\vec{u}^0$, the iteration $\vec{u}' \mapsto \vec{u}$ is given by

$$\vec{u} = \vec{u}^0 - U(t)B(\vec{u}(0), \vec{u}(0)) + B(\vec{u}', \vec{u}') + Q(\vec{u}', \vec{u}') + T(\vec{u}', \vec{u}', \vec{u}'), \tag{4.22}$$

where the bilinear forms B, Q and the trilinear form T are defined by

$$\begin{aligned} B(\vec{u}_1, \vec{u}_2) :=&(\Omega(N_1, u_2), D\widetilde{\Omega}(u_1, \overline{u}_2)), \\ Q(\vec{u}_1, \vec{u}_2) :=& \int_0^t U(t-s)((n_1u_2)_{LH\cup RL}, D(u_1\overline{u}_2)_{HH\cup RR})(s)ds, \\ T(\vec{u}_1, \vec{u}_2, \vec{u}_3) :=& \int_0^t U(t-s)(\Omega(D(u_1\overline{u}_2), u_3) + \Omega(N_1, n_2u_3), D\widetilde{\Omega}(u_1, n_2u_3))(s)ds. \end{aligned}$$

For brevity, we denote

$$\begin{aligned} NL(\vec{u}_1,\vec{u}_2,\vec{u}_3) :=& B(\vec{u}_1,\vec{u}_2)+Q(\vec{u}_1,\vec{u}_2)+T(\vec{u}_1,\vec{u}_2,\vec{u}_3), \quad NL(\vec{u}) := NL(\vec{u},\vec{u}), \\ B(\vec{u}) :=& B(\vec{u},\vec{u}), \quad Q(\vec{u}) := Q(\vec{u},\vec{u}), \quad T(\vec{u}) := T(\vec{u},\vec{u},\vec{u}). \end{aligned}$$

We can estimate each term in the Duhamel formula using some powers of Strichartz norms with non-sharp exponents. For brevity of Hölder-type estimates, we denote the space-time norms by

$$\begin{aligned} &(b,d,s) := L_t^{1/b}\dot{B}^s_{1/d,2}, \\ &(b,d\pm\varepsilon,s)_+ := (b,d+\varepsilon,s)+(b,d-\varepsilon,s), \\ &(b,d\pm\varepsilon,s)_\cap := (b,d+\varepsilon,s)\cap(b,d-\varepsilon,s). \end{aligned} \tag{4.23}$$

Using the above notation, we introduce nearly full sets of the radial Strichartz norms for the Schrödinger and the wave equations (cf. [8]). Fix small numbers

$$0<\kappa \ll \varepsilon \ll 1, \tag{4.24}$$

and let

$$\begin{aligned} &SS := \langle D\rangle^{-1}[(0,\frac{1}{2},0)\cap(\frac{1}{2},\frac{3}{10}-\frac{\kappa}{3},\frac{2}{5}-\kappa)], \\ &SW := (0,\frac{1}{2},0)\cap(\frac{1}{2},\frac{1}{4}-\frac{\kappa}{3},-\frac{1}{4}-\kappa), \quad S := SS\times SW. \end{aligned} \tag{4.25}$$

Also we denote the smallness in (4.7) by using

$$\|u\|_X := \|u\|_{L_t^\infty(B_\infty^{-\frac{1}{2}-\delta})}, \quad \|n\|_Y := \|n\|_{L_t^\infty(\dot{B}_\infty^{-\frac{3}{2}-\delta}+\dot{B}_\infty^{-\frac{3}{2}+\delta})}, \quad Z := X\times Y. \tag{4.26}$$

In the nonlinear terms, we should choose appropriate Strichartz exponents so that all can be controlled by interpolation between S and Z. For that purpose, we will choose (b,d,s) for u and N respectively to be H^s admissible with $0<s<1$ and L^2 admissible for radial functions. Moreover, $b<1/2$ and $(b,d)\neq(0,1/2)$. Besides that, we will use the sum space[6] with small $\varepsilon>0$ for N and the intersection for u, so that we can dispose of very low or high frequencies, and sum over the dyadic decomposition without any difficulty.

4.2.2. *Bare bilinear terms.* First consider the bilinear terms which do not contain the time integration, namely the boundary term in the transform. In the equation for u, $\Omega(n,u)$ is roughly like $\langle D\rangle^{-1}(D^{-1}n)u$ for each dyadic piece.

Lemma 4.4. *(a) There exists $\theta>0$ such that for any N and u, we have*

$$\|\Omega(n,u)\|_{L^\infty H^1} \lesssim 2^{-\theta\beta}\|u\|_{SS}^{1-\theta}\|n\|_{SW}^{1-\theta}\|n\|_Y^\theta\|u\|_X^\theta, \tag{4.27}$$

$$\|\Omega(n,u)\|_{SS} \lesssim 2^{-\theta\beta}\|u\|_{SS}^{1-\theta}\|n\|_{SW}^{1-\theta}\|n\|_Y^\theta\|u\|_X^\theta. \tag{4.28}$$

(b) There exists $\theta>0$ such that for any u and u', we have

$$\|D\widetilde{\Omega}(u,u')\|_{L^\infty L^2} \lesssim 2^{-\theta\beta}\|u\|_{SS}^{1-\theta}\|u'\|_{SS}^{1-\theta}\|u\|_X^\theta\|u'\|_X^\theta, \tag{4.29}$$

$$\|D\widetilde{\Omega}(u,u')\|_{SW} \lesssim 2^{-\theta\beta}\|u\|_{SS}^{1-\theta}\|u'\|_{SS}^{1-\theta}\|u\|_X^\theta\|u'\|_X^\theta. \tag{4.30}$$

[6]This is because $N(0)\in L^2$ while $u(0)\in H^1=L^2\cap\dot{H}^1$.

Proof. (a) By the Coifman-Meyer-type bilinear estimate on dyadic pieces (see [7, Lemma 3.5]), we have for $(j,k) \in XL$,

$$\begin{aligned}\|\Omega(n_j,u_k)\|_{L^\infty H^1} &\lesssim \|D^{-1}n_j\|_{(0,\frac{1}{5}\pm\varepsilon,0)_+}\|u_k\|_{(0,\frac{3}{10}\pm\varepsilon,0)_\cap} \\ &\lesssim 2^{-\beta/10}\|D^{-1}n_j\|_{(0,\frac{1}{5}\pm\varepsilon,\frac{1}{10})_+}\|u_k\|_{(0,\frac{3}{10}\pm\varepsilon,0)_\cap},\end{aligned}$$

and for $(j,k) \in LL$,

$$\begin{aligned}\|\Omega(n_j,u_k)\|_{L^\infty H^1} &\lesssim \|D^{-1}n_j\|_{(0,\frac{2}{15}\pm\varepsilon,0)_+}\|u_k\|_{(0,\frac{11}{30}\pm\varepsilon,0)_\cap} \\ &\lesssim 2^{-\beta/10}\|D^{-1}n_j\|_{(0,\frac{2}{15}\pm\varepsilon,-\frac{1}{10})_+}\|u_k\|_{(0,\frac{11}{30}\pm\varepsilon,0)_\cap}.\end{aligned}$$

Since the right hand side is bounded by $\|n\|_{L^\infty L^2}\|u\|_{L^\infty H^1}$ via non-sharp Sobolev embedding, we obtain, after summation over dyadic decomposition,

$$\|\Omega(n,u)\|_{L^\infty H^1} \lesssim 2^{-\beta/10}\|u\|_{SS}^{1-\theta}\|n\|_{SW}^{1-\theta}\|n\|_Y^\theta\|u\|_X^\theta, \tag{4.31}$$

for some small $\theta > 0$. Similarly we have, for $(j,k) \in XL$,

$$\begin{aligned}&\|\Omega(n_j,u_k)\|_{\langle D\rangle^{-1}(\frac{1}{2},\frac{3}{10}-\frac{\kappa}{3},\frac{2}{5}-\kappa)} \lesssim \|D^{-1}n_j\|_{(\frac{1}{4},\frac{7}{30}-\frac{\kappa}{3}\pm\varepsilon,\frac{2}{5}-\kappa)_+}\|u_k\|_{(\frac{1}{4},\frac{1}{15}\pm\varepsilon,0)_\cap} \\ &\lesssim 2^{-\beta/20}\|D^{-1}n_j\|_{(\frac{1}{4},\frac{7}{30}-\frac{\kappa}{3}\pm\varepsilon,\frac{9}{20}-\kappa)_+}\|u_k\|_{(\frac{1}{4},\frac{1}{15}\pm\varepsilon,0)_\cap},\end{aligned} \tag{4.32}$$

and for $(j,k) \in LL$,

$$\begin{aligned}&\|\Omega(n_j,u_k)\|_{\langle D\rangle^{-1}(\frac{1}{2},\frac{3}{10}-\frac{\kappa}{3},0)} \lesssim \|D^{-1}n_j\|_{(\frac{1}{4},\frac{1}{15}-\frac{\kappa}{3}\pm\varepsilon,0)_+}\|u_k\|_{(\frac{1}{4},\frac{7}{30}\pm\varepsilon,0)_\cap} \\ &\lesssim 2^{-\beta/20}\|D^{-1}n_j\|_{(\frac{1}{4},\frac{1}{15}-\frac{\kappa}{3}\pm\varepsilon,-\frac{1}{20})_+}\|u_k\|_{(\frac{1}{4},\frac{7}{30}\pm\varepsilon,0)_\cap}\end{aligned} \tag{4.33}$$

Hence in either case we can control by non-sharp norms, so

$$\|\Omega(n,u)\|_{SS} \lesssim 2^{-\beta/20}\|u\|_{SS}^{1-\theta}\|n\|_{SW}^{1-\theta}\|n\|_Y^\theta\|u\|_X^\theta. \tag{4.34}$$

(b) We may assume $(j,k) \in XL$, since the other case LX is treated in the same way. Similarly to the above, we have $D\widetilde{\Omega}(f_j,g_k) \sim \langle D\rangle^{-1}(f_jg_k)$, so

$$\begin{aligned}\|D\widetilde{\Omega}(u_j,u'_k)\|_{L^\infty L^2} &\lesssim \|\langle D\rangle D\widetilde{\Omega}(u_j,u'_k)\|_{L^\infty(L^2+L^{6/5})} \\ &\lesssim \|u_j\|_{(0,\frac{1}{3}\pm\varepsilon,0)_+}\|u'_k\|_{(0,\frac{1}{3}\pm\varepsilon,0)_+} \\ &\lesssim 2^{-\beta/10}\|u_j\|_{(0,\frac{1}{3}\pm\varepsilon,\frac{1}{10})_+}\|u'_k\|_{(0,\frac{1}{3}\pm\varepsilon,0)_+},\end{aligned} \tag{4.35}$$

hence

$$\|D\widetilde{\Omega}(u,u')\|_{L^\infty L^2} \lesssim 2^{-\beta/10}\|u\|_{SS}^{1-\theta}\|u'\|_{SS}^{1-\theta}\|u\|_X^\theta\|u'\|_X^\theta. \tag{4.36}$$

Similarly,

$$\begin{aligned}\|D\widetilde{\Omega}(u_j,u'_k)\|_{(\frac{1}{2},\frac{1}{4}-\frac{\kappa}{3},-\frac{1}{4}-\kappa)} &\lesssim \|\langle D\rangle D\widetilde{\Omega}(u_j,u'_k)\|_{(\frac{1}{2},\frac{2}{3},0)} \\ &\lesssim 2^{-\beta/10}\|u_j\|_{(\frac{1}{4},\frac{1}{3},\frac{1}{10})}\|u'_k\|_{(\frac{1}{4},\frac{1}{3},0)},\end{aligned} \tag{4.37}$$

and so

$$\|D\widetilde{\Omega}(u,u')\|_{SW} \lesssim 2^{-\beta/10}\|u\|_{SS}^{1-\theta}\|u'\|_{SS}^{1-\theta}\|u\|_X^\theta\|u'\|_X^\theta. \tag{4.38}$$

Thus the proof is completed. □

4.2.3. *Duhamel bilinear terms.* Next we consider the remaining bilinear terms in the Duhamel form after the normal form transform. Here we have to use the radial improvement of the Strichartz norms. For brevity, we denote the integrals in the Duhamel formula by

$$I_u f := \int_0^t e^{-i(t-s)\Delta} f(s)ds, \quad I_N f := \int_0^t e^{i(t-s)\alpha D} f(s)ds. \tag{4.39}$$

Lemma 4.5. *(a) There exists $\theta > 0$ and $C(\beta) > 1$ such that for any N and u, we have*

$$\begin{aligned}\|I_u(nu)_{LH}\|_{SS} \leq& C(\beta)\|u\|_{SS}^{1-\theta}\|n\|_{SW}^{1-\theta}\|n\|_Y^\theta\|u\|_X^\theta,\\ \|I_u(nu)_{RL}\|_{SS} \leq& C(\beta)\|u\|_{SS}^{1-\theta}\|n\|_{SW}^{1-\theta}\|n\|_Y^\theta\|u\|_X^\theta.\end{aligned}$$

(b) There exists $\theta > 0$ and $C(\beta) > 1$ such that for any u and u', we have

$$\begin{aligned}\|I_N D(uu')_{HH}\|_{SW} \leq& C(\beta)\|u\|_{SS}^{1-\theta}\|u'\|_{SS}^{1-\theta}\|u\|_X^\theta\|u'\|_X^\theta,\\ \|I_N D(uu')_{RR}\|_{SW} \leq& C(\beta)\|u\|_{SS}^{1-\theta}\|u'\|_{SS}^{1-\theta}\|u\|_X^\theta\|u'\|_X^\theta.\end{aligned}$$

Proof. In this proof we ignore the dependence of the constants on β.

(a) For $(j,k) \in LH$, we have for $0 \leq s \leq 1$,

$$\begin{aligned}\|n_j u_k\|_{(1-2\varepsilon,\frac12+2\varepsilon,s+2\varepsilon)} \lesssim& \|n_j\|_{(\frac12-\varepsilon,\frac14\pm\frac{\varepsilon}{3},-\frac14-\varepsilon)_+}\|u_k\|_{(\frac12-\varepsilon,\frac14+2\varepsilon\pm\frac{\varepsilon}{3},s+\frac14+3\varepsilon)_\cap}\\ \lesssim& \|n_j\|_{(\frac12-\varepsilon,\frac14\pm\frac{\varepsilon}{3},-\frac14-\varepsilon)_+}\|u_k\|_{(\frac12-\varepsilon,\frac14+2\varepsilon\pm\frac{\varepsilon}{3},\frac54+3\varepsilon)_\cap},\end{aligned} \tag{4.40}$$

where in the second inequality we used that k is bounded from below. Since the left hand side is $\dot H^s$-admissible norm for the Strichartz estimate (without the radial symmetry), we obtain the full Strichartz bound in H^1.

For $(j,k) \in RL$, we may neglect the regularity of n_j and the product, since their frequencies are bounded from above and below. Using the radial improved Strichartz [8], the full H^1 Strichartz norm is bounded by

$$\|n_j u_k\|_{(\frac12+2\varepsilon,\frac34-3\varepsilon,0)} \lesssim \|n_j\|_{(\frac12-\varepsilon,\frac14,0)}\|u_k\|_{(3\varepsilon,\frac12-3\varepsilon,0)}. \tag{4.41}$$

Summing these estimates over dyadic pieces in the specified regions, and using non-sharp Sobolev embedding and interpolation, we obtain

$$\begin{aligned}\|I_u(nu)_{LH}\|_{SS} \lesssim& \|u\|_{SS}^{1-\theta}\|n\|_{SW}^{1-\theta}\|n\|_Y^\theta\|u\|_X^\theta,\\ \|I_u(nu)_{RL}\|_{SS} \lesssim& \|u\|_{SS}^{1-\theta}\|n\|_{SW}^{1-\theta}\|n\|_Y^\theta\|u\|_X^\theta.\end{aligned} \tag{4.42}$$

(b) We consider only the case $j \geq k$ for $u_j u'_k$, since the other case is treated in the same way. For $(j,k) \in HH$,

$$\|u_j u'_k\|_{(1-\varepsilon,\frac12+\frac23\varepsilon,1+\varepsilon)} \lesssim \|u_j\|_{(\frac12-\frac{\varepsilon}{2},\frac14+\frac{\varepsilon}{3},\frac12+\frac{\varepsilon}{2})}\|u'_k\|_{(\frac12-\frac{\varepsilon}{2},\frac14+\frac{\varepsilon}{3},\frac12+\frac{\varepsilon}{2})}, \tag{4.43}$$

and in the case $(j,k) \in RR$, since j is bounded from above,

$$\begin{aligned}\|u_j u'_k\|_{(\frac12+\varepsilon,\frac34,\frac54+\varepsilon)} \lesssim& \|u_j\|_{(\frac12-\varepsilon,\frac14+2\varepsilon,\frac54+\varepsilon)}\|u'_k\|_{(2\varepsilon,\frac12-2\varepsilon,0)}\\ \lesssim& \|u_j\|_{(\frac12-\varepsilon,\frac14+2\varepsilon,\frac12)}\|u'_k\|_{(2\varepsilon,\frac12-2\varepsilon,0)}.\end{aligned} \tag{4.44}$$

Hence

$$\begin{aligned}\|D(uu')_{HH}\|_{(1-\varepsilon,\frac12+\frac23\varepsilon,\varepsilon)} \lesssim& \|u\|_{SS}^{1-\theta}\|u'\|_{SS}^{1-\theta}\|u\|_X^\theta\|u'\|_X^\theta,\\ \|D(uu')_{RR}\|_{(\frac12+\varepsilon,\frac34,\frac14+\varepsilon)} \lesssim& \|u\|_{SS}^{1-\theta}\|u'\|_{SS}^{1-\theta}\|u\|_X^\theta\|u'\|_X^\theta.\end{aligned} \tag{4.45}$$

The left hand sides are L^2-admissible norms for radial functions. Thus the proof is completed by the radial improved Strichartz [8]. □

4.2.4. *Duhamel trilinear terms.* Finally we estimate the trilinear terms which appear after the normal transform. These are supposedly the easiest, but there is a small complication due to the fact that we have to use negative Sobolev spaces for N in some of the products:

$$\begin{aligned}&\|fg\|_{\dot B^{-s}_{r,2}} \lesssim \|f\|_{\dot B^{-s}_{p,2}}\|g\|_{\dot B^{s}_{q,2}}\\ &0 \le s < 3/q,\ 1/r = 1/p + 1/q - s/3.\end{aligned} \tag{4.46}$$

In the next lemma, the constant may decay as $\beta \to \infty$, but we do not need it.

Lemma 4.6. *(a) There exists $\theta > 0$ such that for any u, v, w, n, n', we have*

$$\begin{aligned}\|I_u\Omega(D(uv), w)\|_{SS} \lesssim& \|u\|_{SS}^{1-\theta}\|v\|_{SS}^{1-\theta}\|w\|_{SS}^{1-\theta}\|u\|_X^\theta\|v\|_X^\theta\|w\|_X^\theta.\\ \|I_u\Omega(n, n'u)\|_{SS} \lesssim& \|n\|_{SW}^{1-\theta}\|n'\|_{SW}^{1-\theta}\|u\|_{SS}^{1-\theta}\|n\|_Y^\theta\|n'\|_Y^\theta\|u\|_X^\theta.\end{aligned}$$

(b) There exists $\theta > 0$ such that for any n, u, u', we have

$$\|I_N D\widetilde{\Omega}(nu, u')\|_{SW} + \|I_N D\widetilde{\Omega}(u, nu')\|_{SW} \lesssim \|n\|_{SW}^{1-\theta}\|u\|_{SS}^{1-\theta}\|u'\|_{SS}^{1-\theta}\|n\|_Y^\theta\|u\|_X^\theta\|u'\|_X^\theta.$$

Proof. (a) Since $\Omega(D(uv)_j, w_k) \sim \langle D\rangle^{-1}((uv)_j w_k)$,

$$\|\Omega(D(uv)_j, w_k)\|_{L^1H^1} \lesssim \|u\|_{L^3L^6}\|v\|_{L^3L^6}\|w\|_{L^3L^6}, \tag{4.47}$$

and by non-sharp Sobolev embedding and interpolation,

$$\|\Omega(D(uv), w)\|_{L^1H^1} \lesssim \|u\|_{SS}^{1-\theta}\|v\|_{SS}^{1-\theta}\|w\|_{SS}^{1-\theta}\|u\|_X^\theta\|v\|_X^\theta\|w\|_X^\theta. \tag{4.48}$$

For $\Omega(n_j, (n'u)_k)$, we have either $2^j \gg 2^k$ or $2^j + 2^k \ll 1$. In the first case, we have

$$\begin{aligned}&\|\Omega(n_j, (n'u)_k)\|_{L^1H^1} \lesssim \|D^{-1+5\varepsilon}n_j\|_{(\frac12-\varepsilon, 2\varepsilon\pm\frac{\varepsilon}{6}, 0)_+}\|D^{-5\varepsilon}(n'u)_k\|_{(\frac12+\varepsilon, \frac12-2\varepsilon\pm\frac{\varepsilon}{6}, 0)_\cap}\\ &\lesssim \|n_j\|_{(\frac12-\varepsilon, 2\varepsilon\pm\frac{\varepsilon}{6}, -1+5\varepsilon)_+}\|n'\|_{(2\varepsilon, \frac12-\frac73\varepsilon\pm\frac{\varepsilon}{6}, -5\varepsilon)_+}\|u\|_{(\frac12-\varepsilon, 2\varepsilon\pm\frac{\varepsilon}{3}, 5\varepsilon)_\cap},\end{aligned} \tag{4.49}$$

where we used the product estimate for negative Sobolev spaces for $n'u$. In the second case $2^j + 2^k \ll 1$, we have

$$\begin{aligned}&\|\Omega(n_j, (n'u)_k)\|_{L^1H^1} \lesssim \|D^{-1}n_j\|_{(\frac12-\varepsilon, \frac{\varepsilon}{3}\pm\frac{\varepsilon}{6}, 0)_+}\|(n'u)_k\|_{(\frac12+\varepsilon, \frac12-\frac{\varepsilon}{3}\pm\frac{\varepsilon}{6}, 0)_\cap}\\ &\lesssim \|n_j\|_{(\frac12-\varepsilon, \frac{\varepsilon}{3}\pm\frac{\varepsilon}{6}, -1)_+}\|n'\|_{(2\varepsilon, \frac12-\frac73\varepsilon\pm\frac{\varepsilon}{6}, -5\varepsilon)_+}\|u\|_{(\frac12-\varepsilon, \frac{11}{3}\varepsilon\pm\frac{\varepsilon}{3}, 5\varepsilon)_\cap}.\end{aligned} \tag{4.50}$$

Hence, by non-sharp Sobolev embedding and interpolation,

$$\|\Omega(n, n'u)\|_{L^1H^1} \lesssim \|n\|_{SW}^{1-\theta}\|n'\|_{SW}^{1-\theta}\|u\|_{SS}^{1-\theta}\|n\|_Y^\theta\|n'\|_Y^\theta\|u\|_X^\theta. \tag{4.51}$$

(b) We have $D\widetilde{\Omega} \sim \langle D\rangle^{-1}$ on each dyadic piece, so

$$\begin{aligned}&\|D\widetilde{\Omega}((nu)_j, u'_k)\|_{L^1L^2} \lesssim \|\langle D\rangle D\widetilde{\Omega}((nu)_j, u'_k)\|_{(1, \frac56-\frac53\varepsilon, -5\varepsilon)}\\ &\lesssim \|(nu)_j\|_{(\frac12+\varepsilon, \frac23-\varepsilon, -5\varepsilon)}\|u'_k\|_{(\frac12-\varepsilon, \frac16+\varepsilon, 5\varepsilon)}\\ &\lesssim \|n\|_{(2\varepsilon, \frac12-\frac73\varepsilon\pm\frac{\varepsilon}{6}, -5\varepsilon)_+}\|u\|_{(\frac12-\varepsilon, \frac16+\varepsilon\pm\frac{\varepsilon}{6}, 5\varepsilon)_\cap}\|u'_k\|_{(\frac12-\varepsilon, \frac16+\varepsilon, 5\varepsilon)},\end{aligned} \tag{4.52}$$

where we used the product estimate twice, but did not use any restriction on j, k. Hence we have the same estimate on $\widetilde{\Omega}(u_j, (n'u)_k)$, and so

$$\begin{aligned}&\|D\widetilde{\Omega}(nu, u')\|_{L^1L^2} + \|D\widetilde{\Omega}(u, nu')\|_{L^1L^2}\\&\lesssim \|n\|_{SW}^{1-\theta}\|u\|_{SS}^{1-\theta}\|u'\|_{SS}^{1-\theta}\|n\|_Y^{\theta}\|u\|_X^{\theta}\|u'\|_X^{\theta}.\end{aligned} \tag{4.53}$$

Thus, the proof is completed. □

Note that in the above estimates we needed the L^∞_t-type norms only for the bare bilinear terms, but not for the Duhamel terms. Thus we have obtained

Lemma 4.7. *There exist $\theta > 0$, $\eta > 0$ and $C(\beta) > 1$ such that for each $\beta \gg 1$ and any $\vec{u}_1, \vec{u}_2, \vec{u}_3$, we have*

$$\begin{aligned}&2^{\theta\beta}\|B(\vec{u}_1, \vec{u}_2)\|_S + \|Q(\vec{u}_1, \vec{u}_2)\|_S/C(\beta) \lesssim \|\vec{u}_1\|_S^{1-\theta}\|\vec{u}_2\|_S^{1-\theta}\|\vec{u}_1\|_Z^{\theta}\|\vec{u}_2\|_Z^{\theta},\\&\|T(\vec{u}_1, \vec{u}_2, \vec{u}_3)\|_S \lesssim \|\vec{u}_1\|_S^{1-\theta}\|\vec{u}_2\|_S^{1-\theta}\|\vec{u}_3\|_S^{1-\theta}\|\vec{u}_1\|_Z^{\theta}\|\vec{u}_2\|_Z^{\theta}\|\vec{u}_3\|_Z^{\theta}.\end{aligned} \tag{4.54}$$

For the Duhamel terms we have also

$$\begin{aligned}&\|Q(\vec{u}_1, \vec{u}_2)\|_S \lesssim C(\beta)\|\vec{u}_1\|_{\widetilde{S}}\|\vec{u}_2\|_{\widetilde{S}},\\&\|T(\vec{u}_1, \vec{u}_2, \vec{u}_3)\|_S \lesssim \|\vec{u}_1\|_{\widetilde{S}}\|\vec{u}_2\|_{\widetilde{S}}\|\vec{u}_3\|_{\widetilde{S}},\end{aligned} \tag{4.55}$$

where

$$\begin{aligned}&\widetilde{S} := \widetilde{SS} \times \widetilde{SW},\\&\widetilde{SS} := \langle D\rangle^{-1}[(\eta, \frac{1}{2} - \frac{2}{5}\eta, \frac{4}{5}\eta) \cap (\frac{1}{2}, \frac{3}{10} - \frac{\kappa}{3}, \frac{2}{5} - \kappa)],\\&\widetilde{SW} := (\eta, \frac{1}{2} - \frac{1}{2}\eta, -\frac{1}{4}\eta) \cap (\frac{1}{2}, \frac{1}{4} - \frac{\kappa}{3}, -\frac{1}{4} - \kappa).\end{aligned} \tag{4.56}$$

4.3. **Nonlinear profile approximation.** We will prove Lemma 4.3 by the following two lemmas.

Lemma 4.8 (Stability)**.** *For any $A > 0$ and $\sigma > 0$, there exists $\varsigma > 0$ with the following property: Suppose that $\vec{u}_a$ satisfies $\|\vec{u}_a\|_{S(0,\infty)} \le A$ and approximately solves the Zakharov system in the sense that*

$$\vec{u}_a = U(t)\vec{u}_a(0) - U(t)B(\vec{u}_a(0)) + NL(\vec{u}_a) + \vec{e}$$

and $\|\vec{e}\|_{S(0,\infty)} \le \varsigma$. Then for any initial data $\vec{u}(0)$ satisfying $\|\vec{u}(0) - \vec{u}_a(0)\|_{H^1\times L^2} < \varsigma$, there is a unique global solution $\vec{u}$ satisfying $\|\vec{u} - \vec{u}_a\|_{S(0,\infty)} < \sigma$.

Proof. Denote $\vec{u}_\triangleright = \vec{u}_a - \vec{u}$, then $\|\vec{u}_\triangleright(0)\|_{H^1\times L^2} \le \varsigma$ and

$$\vec{u}_\triangleright = U(t)\vec{u}_\triangleright(0) - U(t)B(\vec{u}_a(0)) + NL(\vec{u}_a) + \vec{e} + U(t)B(\vec{u}(0)) - NL(\vec{u}). \tag{4.57}$$

Thus

$$\|\vec{u}_\triangleright\|_S \lesssim 2\varsigma + \|B(\vec{u}_a) - B(\vec{u})\|_S + \|Q(\vec{u}_a) - Q(\vec{u})\|_S + \|T(\vec{u}_a) - T(\vec{u})\|_S. \tag{4.58}$$

Noting that $Z \supset S$, by (4.54) we have

$$\begin{aligned}\|B(\vec{u}_a) - B(\vec{u})\|_S &\le \|B(\vec{u}_a, \vec{u}_\triangleright)\|_S + \|B(\vec{u}_\triangleright, \vec{u}_a)\|_S + \|B(\vec{u}_\triangleright, \vec{u}_\triangleright)\|_S\\&\lesssim 2^{-\theta\beta}A\|\vec{u}_\triangleright\|_S + 2^{-\theta\beta}\|\vec{u}_\triangleright\|_S^2.\end{aligned}$$

By (4.55), we have

$$\begin{aligned}\|Q(\vec{u}_a)-Q(\vec{u})\|_S \leq& C(\beta)\left(\|\vec{u}_a\|_{\widetilde{S}}\|\vec{u}_\triangleright\|_S+\|\vec{u}_\triangleright\|_S^2\right),\\ \|T(\vec{u}_a)-T(\vec{u})\|_S \leq& C\left(\|\vec{u}_a\|_{\widetilde{S}}^2\|\vec{u}_\triangleright\|_S+\|\vec{u}_a\|_{\widetilde{S}}\|\vec{u}_\triangleright\|_S^2+\|\vec{u}_\triangleright\|_S^3\right).\end{aligned} \tag{4.59}$$

So

$$\begin{aligned}\|\vec{u}_\triangleright\|_S \leq 2\varsigma &+ (2^{-\theta\beta}CA + C(\beta)\|\vec{u}_a\|_{\widetilde{S}} + C\|\vec{u}_a\|_{\widetilde{S}}^2)\|\vec{u}_\triangleright\|_S \\ &+ (2^{-\theta\beta}C + C(\beta) + C\|\vec{u}_a\|_{\widetilde{S}})\|\vec{u}_\triangleright\|_S^2 + C\|\vec{u}_\triangleright\|_S^3.\end{aligned} \tag{4.60}$$

Choose $\beta=\beta(A)$ such that $2^{-\theta\beta}CA<\frac{1}{4}$. Then we subdivide the time interval $[0,\infty)$ into finite subintervals $I_j=[t_j,t_{j+1}]$, $j=1,\cdots,J$, $J=J(A,\beta)$ such that

$$C(\beta)\|\vec{u}_a\|_{\widetilde{S}(I_j)}+C\|\vec{u}_a\|_{\widetilde{S}(I_j)}^2<\frac{1}{4} \tag{4.61}$$

for each j. Let $\varsigma=\varsigma(A,\sigma,\beta,J)$ small such that

$$C(\beta)8^{2J}\varsigma\ll 1,\quad 8^{2J}\varsigma\ll\sigma. \tag{4.62}$$

Then by (4.60) on I_1, we have $\|\vec{u}_\triangleright\|_{S(I_1)}\leq 8\varsigma$ and

$$\begin{aligned}&\|\vec{u}_\triangleright(t_2)\|_{H^1\times L^2}\\ \leq&\|U(t_2-t_1)\vec{u}_\triangleright(t_1)\|_{H^1\times L^2}+\|U(t_2-t_1)B(\vec{u}_a(t_1))-U(t_2-t_1)B(\vec{u}(t_1))\|_{H^1\times L^2}\\ &+\|B(\vec{u}_a(t_2))-B(\vec{u}(t_2))\|_{H^1\times L^2}+\|Q(\vec{u}_a)-Q(\vec{u})\|_{S(I_1)}\\ &+\|T(\vec{u}_a)-T(\vec{u})\|_{S(I_1)}+\|\vec{e}\|_{S(I_1)}\\ \leq&2\varsigma+4\cdot 8\varsigma\leq 8^2\varsigma.\end{aligned}$$

Using the same analysis as above, we can get $\|\vec{u}_\triangleright\|_{S(I_2)}\leq 8^3\varsigma$. Iterating this for $I_2,I_3,\ldots,I_J$, we obtain $\|\vec{u}_a-\vec{u}_1\|_S\lesssim 8^{2J}\varsigma\ll\sigma$, the desired result was obtained. □

With J close to $\bar{J}$ and large n, our approximate solution is given by

$$\vec{u}_n^J=(u_n^J,N_n^J):=\sum_{j=1}^{J}(\mathfrak{u}_n^j,\mathfrak{N}_n^j)+(u_n^{>J},N_n^{>J}). \tag{4.63}$$

To prove Lemma 4.3, we only need to prove that $\vec{u}_n^J$ is an approximate solution of the Zakharov system. In fact, we have

Lemma 4.9. *Suppose that $\|(\mathfrak{u}_n^j,\mathfrak{N}_n^j)\|_S<\infty$ for all $j<\bar{J}$, then*

$$\lim_{J\to\bar{J}}\limsup_{n\to\infty}\|U(t)B(\vec{u}_n^J(0))-NL(\vec{u}_n^J)-\sum_{j=1}^{J}[U(t)B(\vec{\mathfrak{u}}_n^j(0))-NL(\vec{\mathfrak{u}}_n^j)]\|_S=0.$$

Note that $\|(\mathfrak{u}_n^j,\mathfrak{N}_n^j)\|_S$ does not depend on n.

Proof. By triangle inequality, it suffices to show that

$$\begin{aligned}\lim_{n\to\infty}\|\sum_{j\leq J}&[U(t)B(\vec{\mathfrak{u}}_n^j(0))-NL(\vec{\mathfrak{u}}_n^j)]\\ &-[U(t)B(\sum_{j\leq J}\vec{\mathfrak{u}}_n^j(0))-NL(\sum_{j\leq J}\vec{\mathfrak{u}}_n^j)]\|_S=0,\end{aligned} \tag{4.64}$$

and

$$\begin{aligned}\lim_{J\to\bar{J}}\limsup_{n\to\infty}\|&[U(t)B(\vec{u}_n^J(0)) - NL(\vec{u}_n^J)] \\ &- [U(t)B(\vec{u}_n^J(0) - \vec{u}_n^{>\mathrm{J}}(0)) - NL(\vec{u}_n^J - \vec{u}_n^{>\mathrm{J}})]\|_S = 0.\end{aligned} \tag{4.65}$$

In fact,

$$L.H.S\ of\ (4.64) \lesssim \sum_{i\neq j}\left(\|B(\vec{\mathfrak{u}}_n^i,\vec{\mathfrak{u}}_n^j)\|_S + \|Q(\vec{\mathfrak{u}}_n^i,\vec{\mathfrak{u}}_n^j)\|_S\right) + \sum_{i\neq j \text{ or } j\neq k}\|T(\vec{\mathfrak{u}}_n^i,\vec{\mathfrak{u}}_n^j,\vec{\mathfrak{u}}_n^k)\|_S.$$

For each $i\neq j$, we have $|t_n^i - t_n^j|\to\infty$. for the subsequence $t_n^i - t_n^j\to\infty$, we have by (4.54),

$$\begin{aligned}\|B(\vec{\mathfrak{u}}_n^i,\vec{\mathfrak{u}}_n^j)\|_S &\lesssim \|B(\vec{\mathfrak{u}}^i(\cdot - t_n^i),\vec{\mathfrak{u}}^j(\cdot - t_n^j))\|_{S(-\infty,(t_n^i+t_n^j)/2)\cap S((t_n^i+t_n^j)/2,\infty)} \\ &\lesssim \|\vec{\mathfrak{u}}^i\|_S^{1-\theta}\|\vec{\mathfrak{u}}^j\|_S^{1-\theta}\left[\|\vec{\mathfrak{u}}^i\|^\theta_{Z(-\infty,t_n^j-t_n^i)/2)}\|\vec{\mathfrak{u}}^j\|_Z^\theta + \|\vec{\mathfrak{u}}^i\|_Z^\theta\|\vec{\mathfrak{u}}^j\|^\theta_{Z((t_n^i-t_n^j)/2,\infty)}\right].\end{aligned} \tag{4.66}$$

For each j, by the scattering of $\vec{\mathfrak{u}}^j$,

$$\lim_{T\to\infty}\|\vec{\mathfrak{u}}^j\|_{Z(|t|\geq T)} = 0, \tag{4.67}$$

so from the above estimate

$$\|B(\vec{\mathfrak{u}}_n^i,\vec{\mathfrak{u}}_n^j)\|_S\to 0, \tag{4.68}$$

as $t_n^i - t_n^j\to\infty$. The case $t_n^i - t_n^j\to-\infty$ is treated similarly, as well as the other terms Q and T. Thus we obtain

$$\begin{aligned}&\|B(\vec{\mathfrak{u}}_n^i,\vec{\mathfrak{u}}_n^j)\|_S\to 0 \text{ for } i\neq j,\\ &\|Q(\vec{\mathfrak{u}}_n^i,\vec{\mathfrak{u}}_n^j)\|_S\to 0 \text{ for } i\neq j,\\ &\|T(\vec{\mathfrak{u}}_n^i,\vec{\mathfrak{u}}_n^j,\vec{\mathfrak{u}}_n^k)\|_S\to 0 \text{ for } i\neq j \text{ or } i=j\neq k,\end{aligned} \tag{4.69}$$

from which (4.64) follows immediately.

In order to prove (4.65), we need a uniform bound on the approximate solutions $\vec{u}_n^J$ for $J\to\bar{J}$. Note that (4.9) implies that $\|\vec{\mathbf{u}}_n{}^j(0)\|_{H^1\times L^2}\ll 1$ except for a bounded number of j. Let A be the set of j in the latter case. Then for all $j\notin A$, the small data scattering implies that

$$\|(\mathfrak{u}_n^j,\mathfrak{N}_n^j)\|_S \lesssim \|(\mathbf{u}_n^j(0),\mathbf{N}_n^j(0))\|_{H_x^1\times L_x^2}\ll 1. \tag{4.70}$$

Then by the orthogonality in $H_x^1\times L_x^2$ and $|t_n^i - t_n^j|\to\infty$, we deduce

$$\|\sum_{j\notin A}\vec{\mathfrak{u}}_n^j\|_S^2 \lesssim \sum_{j\notin A}\|\vec{\mathfrak{u}}_n^j\|_S^2 \lesssim \sum_{j\notin A}\|\vec{\mathbf{u}}_n^j\|^2_{H_x^1\times L_x^2}\lesssim 1. \tag{4.71}$$

Since the number of the remaining components $j\in A$ are bounded, we obtain

$$\sup_{J<\bar{J}}\sup_n\|\vec{u}_n^J\|_S<\infty. \tag{4.72}$$

The left hand side of (4.65) is bounded by

$$\begin{aligned}&\|B(\vec{u}_n^J) - B(\vec{u}_n^J - \vec{u}_n^{>\mathrm{J}})\|_S + \|Q(\vec{u}_n^J) - Q(\vec{u}_n^J - \vec{u}_n^{>\mathrm{J}})\|_S \\ &+ \|T(\vec{u}_n^J) - T(\vec{u}_n^J - \vec{u}_n^{>\mathrm{J}})\|_S.\end{aligned} \tag{4.73}$$

By (4.54), (4.7) and (4.72),

$$\lim_{J\to\bar{J}}\limsup_{n\to\infty}\|B(\vec{u}_n^J)-B(\vec{u}_n^J-\vec{u}_n^{>\mathrm{J}})\|_S=0. \tag{4.74}$$

One can estimates Q and T similarly. Then (4.65) was proved. □

Proof of Lemma 4.3. By the construction of $\vec{u}_n^J$,

$$\lim_{n\to\infty}\|\vec{u}_n^J(0)-\vec{u}_n(0)\|_{H^1\times L^2}=0. \tag{4.75}$$

By Lemma 4.9 and Lemma 4.8, passing to a subsequence if necessary, we obtain $\|\vec{u}_n^J-\vec{u}_n\|_S\ll 1$ for large J and n. □

Proof of Lemma 4.1. By the definition of E_λ^*, there is a sequence of global solutions (u_n,N_n) in $\mathcal{K}_\lambda^+(a)$ such that

$$\lim_{n\to\infty}\mathscr{E}_\lambda(u_n,N_n)=E_\lambda^*,\ \lim_{n\to\infty}\|(u_n,N_n)\|_{S(-\infty,0)}=\lim_{n\to\infty}\|(u_n,N_n)\|_{S(0,\infty)}=\infty. \tag{4.76}$$

To see this, first note that since (u_n,N_n) are in $\mathcal{K}_\lambda^+(a)$, they are bounded in $H^1\times L^2$ by the energy. Hence if $\|(u_n,N_n)\|_S\to\infty$ then the L_t^2 part must diverge, and we can translate (u_n,N_n) in t so that the norm diverges both on $(-\infty,0)$ and on $(0,\infty)$.

For the sequence $(u_n(0),N_n(0))$, we use the linear profile decomposition. For the associated nonlinear profile $(\mathfrak{u}_n^j,\mathfrak{N}_n^j)$, we must have $K(\mathfrak{u}_n^j(0))\geq 0$ for each j. In fact, if we denote

$$G_\lambda(\varphi)=J_\lambda(\varphi)-\frac{1}{3}K=\left(\frac{1}{2}-\frac{1}{3}\right)\|\nabla u\|_2^2+\frac{\lambda^2}{2}\|u\|_2^2>0, \tag{4.77}$$

then

$$\begin{aligned}J_\lambda(Q_\lambda)&=\inf\{J_\lambda(\varphi)\mid\varphi\neq 0,\ K(\varphi)=0\}\\&=\inf\{G_\lambda(\varphi)\mid\varphi\neq 0,\ K(\varphi)=0\}\\&=\inf\{G_\lambda(\varphi)\mid\varphi\neq 0,\ K(\varphi)\leq 0\}.\end{aligned} \tag{4.78}$$

By the orthogonality,

$$\overline{\lim_{n\to\infty}}\,G_\lambda(u_n(0))=\overline{\lim_{n\to\infty}}\left(\sum_{j=1}^J G_\lambda(\mathfrak{u}_n^j(0))+G_\lambda(u_n^{>\mathrm{J}}(0))\right)\leq\lambda E_\lambda^*<J_\lambda(Q_\lambda). \tag{4.79}$$

Hence, for n sufficiently large, $G_\lambda(\mathfrak{u}_n^j(0))<J_\lambda(Q_\lambda)$; and then by the third line of (4.78), $K(\mathfrak{u}_n^j(0))\geq 0$. Noting that

$$\lim_{n\to\infty}\mathscr{E}_\lambda(u_n(0),N_n(0))-\sum_{j=1}^J\mathscr{E}_\lambda(\mathfrak{u}_n^j(0),\mathfrak{N}_n^j(0))-\mathscr{E}_\lambda(u_n^{>\mathrm{J}}(0),N_n^{>\mathrm{J}}(0))=0, \tag{4.80}$$

we have

$$\sum_{j=1}^J\mathscr{E}_\lambda(\mathfrak{u}_n^j(0),\mathfrak{N}_n^j(0))\leq\lim_{n\to\infty}\mathscr{E}_\lambda(u_n,N_n)=E_\lambda^*. \tag{4.81}$$

If $\mathscr{E}_\lambda(\mathfrak{u}_n^j(0),\mathfrak{N}_n^j(0))<E_\lambda^*$ for all $j<\bar{J}$, then we have $\|(\mathfrak{u}_n^j,\mathfrak{N}_n^j)\|_S<\infty$ for all j, and so by Lemma 4.3,

$$\limsup_{n\to\infty}\|(u_n,N_n)\|_S<\infty, \tag{4.82}$$

which contradicts $\lim_{n\to\infty} \|(u_n, N_n)\|_{S(0,\infty)} = \infty$. Thus, we must have one $j < \overline{J}$ such that

$$\mathscr{E}_\lambda(\mathfrak{u}_n^j(0), \mathfrak{N}_n^j(0)) = E_\lambda^*. \tag{4.83}$$

Without losing generality, we may assume $j = 1$. Comparing this with (4.81), we have

$$(u_n(0), N_n(0)) = U(-t_n)(\mathbf{f}^1, \mathbf{g}^1) + (u_n^{>1}(0), N_n^{>1}(0)) \tag{4.84}$$

and

$$\|(u_n^{>1}(0), N_n^{>1}(0))\|_{H^1\times L^2} \lesssim \mathscr{E}_\lambda(u_n^{>1}(0), N_n^{>1}(0)) \to 0. \tag{4.85}$$

If $t_n \to -\infty$, then we have

$$\|U(t-t_n)(\mathbf{f}^1, \mathbf{g}^1)\|_{Z(0,\infty)} \to 0, \tag{4.86}$$

and hence

$$\begin{aligned} &\|U(t)(u_n(0), N_n(0))\|_{Z(0,\infty)} \\ &\lesssim \|U(t-t_n)(\mathbf{f}^1, \mathbf{g}^1)\|_{Z(0,\infty)} + \|(u_n^{>1}(0), N_n^{>1}(0))\|_{H^1\times L^2} \to 0. \end{aligned} \tag{4.87}$$

By Lemma 4.7,

$$\|U(t)B((u_n(0), N_n(0)))\|_{S(0,\infty)} + \|NL(U(t)(u_n(0), N_n(0)))\|_{S(0,\infty)} \to 0. \tag{4.88}$$

Then using Lemma 4.8 (with $\vec{u}_a := U(t)(u_n(0), N_n(0))$ and $(u_n(0), N_n(0))$ as the initial data), we obtain

$$\lim_{n\to\infty} \|(u_n, N_n)\|_{S(0,\infty)} < \infty. \tag{4.89}$$

which contradicts $\|(u_n, N_n)\|_{S(0,\infty)} \to \infty$.

If $t_n \to +\infty$, the argument is similar and we obtain a contradiction by using $\|(u_n, N_n)\|_{S(-\infty,0)} \to \infty$.

So, the only case left is $t_n \to 0$. In this case,

$$\|(u_n(0), N_n(0)) - (\mathbf{f}^1, \mathbf{g}^1)\|_{H^1\times L^2} \to 0. \tag{4.90}$$

Let (u, N) be the global solution with initial data $(u(0), N(0)) = (\mathbf{f}^1, \mathbf{g}^1)$, then $\mathscr{E}_\lambda(u, N) \le E_\lambda^*$. By stability, we must have

$$\|(u, N)\|_{S(-\infty,0)} = \|(u, N)\|_{S(0,\infty)} = \infty, \tag{4.91}$$

since otherwise (u_n, N_n) should be bounded either in $S(-\infty, 0)$ or in $S(0, \infty)$. By the definition of E_λ^*, $\mathscr{E}_\lambda(u, N) \ge E_\lambda^*$ and hence $\mathscr{E}_\lambda(u, N) = E_\lambda^*$.

Since (u, N) is locally in S, for any $t_n \in \mathbb{R}$, we have

$$\|(u, N)\|_{S(-\infty,t_n)} = \infty = \|(u, N)\|_{S(t_n,\infty)}. \tag{4.92}$$

Applying the above argument to the sequence $(u_n(t), N_n(t)) := (u(t+t_n), N(t+t_n))$, we see that $(u(t+t_n), N(t+t_n))$ is precompact in $H^1 \times L^2$. Thus we obtain the desired result. □

5. Rigidity Theorem

The main purpose of this section is to disprove the existence of critical element that was constructed in the previous section under the assumption $E^*_\lambda < J(Q)$. The main tool is the spatial localization of the virial identity. We prove

Theorem 5.1 (Rigidity Theorem)**.** *Let* (u, N) *be a global solution to* (1.4) *satisfying* $K(u) \geq 0$*, and* $E_Z(u, N) + \lambda^2 M(u) < J_\lambda(Q_\lambda)$ *for some* $\lambda > 0$*. Moreover, assume* $\{(u, N)(t) : t \in \mathbb{R}\}$ *is precompact in* $H^1 \times L^2$*. Then* $u = N \equiv 0$*.*

Proof. By contradiction, we assume $(u, N) \neq (0, 0)$. Then by the compactness we may assume further $u \neq 0$, since otherwise N would be a free wave and dispersive. We divide the proof into the following three steps:

Step 1: Energy trapping.

We claim that

$$c := \inf_{t\in\mathbb{R}} K(u) > 0. \tag{5.1}$$

If not, then there exists $\{t_n\}$ with $t_n \to t_* \in [-\infty, \infty]$, and $K(u(t_n)) \to 0$. By the precompactness of $\{u(t) : t \in \mathbb{R}\}$, we get that up to a sequence $(u(t_n), N(t_n))$ converges to some (f, g) in $H^1 \times L^2$. Then we have $K(f) = 0$, $J_\lambda(f) \leq E_Z(f, g) + \lambda^2 M(f) = E_Z(u, N) + \lambda^2 M(u) < J_\lambda(Q_\lambda)$. By the variational characterization of Q_λ, we get $f \equiv 0$ which contradicts to the $M(f) = M(u) \neq 0$.

Step 2: Uniform small tails.

Let $\nu = \Re N - |u|^2 = n - |u|^2$. We claim that for any $\varepsilon > 0$, there exists $R > 0$ such that at any $t \in \mathbb{R}$, we have

$$\int_{|x|\geq R} \big(|\nabla u|^2 + |u|^2 + |u|^4 + |u|^6 + |\nu|^2 + |D^{-1}\nabla\nu|^2 + \frac{|D^{-1}\nu|^2}{|x|^2}\big)dx < \varepsilon.$$

Indeed, since $\{(u, N)(t) : t \in \mathbb{R}\}$ is precompact in $H^1 \times L^2$, by Sobolev embedding and the L^p-boundedness of $D^{-1}\nabla$, we get that $\{u(t)\}$ is precompact in L^2, L^4, L^6, $\{D^{-1}N(t)\}$ is precompact in $\dot{H}^1$, and $\{D^{-1}\nabla N(t), \nu(t), D^{-1}\nabla\nu\}$ is precompact in L^2. Then the claim follows immediately.

Step 3: Contradiction to the local virial estimates.

We recall the local virial estimates obtained in Section 3. For any $R > 0$

$$V_R(t) := \langle \underline{J}u|(AX + XA)u\rangle. \tag{5.2}$$

where $X = X^*$ be the operator of smooth truncation to $|x| < R$ by multiplication with $\psi_R(x)$. From the proof in Section 3 and Corollary 2.3 we have

$$|V_R(t)| \lesssim R[\|u\|_2\|\nabla u\|_2 + \|N\|_2^2] \lesssim R. \tag{5.3}$$

On the other hand, from Step 2, Step 1, and Lemma 2.4, we get

$$\begin{aligned}
V_R'(t) =&\langle \nu | X\nu\rangle/2 + \langle \nabla\eta | X\nabla\eta\rangle/2 + \langle \eta_r | r\psi_R'\eta_r\rangle - \frac{1}{4}\langle |\eta|^2 | A_1\Delta\psi_R\rangle \\
&- 2\langle \nu | |u|^2\rangle + O(R^{-1}\|\nu\|_2\|u\|_2^{3/2}\|\nabla u\|_2^{1/2}) \\
&+ \langle \nabla u | 4X\nabla u\rangle + 4\langle u_r | \psi_R' r u_r\rangle \\
&- 3\|u\|_4^4 + O(R^{-2}(\|u\|_2^2 + \|u\|_2^3\|\nabla u\|_2)) \quad \text{(obtained in Section 3)} \\
=&4K(u) + \|\nu\|_2^2 - 2\langle \nu | |u|^2\rangle + o(1),\ R\to\infty \quad \text{(by Step 2)} \\
\geq&(1-\frac{2}{\sqrt{6}})K(u) + o(1) \geq c/2,\ R \gg 1. \quad \text{(by Step 1 and Lemma 2.4)}
\end{aligned}$$

Thus we get

$$V_R(t) \geq V_R(0) + ct/2,$$

which contradicts (5.3) for sufficiently large t. □

Appendix A. Construction of wave operators

Here we briefly sketch a proof for the existence of the wave operators, or the solvability of the final state problem. For the construction of a nonlinear profile in the radial setting, we need only to consider a sequence of solutions in the form

$$\vec{u}_n = U(t)\vec{f} + \int_{-t_n}^{t} U(t-s)(nu, \alpha D|u|^2)ds, \tag{A.1}$$

with $t_n \to \pm\infty$, which is normally transformed into such a form as

$$\vec{u}_n = U(t)\vec{f} - U(t+t_n)B(U(-t_n)\vec{f}) + B(\vec{u}_n) + Q_{-t_n}(\vec{u}_n) + T_{-t_n}(\vec{u}_n), \tag{A.2}$$

where Q_{-t_n} and T_{-t_n} denote respectively Q and T with the Duhamel integration $\int_{-t_n}^{t}$, and arbitrarily fixed β, say $\beta = 10$. Below we consider only the case $t_n \to -\infty$, since the other case is similar. The following is the precise statement that we need for the nonlinear profile in this case.

Lemma A.1. *Let $\vec{f} \in H^1 \times L^2$, $\mathbb{R} \ni t_n \to -\infty$, and let $\{\vec{u}_n\}$ be the sequence of solutions to the Zakharov system with the Cauchy data $\vec{u}_n(-t_n) = U(-t_n)\vec{f}$. Then there exist $T \in \mathbb{R}$ and a unique $\vec{u} \in S(T,\infty)$ satisfying*

$$\vec{u} = U(t)\vec{f} + B(\vec{u}) + Q_\infty(\vec{u}) + T_\infty(\vec{u}), \quad \lim_{n\to\infty} \|\vec{u}_n - \vec{u}\|_{S(T,\infty)} = 0, \tag{A.3}$$

as well as the Zakharov system on (T,∞). Moreover, if $\{\vec{u}_n\}$ is bounded in $L^\infty(\mathbb{R}; H^1 \times L^2)$, then $\vec{u}$ is global and the above convergence holds for any $T \in \mathbb{R}$.

Proof. First, we can solve (A.3) on (T,∞) for $T \gg 1$, by the iteration argument similar to [7] in the space

$$X := \{\vec{u} \in C([T,\infty); H^1 \times L^2) \mid \|\vec{u}\|_{S(T,\infty)} \lesssim 1, \|\vec{u}\|_{Z(T,\infty)} \leq \eta\}, \tag{A.4}$$

with $\eta := 2\|U(t)\vec{f}\|_{Z(T,\infty)} \ll 1$, using the estimates similar to (4.54) as well as

$$\|U(t)\vec{f}\|_{Z(T,\infty)} \lesssim \|U(t)\vec{f}\|_{L^\infty_{t>T}(L^6_x \times \dot{H}^{-1}_6)} \to 0 \text{ as } T \to \infty. \tag{A.5}$$

Similar estimates imply that $\vec{u}_n$ are scattering as $t \to \infty$ for large n. Also similarly to (4.54), we have for some $\theta > 0$

$$\begin{aligned}\|U(t+t_n)B(U(-t_n)\vec{f})\|_S &\lesssim \|B(U(-t_n)\vec{f})\|_{H^1\times L^2} \\ &\lesssim \|U(-t_n)\vec{f}\|_{H^1\times L^2}^{1-\theta}\|U(-t_n)\vec{f}\|_{L^6\times \dot{H}_6^{-1}}^{\theta} \to 0.\end{aligned} \tag{A.6}$$

Then by applying (4.54) to the difference equation, we obtain the convergence $\vec{u}_n \to \vec{u}$ in $S(T,\infty)$. Since $\vec{u}_n$ solves the Zakharov system, so does the limit $\vec{u}$. If the former is uniformly bounded in $H^1 \times L^2$, so is the latter, and the convergence is also extended to arbitrary (T,∞) by the local wellposedness. □

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