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“An Extension of the Chaos Expansion Approximation for the Pricing of Exotic Basket Options”

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AN EXTENSION OF THE CHAOS EXPANSION APPROXIMATION FOR THE PRICING OF EXOTIC BASKET OPTIONS

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ABSTRACT. Funahashi and Kijima (2013) have proposed an approximation method based on the Wiener–Ito chaos expansion for the pricing of European-style contingent claims. In this paper, we extend the method to the multi-asset case with general local volatility structure for the pricing of exotic basket options such as Asian basket options. Through ample numerical experiments, we show that the accuracy of our approximation remains quite high even for a complex basket option with long maturity and high volatility.

Keywords: Wiener–Ito chaos expansion, local volatility, average option, basket option, spread option, Asian basket option

1. INTRODUCTION

The aim of this paper is to provide an approximation method for the pricing of European-style Asian basket options and their variants.

Asian options belong to the class of path-dependent options whose payoff functions are determined by the average of underlying asset price process over some pre-determined period of time.\(^1\) Asian options are popular in the foreign exchange and commodities markets, since they can help corporate firms hedge risks arising from their businesses. In addition, Asian options are cheaper than the corresponding vanilla options, and hence they are preferred by practitioners. On the other hand, basket options are also exotic options, whose payoff functions are based on more than one underlying assets. Examples of basket options include index options, spread options, rainbow options and options on a portfolio. This type of options is also popular in the foreign exchange market, because financial corporations with multiple currency exposures can hedge their exposures less expensively by purchasing a basket option than by purchasing vanilla options on each currency individually. Asian basket options are the combination of these popular options. In general, both Asian and basket options are known to be difficult to price analytically and numerically.

A large number of numerical methods have been proposed in the literature for the pricing of Asian and/or basket options. For Asian options, numerical methods based on the partial differential equation (PDE for short) or Monte Carlo methods are proposed by Kemna and Vorst (1990), Dewynne and Wilmott (1993), Lapeyre and Temam (2001), Vecer (2001) and many others, whereas the literature for basket options includes Rubinstein (1991), Pellizzari (2001) and

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\(^1\)Unless stated otherwise, ‘average’ always means arithmetic average in this paper.
Hager et al. (2010). Unfortunately, these methods are in general computationally too burdensome for practical use. In particular, for general basket options, any standard method such as finite difference method and tree method is subject to the ‘curse of dimensionality’ and cannot solve the problem for the high-dimensional case within a reasonable time for practice. While Monte Carlo simulation seems to work for the pricing problem even for the high-dimensional case, this method is in general too computationally expensive to be used for calibration purpose. Therefore, closed-form approximation formulas might be the only feasible solution to practitioners for the pricing of Asian basket options.

When the underlying asset model is restricted to the geometric Brownian motion (GBM for short) case, several effective approximations have been proposed in the literature. Tumbull and Wakeman (1991) and Ritchken et al. (1993) apply a fourth-order Edgeworth series approximation to the log-normal distribution and obtain an analytical approximation for Asian option prices. Levy (1992) modifies the Tumbull–Wakeman approximation to derive another analytic approximation method which is considered to give more accurate results. Milevsky and Posner (1998) propose closed-form approximation formulas for Asian options based on reciprocal Gamma distributions. Posner and Milevsky (1998) use Johnson functions to approximate the state price density by matching the first four moments and use them in integral formulas for pricing Asian options. Ju (2002) approximates the ratio of the characteristic function of the arithmetic average to that of the approximating log-normal random variable by using the sixth-order Taylor expansion around zero and derives a very accurate approximated closed-form solution. It should be noted that these methods can be directly applied for the pricing of basket options (see Ju (2002) and Krekel et al. (2004) for detailed discussions). On the other hand, Zhang (2001) proposes a semi-analytical approach that uses a singularity-removing technique to derive an analytical approximation formula of Asian options and derives the PDE for the correction term between the exact price and the analytical approximation.

It is well known that the Black and Scholes model (1973) cannot consistently price European options in the market, since implied volatility surfaces are usually skew- or smile-shaped. This tendency holds for the case of Asian basket options as well, and the above mentioned approximations are not suitable for practical use because they are based on the GBM assumption. Hence, it is required to develop some approximation method for more general underlying processes. In this regard, Takahashi (1999) applies the Malliavin–Watanabe theory to derive second-order asymptotic approximation formulas for both Asian and basket options under a general class of diffusion processes. Also, Fouque and Han (2003) use perturbation techniques to approximate Asian option prices under a stochastic volatility environment.

In this paper, we propose an approximation method based on the chaos expansion approach, recently proposed by Funahashi and Kijima (2013), for the pricing of Asian basket options. Through ample numerical examples, we show that the accuracy of our approximation remains quite high even for a complex basket option with long maturity and high volatility under various diffusion models. By the comparison with the previous works, we show that our approximation provides highly accurate results over a wide range of data sets even for the GBM case. Moreover, our approximation formulas can capture the skew and smile effects. Also, our approach can not only save computational time without sacrificing much accuracy, but also lead to the effective and/or stable calculation of Greeks.

This paper is organized as follows. After explaining our problem concisely in the next section, we extend the chaos expansion approach of Funahashi and Kijima (2013) to the multi-asset case in Section 3. Each asset price is approximated by a truncated sum of iterated Ito stochastic integrals, and the Asian basket variable is also described, after the change of order of integration, as a sum of iterated Ito stochastic integrals. An approximated formula of Asian basket
options can then be derived in closed form. In Section 4, we consider a special case where each local volatility function depends only on its price, not on the other asset prices. Approximation for the ordinary Black–Scholes setting (1973) is also considered and compared with the previous approximation results mentioned above. Section 5 is devoted to numerical examples. Comparing the closed formulas with Monte Carlo simulation results, it is observed that our approximation remains quite accurate even for a complex basket option with long maturity and high volatility. Finally, Section 6 concludes this paper.

Throughout this paper, \((\Omega, \mathcal{F}, \mathbb{Q}, \{\mathcal{F}_t\}_{t\geq 0})\) will be a filtered probability space where the filtration \(\{\mathcal{F}_t\}_{t\geq 0}\) satisfies the usual conditions. The probability measure \(\mathbb{Q}\) is a risk-neutral measure and the expectation operator under \(\mathbb{Q}\) is denoted by \(\mathbb{E}\).

2. The Setup

In this paper, we consider a financial market with \(N\) risky assets \(\{S_{i,t}\}_{0\leq t\leq T}, i = 1, 2, \ldots, N\), and one risk-free asset \(\{S_0\}_{0\leq t\leq T}\). The risk-free asset is a money-market account with spot interest rate \(r(t)\), that is a deterministic function of time \(t\). On the other hand, the risky assets are modeled by the following stochastic differential equation (SDE for short):

\[
\frac{dS_{i,t}}{S_{i,t}} = r(t)dt + \sigma_i(S_t, t)dW_{i,t}, \quad 0 \leq t \leq T,
\]

under the risk-neutral measure \(\mathbb{Q}\), where \(S_t = (S_{1,t}, \ldots, S_{N,t})\) and the volatilities \(\sigma_i(s, t)\) are deterministic functions of both asset prices \(s = (s_1, \ldots, s_N)\) and time \(t\), and where \(\{W_{i,t}\}_{t\geq 0}\) are standard Brownian motions under \(\mathbb{Q}\) with correlation \(dW_{i,t}dW_{j,t} = \rho_{i,j}dt\). It is assumed throughout that each volatility \(\sigma_i(s, t)\) is an analytic function of \((s, t)\).

For the multivariate local volatility model (2.1), we consider the following exotic basket options. Namely, for weighting (deterministic) functions \(w_{i,t}\), define the random variable

\[
V_T := \sum_{i=1}^{N} \int_{0}^{T} w_{i,t}S_{i,t}dt, \quad V_0 = V,
\]

and consider a call option written on \(V_T\) with exercise price \(K\), i.e.,

\[
C(V, K, T) = \mathbb{E} \left[ e^{-\int_{0}^{T} r(u)du} (V_T - K)^+ \right].
\]

The aim of this paper is to derive the option value \(C(V, K, T)\).

To this end, there may be several approaches that are applicable for the pricing problem. For example, in principle, any standard method such as finite difference method and tree method can be applied to solve the problem. However, these methods are subject to the curse of dimensionality and cannot solve the problem within a reasonable time for practice. On the other hand, Monte Carlo simulation seems to work for the pricing problem even for the high-dimensional case. However, when calibration is required to the market, this method is in general too computationally expensive to be used in practice, because the entire optimization procedure is extremely time-consuming. Therefore, closed-form approximation formulas might be the only feasible solution for practitioners.

In this paper, we apply the chaos expansion approach recently developed by Funahashi and Kijima (2013) to approximate the random variable \(V_T\) by a truncated sum of iterated Ito stochastic integrals. We then derive the probability density function (PDF for short) of the approximated random variable. The value of the call option can be derived in closed form by the approximated PDF.
Before proceeding, we emphasize that the above formulation is a generalization of the following well-known options:

**European Option:** Let \( N = 1 \) and \( w_{1,t} = \delta(T - t) \), where \( \delta(u) \) represents the Dirac delta function. Then, \( V_T = S_{1,T} \) so that the option (2.3) reduces to the plain-vanilla European option.

**Asian Option:** Let \( N = 1 \) and \( w_{1,t} = 1/T \) for all \( 0 \leq t \leq T \). Then, \( V_T = \frac{1}{T} \int_0^T S_{1,t} dt \).

This option is called an average or (arithmetic) Asian option.

**Partial Average Option:** Let \( N = 1 \) and \( w_{1,t} = 1/(T_1 - T_2) \) for all \( T_1 \leq t \leq T_2 \) and \( w_{1,t} = 0 \) otherwise, where \( 0 \leq T_1 < T_2 \leq T \). In this case, we have \( V_T = \frac{1}{(T_2 - T_1)} \int_{T_1}^{T_2} S_{1,t} dt \).

**Basket Option:** Let \( w_{i,t} = w_i \delta(T - t) \), where \( w_i \) are some constants for \( i = 1, 2, \ldots, N \).

In this case, we have \( V_T = \sum_{i=1}^N w_i S_{i,T} \), the so-called basket option.

**Spread Option:** As a special case of basket options, if \( N = 2 \), \( w_1 = 1 \) and \( w_2 = -1 \), then we have \( V_T = S_{1,T} - S_{2,T} \). Hence, the option is reduced to a spread option.

**Asian Basket Option:** Let \( w_{i,t} = 1/T \), \( i = 1, 2, \ldots, N \), for all \( 0 \leq t \leq T \). Then, \( V_T = \sum_{i=1}^N \frac{1}{T} \int_0^T S_{i,t} dt \), and the option is called an Asian basket option.

3. **The Chaos Expansion Approach**

The chaos expansion approach is composed by the following four steps:

1. Represent the underlying asset by Hermite polynomials,
2. Expand the underlying dynamics by means of successive substitution, and approximate it by a truncated sum of iterated Ito stochastic integrals using the Wiener–Ito chaos expansion,
3. Derive the PDF of the approximated underlying variable, and
4. Compute the value of a European contingent claim in closed form by using the PDF.

In the following, we extend the results of Funahashi and Kijima (2013) to the multi-dimensional case. Some of the extensions are straightforward; in that case, we omit our exposition largely.

### 3.1. Approximation of the underlying assets.

First, by applying Ito’s formula to the SDE (2.1), we obtain

\[
S_{i,t} = F_i(0,t) \exp \left[ J_{i,t}(\sigma_i) - \frac{1}{2} \| \sigma_i \|_2^2 \right],
\]

where \( F_i(0,t) = S_{i,0} e^{\int_0^t \rho(u) du} \) is the forward price of the underlying asset with delivery date \( t \), \( J_{i,t}(g) = \int_0^t g(u) dW_{i,u} \) and \( \| g \|_2^2 = \int_0^t g^2(u) du \). It is well known (see, e.g., Chapter 1 of Nualart (2006)) that the above expression can be written as

\[
S_{i,t} = F_i(0,t) \sum_{n=0}^{\infty} \frac{\| \sigma_i \|_n^n}{n!} h_n \left( J_{i,t}(\sigma_i) \right)^n \| \sigma_i \|_n^n
\]

for any \( \sigma_i \in L^2([0,T]) \), where \( h_n(x) \) denotes the Hermite polynomial of order \( n \).

Second, let \( S^{(0)}_{i,t} = F_i(0,t) \) and define \( S^{(m)}_{i,t} \) successively by

\[
S^{(m+1)}_{i,t} = F_i(0,t) \exp \left[ J_{i}(\sigma^{(m)}_i) - \frac{1}{2} \| \sigma^{(m)}_i \|_2^2 \right],
\]
where \( \sigma_i^{(m)}(t) = \sigma_i(S_i^{(m)}, t) \) with \( S_i^{(m)} = (S_{1,i}^{(m)}, \ldots, S_{N,i}^{(m)}) \). As in Funahashi and Kijima (2013), we assume that \( S_i^{(m)} \) converges to \( S_i \) componentwise almost surely as \( m \to \infty \). Then, we have
\[
S_{i,t} = S_{i,t}^{(1)} + \sum_{m=1}^{\infty} \left\{ S_{i,t}^{(m+1)} - S_{i,t}^{(m)} \right\}.
\]

Also, from (3.2) and (3.3), we obtain
\[
\frac{S_{i,t}^{(m+1)}}{F_i(0, t)} = \sum_{n=0}^{\infty} \frac{\|\sigma_i^{(m)}\|_n!}{n!} h_n \left( \frac{J_{i,t}(\sigma_i^{(m)})}{\|\sigma_i^{(m)}\|_n} \right).
\]

It follows that
\[
S_{i,t} = S_{i,t}^{(1)} + F_i(0, t) \sum_{m,n=1}^{m+n \leq 3} I_{i,m,n}(t),
\]
where
\[
I_{i,m,n}(t) = \frac{1}{n!} \left\{ \|\sigma_i^{(m)}\|_n! h_n \left( \frac{J_{i,t}(\sigma_i^{(m)})}{\|\sigma_i^{(m)}\|_n} \right) - \|\sigma_i^{(m-1)}\|_n! h_n \left( \frac{J_{i,t}(\sigma_i^{(m-1)})}{\|\sigma_i^{(m-1)}\|_n} \right) \right\}.
\]

As in Funahashi and Kijima (2013), we approximate \( S_{i,t} \) by a truncated sum at \( m + n \leq 3 \). Namely, our approximation is given by
\[
S_{i,t} \approx S_{i,t}^{(1)} + F_i(0, t) \sum_{m+n \leq 3} I_{i,m,n}(t).
\]

This approximation is justified by Proposition 2.2 of Funahashi and Kijima (2013), when the volatility term is small in the \( L_2 \) sense.

It remains to approximate the remaining terms in (3.5). To this end, we invoke Proposition 1.14 of Nualart (2006) to derive
\[
\frac{S_{i,t}^{(1)}}{F_i(0, t)} = 1 + \sum_{n=1}^{\infty} \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \sigma_i^{(0)}(t_1) \sigma_i^{(0)}(t_2) \cdots \sigma_i^{(0)}(t_n) dW_{i,t_1} \cdots dW_{i,t_n},
\]
where \( \sigma_i^{(0)}(t) = \sigma_i(S_i^{(0)}, t), S_i^{(0)} = (F_1(0, t), \ldots, F_N(0, t)) \), is a deterministic function. According to our strategy, we approximate it as
\[
S_{i,t}^{(1)} \approx F_i(0, t) \left[ 1 + \int_0^t \sigma_i^{(0)}(t_1) dW_{i,t_1} + \int_0^t \int_0^{t_2} \sigma_i^{(0)}(t_1) \sigma_i^{(0)}(t_2) dW_{i,t_1} dW_{i,t_2} 
\right. \\
\left. + \int_0^t \int_0^{t_3} \int_0^{t_2} \sigma_i^{(0)}(t_1) \sigma_i^{(0)}(t_2) \sigma_i^{(0)}(t_3) dW_{i,t_1} dW_{i,t_2} dW_{i,t_3} \right],
\]
the third-order approximation.

In order to approximate \( I_{i,m,n}(t), m + n \leq 3 \), we employ Taylor’s expansion around \( S_i^{(m-1)} \), as in Funahashi and Kijima (2013). Namely, since \( J_{i,t}(\sigma_i^{(m)}) = \int_0^t \sigma_i(S_i^{(m)}, u) dW_{i,u} \), it follows that
\[
J_{i,t}(\sigma_i^{(m)}) \approx J_{i,t}(\sigma_i^{(m-1)}) + \sum_{p=1}^{N} \int_0^t \partial_p \sigma_i^{(m-1)}(u) \{ S_{p,u}^{(m)} - S_{p,u}^{(m-1)} \} dW_{i,u}
\]
\[
+ \frac{1}{2} \sum_{p,q=1}^{N} \int_0^t \partial_{pq} \sigma_i^{(m-1)}(u) \{ S_{p,u}^{(m)} - S_{p,u}^{(m-1)} \} \{ S_{q,u}^{(m)} - S_{q,u}^{(m-1)} \} dW_{i,u},
\]
(3.7)
where $\partial_{p_i} \sigma_i^{(m)}(u)$ denotes the partial derivative of $\sigma_i^{(m)}(u)$ with respect to the $p$th variable and $\partial_{pq} \sigma_i^{(m)}(u)$ represents the second-order partial derivative with respect to the $p$th and $q$th variables. Furthermore, we use the approximation

\[
J_{i,t}^2(\sigma_i^{(m)}) \approx J_{i,t}^2(\sigma_i^{(m-1)}) + 2 J_{i,t}(\sigma_i^{(m-1)}) \sum_{p=1}^{N} \int_0^t \partial_p \sigma_i^{(m-1)}(u) \{ S_{p,u}^{(m)} - S_{p,u}^{(m-1)} \} dW_{i,u}.
\]

Repeated application of the expansion results (3.7) and (3.8) leads to the following. Recall that our strategy is to neglect those terms that produce fourth- or higher-order iterated Itô stochastic integrals. The proof is given in Appendix A.

**Lemma 3.1.** Each term $I_{i;m,u}(t)$ defined by (3.4) is approximated as follows:

\[
I_{i,1,1}(t) \approx \sum_{p=1}^{N} \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0,s) \left( \int_0^s \sigma_p^{(0)}(u) dW_{p,u} \right) dW_{i,s}
\]

\[
+ \sum_{p=1}^{N} \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0,s) \left( \int_0^s \sigma_p^{(0)}(u) \left( \int_0^u \sigma_p^{(0)}(r) dW_{p,r} \right) dW_{p,u} \right) dW_{i,s}
\]

\[
+ \sum_{p,q=1}^{N} \int_0^t \partial_{pq} \sigma_i^{(0)}(s) F_p(0,s) F_q(0,s) \left( \int_0^s \sigma_p^{(0)}(u) \left( \int_0^u \sigma_q^{(0)}(r) dW_{q,r} \right) dW_{p,u} \right) dW_{i,s}
\]

\[
+ \frac{1}{2} \sum_{p,q=1}^{N} \int_0^t \partial_{pq} \sigma_i^{(0)}(s) F_p(0,s) F_q(0,s) \left( \int_0^s \sigma_p^{(0)}(u) \sigma_q^{(0)}(u) du \right) dW_{i,s},
\]

\[
I_{i,1,2}(t) \approx \sum_{p=1}^{N} \int_0^t \sigma_i^{(0)}(s) \left( \int_0^s \partial_p \sigma_i^{(0)}(u) F_p(0,u) \left( \int_0^u \sigma_p^{(0)}(r) dW_{p,r} \right) dW_{p,u} \right) dW_{i,s}
\]

\[
+ \sum_{p=1}^{N} \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0,s) \left( \int_0^s \sigma_i^{(0)}(u) \left( \int_0^u \sigma_p^{(0)}(r) dW_{p,r} \right) dW_{p,u} \right) dW_{i,s}
\]

\[
+ \sum_{p=1}^{N} \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0,s) \left( \int_0^s \sigma_i^{(0)}(u) \left( \int_0^u \sigma_i^{(0)}(r) dW_{i,r} \right) dW_{p,u} \right) dW_{i,s}
\]

\[
+ \sum_{p=1}^{N} \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0,s) \left( \int_0^s \sigma_i^{(0)}(u) \sigma_p^{(0)}(u) du \right) dW_{i,s},
\]

and

\[
I_{i,2,1}(t) \approx \sum_{p,q=1}^{N} \int_0^t \partial_{pq} \sigma_i^{(0)} F_p(0,s) \left( \int_0^s \partial_q \sigma_p^{(0)} F_q(0,s) \left( \int_0^u \sigma_q^{(0)}(r) dW_{q,r} \right) dW_{p,u} \right) dW_{i,s}.
\]

Note that all the integrands in Lemma 3.1 are deterministic functions, since $\sigma_i^{(0)}(t) = \sigma_i(S_i^{(0)}, t)$ with $S_i^{(0)} = F_i(0, t)$ being the forward price of asset $S_i$ that is observed in the market.

We are now in a position to state our approximation result. The next result is obtained by putting above approximation results all together.

**Theorem 3.1.** Each asset price $S_{i,t}$ is approximated as

\[
S_{i,t} \approx F_i(0, t) \left[ 1 + A_{i,t}^1 + A_{i,t}^2 + A_{i,t}^3 \right], \quad i = 1, 2, \ldots, N,
\]
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where

\[ A_{i,t}^1 = \int_0^t P_i^1(s) dW_{i,s}, \]

\[ A_{i,t}^2 = \int_0^t \sigma_i^{(0)}(s) \left( \int_0^s \sigma_i^{(0)}(u) \sigma_p^{(0)}(u) dW_{i,u} \right) dW_{i,s} + \sum_{p=1}^N \int_0^t P_{i,p}^2(s) \left( \int_0^s \sigma_p^{(0)}(u) \sigma_q^{(0)}(u) dW_{p,u} \right) dW_{i,s}, \]

and \( A_{i,t}^3 = \sum_{k=1}^7 A_{i,t}^3(k) \) with \( A_{i,t}^3(k) \) being defined by

\[ A_{i,t}^3(1) = \int_0^t \sigma_i^{(0)}(s) \left( \int_0^s \sigma_i^{(0)}(u) \left( \int_0^u \sigma_i^{(0)}(r) dW_{i,r} \right) dW_{i,u} \right) dW_{i,s}, \]

\[ A_{i,t}^3(2) = \sum_{p=1}^N \int_0^t P_{i,p}^2(s) \left( \int_0^s \sigma_i^{(0)}(u) \left( \int_0^u \sigma_p^{(0)}(r) dW_{p,r} \right) dW_{p,u} \right) dW_{i,s}, \]

\[ A_{i,t}^3(3) = \sum_{p=1}^N \int_0^t \sigma_i^{(0)}(s) \left( \int_0^s \sigma_i^{(0)}(u) \left( \int_0^u \sigma_p^{(0)}(r) dW_{p,u} \right) dW_{p,r} \right) dW_{i,s}, \]

\[ A_{i,t}^3(4) = \sum_{p=1}^N \int_0^t P_{i,p}^2(s) \left( \int_0^s \sigma_i^{(0)}(u) \left( \int_0^u \sigma_p^{(0)}(r) dW_{p,u} \right) dW_{p,r} \right) dW_{i,s}, \]

\[ A_{i,t}^3(5) = \sum_{p=1}^N \int_0^t \sigma_i^{(0)}(s) \left( \int_0^s \sigma_i^{(0)}(u) \left( \int_0^u \sigma_q^{(0)}(r) dW_{q,u} \right) dW_{q,r} \right) dW_{i,s}, \]

\[ A_{i,t}^3(6) = \sum_{p,q=1}^N \int_0^t P_{i,p}^3(s) \left( \int_0^s \sigma_p^{(0)}(u) \left( \int_0^u \sigma_q^{(0)}(r) dW_{q,u} \right) dW_{q,r} \right) dW_{i,s}, \]

\[ A_{i,t}^3(7) = \sum_{p,q=1}^N \int_0^t \sigma_i^{(0)}(s) \left( \int_0^s \sigma_i^{(0)}(u) \left( \int_0^u \sigma_p^{(0)}(r) dW_{p,r} \right) dW_{p,u} \right) dW_{i,s}. \]

Note that \( A_{i,t}^k, \ k = 1, 2, 3, \) corresponds to the \( k \)-th order iterated Ito stochastic integrals. In particular, \( A_{i,t}^1 \) follows a normal distribution with zero mean. Here, we define

\[ P_i^1(s) := \sigma_i^{(0)}(s) + \sum_{p=1}^N \partial_p \sigma_i^{(0)}(s) F_p(0, s) \left( \int_0^s \sigma_i^{(0)}(u) \sigma_p^{(0)}(u) du \right), \]

\[ P_{i,p}^2(s) := \sigma_i^{(0)}(s) F_p(0, s), \quad p = 1, 2, \ldots, N, \]

\[ P_{i,p}^3(s) := \sigma_i^{(0)}(s) F_p(0, s) F_q(0, s), \quad p, q = 1, 2, \ldots, N. \]

Note that \( P_i^k(t) \)'s are all deterministic functions.
Now, from (3.9), we have

\begin{equation}
\int_0^T w_{i,t} S_{i,t} dt = \int_0^T w_{i,t} F_i(0, t) dt + a_{i,1}(T) + a_{i,2}(T) + a_{i,3}(T),
\end{equation}

where

\[ a_{i,k}(T) = \int_0^T w_{i,t} F_i(0, t) A_{i,t}^k dt, \quad k = 1, 2, 3. \]

By changing the order of integration, we obtain

\[ a_{i,1}(T) = \int_0^T \bar{p}_1^i(t, T) dW_{i,t}, \quad \bar{p}_1^i(t, T) := P_1^i(t) \int_t^T w_{i,s} F_i(0, s) ds, \]

and

\[ a_{i,2}(T) = \int_0^T \bar{s}_i(t, T) \left( \int_t^T \sigma_p^0(s) dW_{p,s} \right) dW_{i,t} + \sum_{p=1}^N \int_0^T \bar{p}_{1p}^2(t, T) \left( \int_t^T \sigma_p^0(s) dW_{p,s} \right) dW_{i,t}, \]

where

\[ \bar{s}_i(t, T) := \sigma_i^0(t) \int_t^T w_{i,s} F_i(0, s) ds, \quad \bar{p}_{1p}^2(t, T) := P_{1p}^2(t) \int_t^T w_{i,s} F_i(0, s) ds. \]

Similar representation holds for \( a_{i,3}(T) \). Namely, the first term of \( a_{i,3}(T) \) is given by

\begin{align*}
\bar{a}_{i,3}^1(T) &:= \int_0^T w_{i,t} F_i(0, t) A_{i,t}^3(1) dt \\
&= \int_0^T \bar{s}_i(t, T) \left( \int_0^s \sigma_i^0(u) \left( \int_0^u \sigma_i^0(r) dW_{i,r} \right) dW_{i,u} \right) dW_{i,s}.
\end{align*}

Other terms \( \bar{a}_{i,3}^k(T), k = 2, \ldots, 7 \), are provided in Appendix B. The random variable \( V_T \) in (2.2) is now approximated as

\begin{equation}
V_T \approx \sum_{i=1}^N \int_0^T w_{i,t} F_i(0, t) dt + a_1(T) + a_2(T) + a_3(T),
\end{equation}

where \( a_k(T) = \sum_{i=1}^N a_{i,k}(T) \).

3.2. Option pricing formula. Let \( X_T := V_T - \sum_{i=1}^N \int_0^T w_{i,t} F_i(0, t) dt \) so that

\[ X_t \approx a_1(t) + a_2(t) + a_3(t). \]

Since \( a_1(T) \) is a mixture of normal random variables, \( a_1(t) \) follows a normal distribution with zero mean and variance

\[ \Sigma_t = \sum_{i,j=1}^N \int_0^t \rho_{i,j} \bar{p}_{i,1}(s) \bar{p}_{j,1}(s) ds. \]

It follows that \( a_1(t) \) can be rewritten as

\begin{equation}
a_1(t) = \int_0^t \sqrt{\Lambda_t} dW_t,
\end{equation}

where \( \Lambda_t = \sum_{i,j=1}^N \rho_{i,j} \bar{p}_{i,1}(t) \bar{p}_{j,1}(t) \) and \( dW_t = \sum_{i=1}^N (\bar{p}_{i,1}(t) / \sqrt{\Lambda_t}) dW_{i,t} \). Note that \( W_t \) is considered to be a standard Brownian motion under \( \mathbb{Q} \).
By applying the following result, an approximation of the density function of $X_t$ can be obtained. The proof is found in Funahashi and Kijima (2013). The density function of $X_t$ is denoted by $f_{X_t}(x)$.

**Lemma 3.2.** The density function of $X_t$ is approximated by

$$f_{X_t}(x) \approx n(x; 0, \Sigma_t) - \frac{\partial}{\partial x} \left\{ \mathbb{E}[a_2(t)|a_1(t) = x]n(x; 0, \Sigma_t) \right\}$$

$\quad - \frac{\partial}{\partial x} \left\{ \mathbb{E}[a_3(t)|a_1(t) = x]n(x; 0, \Sigma_t) \right\}$

$\quad + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left\{ \mathbb{E}[a_2(t)^2|a_1(t) = x]n(x; 0, \Sigma_t) \right\},$

where $n(x; a, b)$ denotes the normal density function with mean $a$ and variance $b$.

The conditional expectations in Lemma 3.2 can be evaluated explicitly. Some key results are provided in Appendix C. Using the approximated density function, say $\tilde{f}_{X_t}(x)$, the call option price (2.3) can now be approximated by

$$C(V, K, T) = \mathbb{E} \left[ e^{-\int_0^T r(t)dt} (V(T) - K)^+ \right]$$

$\quad = \mathbb{E} \left[ e^{-\int_0^T r(t)dt} (X_T + \bar{K})^+ \right]$\n
$\quad \approx e^{-\int_0^T r(t)dt} \int_{\bar{K}}^{\infty} (x + \bar{K}) \tilde{f}_{X_T}(x) dx,$

where $\bar{K} := \sum_{i=1}^N \int_0^T w_{i,t} F_i(0, t) dt - K$. The derivation of the approximated option price is tedious but straightforward. The resulting formula is complicated and omitted. The complete derivation and formulas are available from the authors upon request.

### 4. Some Special Cases

So far, we have considered the general local volatility model (2.1) whose volatility value depends not only on its price but also the other asset prices. However, it is usually very difficult to specify the volatility functions in its full generality for practical uses. In this section, as a special case of (2.1), we assume that

$$\frac{dS_{i,t}}{S_{i,t}} = r(t)dt + \sigma_i(S_{i,t}, t)dW_{i,t}, \quad 0 \leq t \leq T,$$

under the risk-neutral measure $\mathbb{Q}$. That is, we assume that the local volatility depends only on the price of its own (not the others), while assuming that the Brownian motions are correlated as $dW_{i,t}dW_{i',t} = \rho_{i,j}dt$. Then, all the cross partial-derivatives in Theorem 3.1 disappear. Namely, in this case, $P^k_i(s)$ in Theorem 3.1 are reduced to

$$P^1_i(s) \quad := \quad \sigma_i^{(0)}(s) + \partial_t \sigma_i^{(0)}(0, s) F_i(0, s) \left( \int_0^s \left\{ \sigma_i^{(0)}(u) \right\}^2 du \right)$$

$\quad + \frac{1}{2} \partial_{t^2} \sigma_i^{(0)}(s) F_i^2(0, s) \left( \int_0^s \left\{ \sigma_i^{(0)}(u) \right\}^2 du \right),$

$$P^2_{i,i}(s) \quad := \quad \partial_i \sigma_i^{(0)}(s) F_i(0, s),$$

$$P^3_{i,i}(s) \quad := \quad \partial_{i^2} \sigma_i^{(0)}(0, s) F_i^2(0, s).$$
Hence, each asset price is approximated as

\[ S_{i,t} \approx F_i(0,t) \left[ 1 + \int_0^t r_{i,1}(s) dW_{i,s} + \int_0^t r_{i,2}(s) \left( \int_0^s \sigma_i^{(0)}(u) dW_{i,u} \right) dW_{i,s} \right. \]

\[ + \int_0^t r_{i,3}(s) \left( \int_0^s \sigma_i^{(0)}(u) \left( \int_0^u \sigma_i^{(0)}(r) dW_{i,r} \right) dW_{i,u} \right) dW_{i,s} \]

\[ + \left. \int_0^t r_{i,4}(s) \left( \int_0^s r_{i,5}(u) \left( \int_0^u \sigma_i^{(0)}(r) dW_{i,r} \right) dW_{i,u} \right) dW_{i,s} \right] , \tag{4.3} \]

where \( \sigma_i^{(0)}(s) := \sigma_i(F_i(0,s), s) \) and

\[ r_{i,1}(s) := P_i^1(s), \]

\[ r_{i,2}(s) := \sigma_i^{(0)}(s) + P_i^2(s), \]

\[ r_{i,3}(s) := \sigma_i^{(0)}(s) + 3P_i^2(s) + P_i^3(s), \]

\[ r_{i,4}(s) := \sigma_i^{(0)}(s) + P_i^2(s), \]

\[ r_{i,5}(s) := P_i^2(s). \]

The iterative integrals \( a_{i,k}(T), k = 1, 2, 3, \) in (3.10) can also be simplified as

\[ a_{i,1}(T) = \int_0^T \bar{r}_{i,1}(t) dW_{i,t}, \]

\[ a_{i,2}(T) = \int_0^T \bar{r}_{i,2}(t) \left( \int_0^t \sigma_i^{(0)}(s) dW_{i,s} \right) dW_{i,t}, \]

\[ a_{i,3}(T) = \int_0^T \bar{r}_{i,3}(t) \left( \int_0^t \sigma_i^{(0)}(s) \left( \int_0^s \sigma_i^{(0)}(u) dW_{i,u} \right) dW_{i,s} \right) dW_{i,t} \]

\[ + \int_0^T \bar{r}_{i,4}(t) \left( \int_0^s r_{i,5}(u) \left( \int_0^u \sigma_i^{(0)}(r) dW_{i,r} \right) dW_{i,u} \right) dW_{i,s}, \tag{4.4} \]

where \( \bar{r}_{i,k}(t) = r_{i,k}(t) \int_t^T w_{i,s} F(0,s) ds \) for \( k = 1, 2, 3, 4. \)

Moreover, the conditional expectations in Lemma 3.2 are derived explicitly as

\[ \mathbb{E}[a_2(t)|a_1(t) = x] = q_1(t) \left( \frac{x^2}{\Sigma_t^2} - \frac{1}{\Sigma_t} \right), \]

\[ \mathbb{E}[a_3(t)|a_1(t) = x] = q_2(t) \left( \frac{x^3}{\Sigma_t^3} - \frac{3x}{\Sigma_t^2} \right), \]

\[ \mathbb{E}[a_4^2(t)|a_1(t) = x] = q_3(t) \left( \frac{x^4}{\Sigma_t^4} - \frac{6x^2}{\Sigma_t^3} + \frac{3}{\Sigma_t^2} \right) + q_4(t) \left( \frac{x^2}{\Sigma_t^2} - \frac{1}{\Sigma_t} \right) + q_5(t), \]
The value of a European call option with maturity $T$ and strike $K$ is approximated as

$$
C(T) \approx \frac{e^{-\int_0^T r(t) dt} n(\bar{K}; 0, \Sigma_t)}{2\Sigma_t^2} \left[ q_3(T) (\bar{K}^4 - 6\bar{K}^2 \Sigma_t + 3\Sigma_t^2) 
+ \Sigma_t^2 (q_4(T) + 2q_2(T)) (\bar{K}^2 - \Sigma_t) 
+ \Sigma_t^4 \left\{ -2q_1(T) \bar{K} + q_5(T) \Sigma_t + 2\Sigma_t^2 \right\} 
+ e^{-\int_0^T r(t) dt} \bar{K} \left( 1 - \Phi(-\bar{K}/\sqrt{\Sigma_t}) \right) \right],
$$

where

$$
\Sigma_t = \sum_{i,j=1}^N \int_0^t \rho_{i,j} \tilde{r}_{i,1}(s) \tilde{r}_{j,1}(s) ds,
$$

$$
q_1(t) = \sum_{i,j,k,l=1}^N \int_0^t \rho_{i,j} \tilde{r}_{i,1}(s) \tilde{r}_{j,2}(s) \left( \int_0^s \rho_{i,k} \sigma_i^{(0)}(u) \tilde{r}_{k,1}(u) du \right) ds,
$$

$$
q_2(t) = \sum_{i,j,k,l=1}^N \int_0^t \rho_{i,j} \tilde{r}_{i,1}(s) \tilde{r}_{j,3}(s) \left( \int_0^s \rho_{i,k} \sigma_i^{(0)}(u) \tilde{r}_{k,1}(u) \left( \int_0^u \rho_{i,s} \sigma_i^{(0)}(r) \tilde{r}_{l,1}(r) dr \right) du \right) ds,
$$

$$
q_3(t) = q_1(t),
$$

$$
q_4(t) = 2 \sum_{i,j,k,l=1}^N \int_0^t \rho_{i,k} \tilde{r}_{k,1}(s) \tilde{r}_{i,2}(s) \left( \int_0^s \rho_{i,j} \tilde{r}_{l,1}(u) \tilde{r}_{j,2}(u) \left( \int_0^u \rho_{i,s} \sigma_i^{(0)}(r) \sigma_j(0) dr \right) du \right) ds,
$$

By substituting these results into Lemma 3.2, we can obtain the following results.

**Theorem 4.1.** The probability density function of $X_t$ is approximated as

$$
f_{X_t}(x) \approx \frac{1}{2} n(x; 0, \Sigma_t) \left[ \frac{q_3(t)}{\Sigma_t^3} h_6 \left( \frac{x}{\sqrt{\Sigma_t}} \right) + \frac{(2q_2(t) + q_4(t))}{\Sigma_t^2} h_4 \left( \frac{x}{\sqrt{\Sigma_t}} \right) 
+ \frac{2q_1(t)}{(\sqrt{\Sigma_t})^2} h_3 \left( \frac{x}{\sqrt{\Sigma_t}} \right) + \frac{q_5(t)}{\Sigma_t} h_2 \left( \frac{x}{\sqrt{\Sigma_t}} \right) + 2 \right],
$$

where $\Sigma_t$ and $q_0(t)$ are defined above and $n(x; a, b)$ denotes the normal density function with mean $a$ and variance $b$.

**Theorem 4.2.** The value of a European call option with maturity $T$ and strike $K$ is approximated as

$$
C(T) \approx \frac{e^{-\int_0^T r(t) dt} n(\bar{K}; 0, \Sigma_t)}{2\Sigma_t^2} \left[ q_3(T) (\bar{K}^4 - 6\bar{K}^2 \Sigma_t + 3\Sigma_t^2) 
+ \Sigma_t^2 (q_4(T) + 2q_2(T)) (\bar{K}^2 - \Sigma_t) 
+ \Sigma_t^4 \left\{ -2q_1(T) \bar{K} + q_5(T) \Sigma_t + 2\Sigma_t^2 \right\} 
+ e^{-\int_0^T r(t) dt} \bar{K} \left( 1 - \Phi(-\bar{K}/\sqrt{\Sigma_t}) \right) \right],
$$
where $\Phi(x)$ is the cumulative distribution function of the standard normal distribution.

### 4.1. The GBM case.

As in the previous papers, we now assume that the underlying assets follow ordinary geometric Brownian motions, i.e.,

$$dS_{i,t} = r_i dt + \sigma_i dW_{i,t}, \quad 0 \leq t \leq T,$$

where $r$ and $\sigma_i$ are some constants with $dW_{i,t}dW_{i,t} = \rho_{i,j} dt$. Then, by definition, we have $\sigma_i^{(0)}(s) = \sigma_i$ so that $P_{i,k}^1(s)$ in (4.2) are reduced to

$$P_{i,1}^1(s) = \sigma_i, \quad P_{i,2}^2(s) = P_{i,i,i}^3(s) = 0.$$

It follows from (4.3) that each asset price is approximated as

$$S_{i,t} \approx S_{i,0}e^{rt} \left[ 1 + \int_0^t \sigma_i dW_{i,s} + \int_0^t \sigma_i \left( \int_0^s \sigma_i dW_{i,u} \right) dW_{i,u} \right],$$

since $\sigma_i^{(0)}(s) = r_{i,k}(s) = \sigma_i$, $k = 1, \ldots, 4$, and $r_{i,5}(s) = 0$ in this case. The iterative integrals $a_{i,k}(T)$ in (4.4) are further simplified as

$$a_{i,1}(T) = \int_0^T \bar{r}_i(t) dW_{i,t},$$

$$a_{i,2}(T) = \int_0^T \bar{r}_i(t) \left( \int_0^t \sigma_i dW_{i,s} \right) dW_{i,t},$$

$$a_{i,3}(T) = \int_0^T \bar{r}_i(t) \left( \int_0^t \sigma_i \left( \int_0^s \sigma_i dW_{i,u} \right) dW_{i,u} \right) dW_{i,t},$$

where $\bar{r}_i(t) = S_{i,0} \sigma_i \int_t^T w_{i,u} e^{rs} ds$. The option price formula in Theorem 4.2 is simplified accordingly.

### 4.2. Numerical comparison for the GBM case.

In this subsection, we calculate ordinary Asian option and Basket option prices by using our method and compare them with those obtained by several approximation methods previously proposed in the literature under the Black–Scholes setting (1973). In order to avoid intentional choice of parameter values, we adopt the same parameter setting as those used in Table 2 of Ju (2002).

Table 1 shows the Asian option prices calculated by the semi-analytic method (SA (Exact)) of Zhang (2001), the log-normal approximation (LN) of Levy (1992), the Edgeworth expansion method (EW) of Tumbull and Wakeman (1991) and Ritchken et al. (1993), the reciprocal gamma approximation (RG) of Milevsky and Posner (1998), the 4th-order moment approximation (FM4) of Posner and Milevsky (1998), the 6th-order Taylor expansion approximation (TE6) of Ju (2002) and our approximation (WIC). We use SA (Exact) to be the benchmark values for our comparison purposes. In the table, volatility $\sigma$ varies from 5% to 50% and strike $K$ from 95 to 105. The other parameters are set as $S_0 = 100$, $r = 0.09$, and $T = 3$. In order to check the accuracy, the residual mean squared error (labeled by RMSE) as well as the maximum absolute error (labeled by MAE) from the SA (Exact) is considered. The results of SA, LN, EW, RG, FM4 and TE6 are quoted from Table 2 of Ju (2002). The ranking based on the
Table 1: Asian option prices for the GBM case. The results of SA, LN, EW, RG, FM and TE6 are quoted from Table 2 of Ju (2002), where SA (Exact) stands for the semi-analytic method of Zhang (2001), LN the log-normal approximation of Levy (1992), EW the Edgeworth expansion method of Tumbull and Wakeman (1991) and Ritchken et al. (1993), RG the reciprocal gamma approximation of Milevsky and Posner (1998), FM4 the 4th-order moment approximation of Posner and Milevsky (1998), TE6 the 6th-order Taylor expansion approximation of Ju (2002), and WIC the approximation method proposed in this paper. The other parameters are set as $S = 100$, $r = 0.09$, and $T = 3$. The CPU time to calculate our results (WIC) was 0.001 second for each case in average.

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<th>LN</th>
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</table>

RMSE (also MAE) is also appended in the table. The CPU time to calculate our results (WIC) was 0.001 second for each case in average.\(^2\)

On the other hand, Table 2 shows the basket option prices calculated by the Monte Carlo simulation (MC), LN, RG, FM4 TE6, the geometric conditioning method (GC) of Curran (1994) and our approximation method WIC, where each basket consists of five homogeneous stocks. In the table, strike $K$ varies from 90 to 110, short rate $r$ from 0.05 to 0.1, volatility $\sigma$ from 20% to 50%, and correlation $\rho$ from 0 to 0.5. The weights are $w_1 = 0.05$, $w_2 = 0.15$, $w_3 = 0.2$, $w_4 = 0.25$ and $w_5 = 0.35$. The other parameters are set as $S_{i,0} = 100$ and $T = 3$. The results of MC, LN, RG, FM4, GC and TE6 are quoted from Table 6 of Ju (2002). As in Table 1, the residual mean squared error (RMSE) as well as the maximum absolute error (MAE) from Monte Carlo simulation (MC) and the ranking are also appended. The CPU time to calculate our results (WIC) was 0.826 second in average.

From these tables, it is explicitly observed that our approximation (WIC) is quite comparable to the existing accurate methods even for the GBM case. In particular, even in the case of high volatility, we can see that the accuracy of our approximation remains quite high. Note that we have made no intention for the parameter choice. Also, our approximation is not restricted to

\(^2\)The CPU time is measured by the time function equipped by C++. Note that $q_i(T)$ in the option pricing formula (Theorem 4.2) are evaluated by numerical integration.
Table 2: Basket option prices for the GBM case. The results of MC, LN, RG, FM4, GC and TE6 are quoted from the Table 6 of Ju (2002), where MC stands for the Monte Carlo simulation, LN the log-normal approximation of Levy (1992), RG the reciprocal gamma approximation of Milevsky and Posner (1998), FM4 the 4th-order moment approximation of Posner and Milevsky (1998), TE6 the 6th-order Taylor expansion approximation of Ju (2002), GC the geometric conditioning method of Curran (1994) and WIC the proposed method in this paper. Here, each basket consists of five homogeneous stocks with initial price $S_i, 0 = 100$. The option maturity is $T = 3$ years, and the weights are $w_1 = 0.05$, $w_2 = 0.15$, $w_3 = 0.2$, $w_4 = 0.25$ and $w_5 = 0.35$. The CPU time to calculate our results (WIC) was 0.826 second in average.

5. NUMERICAL EXAMPLES

In this section, we examine the accuracy of our approximation method under more general settings. Since there are no closed form solutions for Asian basket options, we compare our approximation results with Monte Carlo simulation.

In the following, we investigate three settings. First, we test the so-called constant elasticity of variance (CEV) model. Second, we examine a complex nonlinear volatility model and test the performance of our approximation formulas by calibrating it to the real market. Finally, we consider a general Asian basket option.

5.1. The CEV Model. In this subsection, we suppose that the underlying asset price follows the CEV model. To be more specific, we assume that the volatility in the SDE (4.1) is specified...
for each $i$ as

$$
\sigma_i(S_{i,t}, t) := \sigma_i S_{i,t}^{\beta_i - 1}, \quad t \geq 0,
$$

where $\sigma_i$ and $\beta_i$ are some constants. In particular, if $\beta_i = 1$ then the model becomes the Black-Scholes setting, whereas it is called the square-root model if $\beta_i = 0.5$.

We show in Figures 1 and 2 the differences between the option prices calculated by our approximation formula in Theorem 4.2 and the corresponding Monte Carlo simulation results for short maturity (6 months, left-hand-side panel) and long maturity (5 years, right-hand-side panel) cases.

Figure 1 reports the Asian option prices under the square-root model ($\beta = 0.5$). The parameters are set as $S_0 = 100$, $r = 0.03$, and $\sigma = 1.33$.

On the other hand, in Figure 2, we consider a European basket option that consists of 2 asymmetric stocks. The parameters are set as $r = 0.03$, $S_{1,0} = S_{2,0} = 100$, $\beta_1 = 1$, $\beta_2 = 0.5$, $\sigma_1 = 0.15$ and $\sigma_2 = 1.33$. Thus, while asset 1 follows the square-root process, asset 2 follows the GBM, in order to make the underlying assets asymmetric. Figure 2 reports the results for the cases of $\rho = 0.75$, $\rho = 0$ and $\rho = -0.75$ from the top two panels to the bottom two panels, respectively. The left-hand-side panel corresponds to the short maturity (6 months) case, whereas the right-hand-side panel corresponds to the long maturity (5 years) case.

From these figures, for both Asian and Basket options, we observe that the error becomes slightly large for long maturity and/or far in-the- and out-of-the-money strikes. But, the errors of our approximation are small enough for practical uses.

### 5.2. Calibration to the market

We examine the performance of our approximation formula in Theorem 4.2 by testing it on real market data. For this purpose, we consider two examples of volatility surface observed in the currency options market of 5 year JPY/USD and JPY/AUD options.

To this end, we need to slightly modify the SDE (4.1) in order to apply our results to the FX options market. Namely, we assume that the $i$-th spot exchange rate $S_{i,t}$ follows the SDE

$$
\frac{dS_{i,t}}{S_{i,t}} = (r_d(t) - r_{i,f}(t))dt + \sigma_i(S_{i,t}, t)dW_{i,t}, \quad i = 1, 2,
$$

Figure 1: Asian option prices for the square-root model. The left-hand side panel corresponds to the short maturity ($T = 6$ months) case and the right-hand side panel corresponds to the long maturity ($T = 5$ years) case. MC means the option prices calculated by Monte Carlo simulation, while WIC indicates the approximate prices calculated by the formula given in Theorem 4.2. The parameters are set as $S_0 = 100$, $r = 0.03$, and $\sigma = 1.33$. For each $i$ as

$$
\sigma_i(S_{i,t}, t) := \sigma_i S_{i,t}^{\beta_i - 1}, \quad t \geq 0,
$$

where $\sigma_i$ and $\beta_i$ are some constants. In particular, if $\beta_i = 1$ then the model becomes the Black-Scholes setting, whereas it is called the square-root model if $\beta_i = 0.5$.

We show in Figures 1 and 2 the differences between the option prices calculated by our approximation formula in Theorem 4.2 and the corresponding Monte Carlo simulation results for short maturity (6 months, left-hand-side panel) and long maturity (5 years, right-hand-side panel) cases.

Figure 1 reports the Asian option prices under the square-root model ($\beta = 0.5$). The parameters are set as $S_0 = 100$, $r = 0.03$, and $\sigma = 1.33$.

On the other hand, in Figure 2, we consider a European basket option that consists of 2 asymmetric stocks. The parameters are set as $r = 0.03$, $S_{1,0} = S_{2,0} = 100$, $\beta_1 = 1$, $\beta_2 = 0.5$, $\sigma_1 = 0.15$ and $\sigma_2 = 1.33$. Thus, while asset 1 follows the square-root process, asset 2 follows the GBM, in order to make the underlying assets asymmetric. Figure 2 reports the results for the cases of $\rho = 0.75$, $\rho = 0$ and $\rho = -0.75$ from the top two panels to the bottom two panels, respectively. The left-hand-side panel corresponds to the short maturity (6 months) case, whereas the right-hand-side panel corresponds to the long maturity (5 years) case.

From these figures, for both Asian and Basket options, we observe that the error becomes slightly large for long maturity and/or far in-the- and out-of-the-money strikes. But, the errors of our approximation are small enough for practical uses.
Figure 2: Basket option prices for the CEV model. The left-hand side panel corresponds to the short maturity \((T = 6\) months) case and the right-hand side panel corresponds to the long maturity \((T = 5\) years) case. The top two panels report the results for the case of \(\rho = 0.75\), the middle panels \(\rho = 0\) and the bottom panels \(\rho = -0.75\), respectively. MC means the option prices calculated by Monte Carlo simulation, while WIC indicates the approximate prices calculated by the formula given in Theorem 4.2. The parameters are set as \(S_{1,0} = S_{2,0} = 100, r = 0.03, \beta_1 = 1, \beta_2 = 0.5, \sigma_1 = 0.15\) and \(\sigma_2 = 1.33\).
where $r_d(t)$ and $r_{i,f}(t)$ represent the domestic and $i$th foreign short rates, respectively. In this setting, the forward price in Theorem 4.2 is modified as $F_i(0, t) = \exp\{\int_0^t (r_d(s) - r_{i,f}(s))ds\}$.

In order to capture the volatility smile in the FX options market, we assume that the volatility in (5.1) is given by

\begin{equation}
\sigma_i(S_{i,t}, t) := \left(\alpha_i + \beta_i x + \gamma_i x^2 + \delta_i x^3\right) e^{-\epsilon_i x}, \quad t \geq 0,
\end{equation}

where $x = S_{i,t}/F_i(0, t)$. The idea behind this specification is found in Funahashi and Kijima (2013).

First, we examine Asian options under the above setting. We calibrate our model to the 5 year JPY/USD option market dated on November 11 2011. The calibrated parameters are given as

\begin{align*}
\alpha_1 &= 1.29531, \\
\beta_1 &= -1.99897, \\
\gamma_1 &= -0.00018, \\
\delta_1 &= 1.014743, \\
\epsilon_1 &= 0.809651
\end{align*}

Figure 3 reports the prices of 3-year Asian options computed from these calibrated parameters. In the figure, MC indicates the prices calculated by Monte Carlo simulation and WIC means the approximate prices calculated by the formula given in Theorem 4.2. The other parameters are set as $r_d(t) = 0.00977$, $r_{1,f}(t) = 0.0222$, and $S_{1,0} = 77.54$.

Second, we examine the basket options on two currencies, JPY/USD and JPY/AUD. In addition to the JPY/USD case, we calibrate our model to the 5 year JPY/AUD option market on the same date (November 11, 2011). The calibration results are given as

\begin{align*}
\alpha_2 &= 0.853855, \\
\beta_2 &= -0.699724, \\
\gamma_2 &= -0.001258, \\
\delta_2 &= 0.221596, \\
\epsilon_2 &= 0.627338
\end{align*}
Figure 4: Basket option prices for the calibrated parameters. MC and WIC indicate Monte Carlo simulation and our approximation results, respectively. The other parameters are set as $r_d(t) = 0.00977$, $r_{1,f}(t) = 0.0222$, $r_{2,f}(t) = 0.04848$, $S_{1,0} = 77.54$, $S_{2,0} = 78.60$, $T = 3$ and $w_1 = w_2 = 0.5$. The correlation between them is set as $\rho = 0.5$.

Figure 4 reports the prices of 3-year basket options computed from these parameters. In the figure, MC and WIC indicate Monte Carlo simulation and our approximation results, respectively. The other parameters are set as $r_d(t) = 0.00977$, $r_{1,f}(t) = 0.0222$, $r_{2,f}(t) = 0.04848$, $S_{1,0} = 77.54$, $S_{2,0} = 78.60$, and $w_1 = w_2 = 0.5$. The correlation between them is set as $\rho = 0.5$.

From these figures, we observe that the accuracy of our approximation method is satisfactory for the practical cases in the wide ranges of strikes and maturities.

5.3. Valuation of complex basket options. In this last subsection, we consider the valuation of a fictitious, complex basket option that consists of 6 underlying assets. By complex, we mean that the basket consists of inhomogeneous underlyings with different asset dynamics, different weighting functions, and different correlation structures.

The asset dynamics are modeled by the SDE (4.1) with volatility functions $\sigma_i(x, t), i = 1, 2$, given in (5.2) and

$$\sigma_i(S_i, t) = \begin{cases} 
\sigma_i, & i = 3, 4, \\
\sigma_i/\sqrt{S_i}, & i = 5, 6,
\end{cases}$$

with $\sigma_3 = 0.1$, $\sigma_4 = 0.2$, $\sigma_5 = 1$ and $\sigma_6 = 2$.\(^3\) The initial prices are assumed to be $S_{i,0} = 100$ for all $i$. The spot interest rate $r(t)$ is assumed to be a constant, say $r = 0.02$.

\(^3\)In order to keep the asset volatilities similar levels, we set $\sigma_3 = 0.1$ and $\sigma_5 = 1$, because $S_{4,0} = S_{5,0} = 100$ so that $\sigma_3(S_{4,0}, 0) = \sigma_5(S_{5,0}, 0) = 0.1$. The same reasoning applies for assets 4 and 6.
The weighting functions are assumed to be given by

\[
\begin{cases}
0.1\delta(T - t) & (maturity \ price), \\
0.2\frac{1}{T_2 - T_1}1_{\{T_1 \leq t \leq T_2\}} & (partial \ average), \\
0.2\frac{\eta}{1 - e^{-\eta(T - t)}} & (exponential \ weighting),
\end{cases}
\]

where \( T \) denotes the maturity of the complex basket option and \( 0 < T_1 < T_2 \leq T \). Then, \( \bar{r}_{i,k}(t) \) defined in (4.4) become

\[
\bar{r}_{i,k}(t) = \begin{cases}
0.1r_{i,k}(t)S_{i,0}e^{rT}, \\
0.2r_{i,k}(t)S_{i,0}\left(\frac{e^{rT_2} - e^{rT_1}}{r(T_2 - T_1)}1_{\{t < T_1\}} + \frac{e^{rT_2} - e^{rt}}{r(T_2 - T_1)}1_{\{T_1 \leq t < T_2\}}\right), \\
0.2r_{i,k}(t)S_{i,0}\frac{\eta(e^{rT} - e^{r+\eta(t-\eta T)})}{(1 - e^{-\eta T})(r + \eta)},
\end{cases}
\]

where \( i = 1, 4 \) for \( i = 2, 5 \)

In this numerical example, we assume that \( T = 3, T_1 = 1, T_2 = 2 \) and \( \eta = 0.95 \).

Finally, we assume that the correlations between the assets are given by

\[
\rho_{jk} = \begin{pmatrix}
1 & 0.7 & 0.5 & 0.3 & 0.2 & 0.1 \\
0.7 & 1 & 0.6 & 0.4 & 0.3 & 0.2 \\
0.5 & 0.6 & 1 & 0.5 & 0.4 & 0.3 \\
0.3 & 0.4 & 0.5 & 1 & 0.5 & 0.4 \\
0.2 & 0.3 & 0.4 & 0.5 & 1 & 0.5 \\
0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 1
\end{pmatrix}.
\]

Note that the correlation matrix needs to be positive definite.

The option prices calculated by our formula are compared with those calculated by Monte Carlo simulation. The results are depicted in Figure 5. The CPU time to calculate our results (WIC) was 2.612 seconds for each case in average. Even such a complex basket product can be evaluated reasonably fast.

We can see from Figure 5 that the errors of our approximation are very small even in this complex setting. Hence, we conclude that our approximation formula is flexible enough for the pricing of exotic basket derivatives.

6. Conclusion

In this paper, we propose an approximation method based on the chaos expansion approach proposed by Funahashi and Kijima (2013) for the pricing of complex basket options such as Asian basket options, which are known to be difficult to price both analytically and numerically. Our approximation is not restricted to the case of geometric Brownian motions, and can be applied to the multi-dimensional local volatility model. Also, it is possible to extend our approach to include stochastic volatility models by following the idea of Funahashi (2012). Through ample numerical examples, we show that the accuracy of our approximation remains quite high even for the long maturity and/or the high volatility cases under various diffusion models.

We note that differentiation calculated from the closed-form approximation provides a good approximation for the option delta. Higher-order differentiation (maybe numerical) provides an approximated gamma, and the other Greeks are evaluated similarly.

As future works, we will extend our approach to include stochastic volatility models that are modeled by fractional Brownian motions as studied by Comte and Renault (1998). Applications...
For the valuation problems of other financial contingent claims such as barrier options are also considered.

**APPENDIX A. PROOF OF LEMMA 3.1**

In this appendix, we approximate each $I_{i,m,n}(t)$ by using the approximations (3.7) and (3.8). The strategy is to neglect the terms of iterative integrals of higher than the third order. We call such terms “higher terms”. The proof is similar to the one given by Funahashi and Kijima (2013). However, we provide a concise proof for the reader’s convenience.

**A.1. Approximation of $I_{i:1,1}(t)$**. By definition, $I_{i:1,1}(t) = J_{i,t}(\sigma_1^{(1)}) - J_{i,t}(\sigma_1^{(0)})$ and so, from (3.7), we have

$$I_{i:1,1}(t) \approx \sum_{p=1}^{N} \int_0^t \partial_p \sigma_1^{(0)}(u) \{S_{p,u}^{(1)} - S_{p,u}^{(0)}\} dW_{i,u}$$

$$+ \frac{1}{2} \sum_{p,q=1}^{N} \int_0^t \partial_{pq} \sigma_1^{(0)}(u) \{S_{p,u}^{(1)} - S_{p,u}^{(0)}\} \{S_{q,u}^{(1)} - S_{q,u}^{(0)}\} dW_{i,u}.$$
where we have applied (3.6) for the further approximation.

\[ (A.1) \quad I_{i:1,1}(t) \]

\[ \approx \sum_{p=1}^{N} \int_{0}^{t} \int_{0}^{s} \sigma_p\sigma_i(s) F_p(0, s) \left( \int_{0}^{u} \sigma_p(u) dW_{p,u} \right) dW_{i,s} \]

\[ + \sum_{p=1}^{N} \int_{0}^{t} \int_{0}^{s} \sigma_p\sigma_i(s) F_p(0, s) \left( \int_{u}^{s} \sigma_p(r) dW_{p,r} \right) dW_{i,s} \]

\[ + \frac{1}{2} \sum_{p,q=1}^{N} \int_{0}^{t} \int_{0}^{s} \sigma_p\sigma_i(s) F_p(0, s) F_q(0, s) \left( \int_{0}^{u} \sigma_p(u) dW_{p,u} \right) \left( \int_{0}^{u} \sigma_q(u) dW_{q,u} \right) dW_{i,s}. \]

Further, by Ito's formula, we get

\[ \left( \int_{0}^{t} \sigma_p(s) dW_{p,s} \right) \left( \int_{0}^{t} \sigma_q(s) dW_{q,s} \right) = \int_{0}^{t} \sigma_q(s) \left( \int_{0}^{u} \sigma_p(u) dW_{p,u} \right) dW_{q,s} \]

\[ + \int_{0}^{t} \sigma_p(s) \left( \int_{0}^{u} \sigma_q(u) dW_{q,u} \right) dW_{p,s} \]

\[ + \int_{0}^{t} \sigma_p(s)\sigma_q(s) ds. \]

(A.2)

Finally, substitution of (A.2) into (A.1) yields the result.

A.2. Approximation of \( I_{i:1,2}(t) \). By the definition of Hermite polynomials, we have

\[ I_{i:1,2}(t) = \frac{1}{2} \left\{ \left( J_{i,1}^2(\sigma_i^{(1)}) - J_{i,1}^2(\sigma_i^{(0)}) \right) - \left( \|\sigma_i^{(1)}\|_i^2 - \|\sigma_i^{(0)}\|_i^2 \right) \right\} \]

\[ \approx J_{i,t}(\sigma_i^{(0)}) \left( \sum_{i=1}^{N} \int_{0}^{t} \partial_p\sigma_i^{(0)}(u) \{ S_p^{(1)} - S_p^{(0)} \} dW_{p,u} \right) - \frac{1}{2} \left( \|\sigma_i^{(1)}\|_i^2 - \|\sigma_i^{(0)}\|_i^2 \right), \]

where we have used (3.7) for the approximation. Hence, since \( J_{i,t}(\sigma_i^{(0)}) = \int_{0}^{t} \sigma_i^{(0)}(u) dW_{u} \) and \( \sigma_i^{(0)}(t) = \sigma_i(S_i^{(0)}, t), S_i^{(0)} = (F_1(0, t), \ldots, F_N(0, t)) \), by ignoring the higher terms, we obtain

\[ I_{i:1,2}(t) \approx \left( \int_{0}^{t} \sigma_i^{(0)}(s) dW_{i,s} \right) \left( \sum_{p=1}^{N} \int_{0}^{t} \partial_p\sigma_i^{(0)}(s) F_p(0, s) \left( \int_{0}^{s} \sigma_p(u) dW_{p,u} \right) dW_{i,s} \right) \]

\[ - \frac{1}{2} \left( \|\sigma_i^{(1)}\|_i^2 - \|\sigma_i^{(0)}\|_i^2 \right), \]

(A.3)

where we have applied (3.6) for the further approximation.
Now, by Itô’s formula, the first term in (A.3) is rewritten as
\[
\left( \int_0^t \sigma_i^{(0)}(s) dW_{i,s} \right) \left( \sum_{p=1}^N \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0, s) \left( \int_0^s \sigma_p^{(0)}(u) dW_{p,u} \right) dW_{i,s} \right)
\]
\[
= \sum_{p=1}^N \int_0^t \sigma_i^{(0)}(s) \left( \int_0^s \partial_p \sigma_p^{(0)}(u) F_p(0, s) \left( \int_0^u \sigma_p^{(0)}(r) dW_{p,r} \right) dW_{p,u} \right) dW_{i,s}
+ \sum_{p=1}^N \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0, s) \left( \int_0^s \sigma_i^{(0)}(u) dW_{i,u} \right) \left( \int_0^s \sigma_p^{(0)}(u) dW_{p,u} \right) dW_{i,s}
+ \sum_{p=1}^N \int_0^t \sigma_i^{(0)}(s) \partial_p \sigma_p^{(0)}(s) F_p(0, s) \left( \int_0^s \sigma_p^{(0)}(u) dW_{p,u} \right) ds.
\]

On the other hand, the second term in (A.3) is approximated as
\[
\| \sigma_i^{(1)} \|_t^2 - \| \sigma_i^{(0)} \|_t^2 = \int_0^t \left\{ (\sigma_i^{(1)}(s))^2 - (\sigma_i^{(0)}(s))^2 \right\} ds
\approx 2 \sum_{p=1}^N \int_0^t \sigma_i^{(0)}(u) \partial_p \sigma_i^{(0)}(u) \{ S_p^{(1)} - S_p^{(0)} \} ds,
\]
by Taylor’s expansion around \( S_i^{(0)} \).

Similarly, we get
\[
\sum_{p=1}^N \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0, s) \left( \int_0^s \sigma_p^{(0)}(u) dW_{p,u} \right)^2 dW_{p,s}
= \sum_{p=1}^N \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0, s) \left( \int_0^s \sigma_p^{(0)}(u) \left( \int_0^u \sigma_p^{(0)}(r) dW_{p,r} \right) dW_{p,u} \right) dW_{i,s}
+ \sum_{p=1}^N \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0, s) \left( \int_0^s \sigma_i^{(0)}(u) \left( \int_0^u \sigma_p^{(0)}(r) dW_{p,r} \right) dW_{i,u} \right) dW_{i,s}
+ \sum_{p=1}^N \int_0^t \partial_p \sigma_i^{(0)}(s) F_p(0, s) \left( \int_0^s \sigma_p^{(0)}(u) \sigma_i^{(0)}(u) du \right) dW_{i,s}.
\]

Finally, we put these results together to obtain the approximation result for \( I_{i:1,2}(t) \).

A.3. Approximation of \( I_{i:2,1}(t) \). By definition, \( I_{i:2,1}(t) = J_{i,t}(\sigma_i^{(2)}) - J_{i,t}(\sigma_i^{(1)}) \) and so, from (3.7), we have
\[
I_{i:2,1}(t) \approx \sum_{p=1}^N \int_0^t \partial_p \sigma_i^{(1)}(u) (S_p^{(2)} - S_p^{(1)}) dW_{i,s}.
\]
Since \( S_i^{(2)} - S_i^{(1)} = F_i(0, t) I_{i:1,1}(t) \), and since \( I_{i:1,1}(t) = J_{i,t}(\sigma_i^{(1)}) - J_{i,t}(\sigma_i^{(0)}) \), we have
\[
I_{i:2,1}(t) \approx \sum_{p=1}^N \int_0^t \partial_p \sigma_i^{(1)}(u) F_p(0, t) I_{i:1,1}(t) dW_{i,s}.
\]
AN EXTENSION OF THE CHAOS EXPANSION APPROXIMATION FOR THE PRICING OF EXOTIC BASKET OPTIONS

Hence, from (A.1), by ignoring the higher terms, we obtain

\[ I_{i,2,1}(t) \approx \sum_{p,q=1}^{N} \int_{0}^{t} \partial_{p} \sigma_{i}^{(1)}(u) F_{p}(0, t) \left( \int_{0}^{t} \partial_{q} \sigma_{p}^{(0)}(s) F_{q}(0, s) \left( \int_{0}^{s} \sigma_{q}^{(0)}(u) dW_{q,u} \right) dW_{p,s} \right) W_{i,s}. \]

Now, we apply Taylor’s expansion to \( \partial_{p} \sigma_{i}^{(1)}(t) \) around \( S_{t}^{(0)} \). It follows by ignoring the higher terms again that

\[ I_{i,2,1}(t) \approx \sum_{p,q,r=1}^{N} \int_{0}^{t} \left\{ \partial_{p} \sigma_{i}^{(0)}(u) + \partial_{pp} \sigma_{i}^{(0)}(u) \{ S_{r,u}^{(1)} - S_{r,u}^{(0)} \} \right\} \times F_{p}(0, t) \left( \int_{0}^{t} \partial_{q} \sigma_{p}^{(0)}(s) F_{q}(0, s) \left( \int_{0}^{s} \sigma_{q}^{(0)}(u) dW_{q,u} \right) dW_{p,s} \right) W_{i,s}. \]

By applying (3.6) and ignoring the higher terms again, we finally get the result.

**APPENDIX B. EXPLICIT FORMULAS OF \( a_{i,3}^{k}(T) \)**

From Theorem 3.1, the second term of \( a_{i,3}(T) \) is given by

\[ a_{i,3}^{2}(T) = \sum_{p=1}^{N} \int_{0}^{T} w_{i,p} F_{i}(0, t) \int_{0}^{t} P_{i,p}^{2}(s) \left( \int_{0}^{s} \sigma_{p}^{(0)}(u) \left( \int_{0}^{u} \sigma_{p}^{(0)}(r) dW_{p,r} \right) dW_{p,u} \right) dW_{i,s} dt. \]

By changing the order of integration, we get

\[ a_{i,3}^{2}(T) = \sum_{p=1}^{N} \int_{0}^{T} \bar{P}_{i,p}^{3}(t, T) \left( \int_{0}^{s} \sigma_{p}^{(0)}(u) \left( \int_{0}^{u} \sigma_{p}^{(0)}(r) dW_{p,r} \right) dW_{p,u} \right) dW_{i,s}, \]

where

\[ \bar{P}_{i,p}^{3}(t, T) := P_{i,p}^{2}(t) \int_{t}^{T} w_{i,s} F_{i}(0, s) ds. \]

Similarly, we obtain

\[ a_{i,3}^{3}(T) = \sum_{p=1}^{N} \int_{0}^{T} \bar{s}_{i}(t, T) \left( \int_{0}^{s} P_{i,p}^{2}(u) \left( \int_{0}^{u} \sigma_{p}^{(0)}(r) dW_{p,r} \right) dW_{p,u} \right) dW_{i,s}, \]

\[ a_{i,3}^{4}(T) = \sum_{p=1}^{N} \int_{0}^{T} \bar{P}_{i,p}^{2}(t, T) \left( \int_{0}^{s} \sigma_{i}^{(0)}(u) \left( \int_{0}^{u} \sigma_{i}^{(0)}(r) dW_{i,r} \right) dW_{i,u} \right) dW_{i,s}, \]

\[ a_{i,3}^{5}(T) = \sum_{p=1}^{N} \int_{0}^{T} \bar{P}_{i,p}^{2}(t, T) \left( \int_{0}^{s} \sigma_{p}^{(0)}(u) \left( \int_{0}^{u} \sigma_{i}^{(0)}(r) dW_{i,r} \right) dW_{p,u} \right) dW_{i,s}, \]

\[ a_{i,3}^{6}(T) = \sum_{p,q=1}^{N} \int_{0}^{T} \bar{P}_{i,p,q}(t, T) \left( \int_{0}^{s} \sigma_{p}^{(0)}(u) \left( \int_{0}^{u} \sigma_{q}^{(0)}(r) dW_{q,r} \right) dW_{p,u} \right) dW_{i,s}, \]

and

\[ a_{i,3}^{7}(T) = \sum_{p,q=1}^{N} \int_{0}^{T} \bar{P}_{i,p,q}(t, T) \left( \int_{0}^{s} \sigma_{p}^{(0)}(u) \left( \int_{0}^{u} \sigma_{q}^{(0)}(r) dW_{q,r} \right) dW_{p,u} \right) dW_{i,s}. \]
Here, we define

$\bar{p}^4_{i,p,q}(t, T) := P^3_{i,p,q}(t) \int_t^T w_{i,s} F_i(0, s) ds.$

**APPENDIX C. FORMULAS FOR CONDITIONAL EXPECTATIONS**

Let $W_i^i, i = 1, \ldots, 5$, be standard Brownian motions with correlation $dW_i^i dW_j^i = \eta_{i,j} dt$, and let $y_i(x), i = 1, \ldots, 5$, be some deterministic functions. Moreover, let $\Sigma := \int_0^T y_i^2(t) dt$, and denote $J_T(y_1) = \int_0^T y_1(t) dW_1^1$. In order to derive the conditional expectations of Lemma 3.2, the following well-known results are sufficient:

\begin{equation}
E \left[ \int_0^T y_3(t) \left( \int_0^t y_2(s) dW_s^2 \right) dW_t^3 | J_T(y_1) = x \right] = \frac{x^2}{\Sigma^2} - \frac{1}{\Sigma},
\end{equation}

where

$v_1 = \int_0^T \eta_{1,3} y_3(t) y_1(t) \left( \int_0^t \eta_{1,2} y_2(s) y_1(s) ds \right) dt$,

\begin{equation}
E \left[ \int_0^T y_4(t) \left( \int_0^t y_3(s) \left( \int_0^s y_2(u) dW_u^2 \right) dW_s^3 \right) dW_t^4 | J_T(y_1) = x \right] = \frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2},
\end{equation}

where

$v_2 = \int_0^T \eta_{1,4} y_4(t) y_1(t) \left( \int_0^t \eta_{1,3} y_3(s) y_1(s) \left( \int_0^s \eta_{1,2} y_2(u) y_1(u) du \right) ds \right) dt$,

and

\begin{equation}
E \left[ \left( \int_0^T y_3(t) \left( \int_0^t y_2(s) dW_s^2 \right) dW_t^3 \right) \left( \int_0^T y_5(t) \left( \int_0^t y_4(s) dW_s^2 \right) dW_t^3 \right) | J_T(y_1) = x \right] = \frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} - \frac{3}{\Sigma^2} + v_4 \left( \frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) + v_5,
\end{equation}

Formulas (C.1), (C.2) and (C.3) are one-dimensional versions of Lemma 2.1 in Takahashi (1999). See also Yoshida (1992) for detailed discussions on the conditional expectations.
where

\[ v_3 = \left( \int_0^T \eta_{1,3}y_3(t)y_1(t) \left( \int_0^t \eta_{1,2}y_2(t)y_1(t) \, ds \right) \, dt \right) \times \left( \int_0^T \eta_{1,5}y_5(t)y_1(t) \left( \int_0^t \eta_{1,4}y_4(t)y_1(t) \, ds \right) \, dt \right), \]

\[ v_4 = \int_0^T \eta_{1,3}y_3(t)y_1(t) \left( \int_0^t \eta_{1,5}y_5(s)y_1(s) \left( \int_0^s \eta_{2,4}y_4(u)y_2(u) \, du \right) \, ds \right) \, dt \]

\[ + \int_0^T \eta_{1,5}y_5(t)y_1(t) \left( \int_0^t \eta_{1,3}y_3(s)y_3(s) \left( \int_0^s \eta_{2,4}y_4(u)y_2(u) \, du \right) \, ds \right) \, dt \]

\[ + \int_0^T \eta_{1,3}y_3(t)y_1(t) \left( \int_0^t \eta_{2,5}y_2(s)y_5(s) \left( \int_0^s \eta_{1,4}y_4(u)y_1(u) \, du \right) \, ds \right) \, dt \]

\[ + \int_0^T \eta_{1,5}y_5(t)y_1(t) \left( \int_0^t \eta_{3,4}y_4(s)y_4(s) \left( \int_0^s \eta_{1,2}y_2(u)y_1(u) \, du \right) \, ds \right) \, dt \]

\[ + \left\{ \int_0^T \eta_{3,5}y_5(t)y_3(t) \left( \int_0^t \eta_{1,2}y_2(s)y_1(s) \left( \int_0^t \eta_{1,4}y_4(s)y_1(s) \, ds \right) \, dt \right) \right\}, \]

and

\[ v_5 = \int_0^T \eta_{3,5}y_5(t)y_3(t) \left( \int_0^t \eta_{2,4}y_4(u)y_2(u) \, du \right) \, dt. \]

Namely, for the conditional expectation \( E[a_2(t) \mid a_1(t) = x] \), since \( a_2(t) = \sum_{i=1}^{N} a_{i,2}(T) \) and

\[ a_{i,2}(T) = \int_0^T \bar{s}_i(t, T) \left( \int_0^t \sigma_i^{(0)}(s) \, dW_{i,s} \right) \, dW_{i,t} \]

\[ + \sum_{p=1}^{N} \int_0^T \bar{p}_{ip}(t, T) \left( \int_0^t \sigma_p^{(0)}(s) \, dW_{p,s} \right) \, dW_{i,t}, \]

we can apply (C.1) to calculate \( E \left[ \int_0^T \bar{s}_i(t, T) \left( \int_0^t \sigma_p^{(0)}(s) \, dW_{p,s} \right) \, dW_{i,t} \bigg| a_1(t) = x \right] \). To this end, we set \( y_1(x) = \sqrt{\Lambda}, \ y_2(x) = \sigma_p^{(0)}(x), \ y_3(x) = \bar{s}_i(x, T) \), and note that

\[ \eta_{1,2} = dW_{p,t} \, dW_{\bar{t}} = \sum_{k=1}^{N} \rho_{kp} \left( \bar{p}_{k,1}(t) / \sqrt{\Lambda_t} \right) \, dt, \]

\[ \eta_{1,3} = dW_{i,t} \, dW_{\bar{t}} = \sum_{k=1}^{N} \rho_{ik} \left( \bar{p}_{k,1}(t) / \sqrt{\Lambda_t} \right) \, dt. \]

It follows that

\[ E \left[ \int_0^T \bar{s}_i(t, T) \left( \int_0^t \sigma_p^{(0)}(s) \, dW_{p,s} \right) \, dW_{i,t} \bigg| a_1(t) = x \right] = v_1 \left( \frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right), \]
Other terms are similarly calculated.

The calculation of the conditional expectations $E[a_3(t)|a_1(t) = x]$ and $E[a_2(t)^2|a_1(t) = x]$ are rather complicated and tedious, but straightforward. We omit the details. The detailed results are available from the authors upon request.

REFERENCES
