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<td>Author(s)</td>
<td>Ozawa, Narutaka</td>
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<tr>
<td>Citation</td>
<td>Journal of Mathematical Physics (2013), 54(3)</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2013-03</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/173756">http://hdl.handle.net/2433/173756</a></td>
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<td>Rights</td>
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<td>Type</td>
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Tsirelson’s problem and asymptotically commuting unitary matrices

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(Received 16 November 2012; accepted 28 February 2013; published online 15 March 2013)

In this paper, we consider quantum correlations of bipartite systems having a slight interaction, and reinterpret Tsirelson’s problem (and hence Kirchberg’s and Connes’s conjectures) in terms of finite-dimensional asymptotically commuting positive operator valued measures. We also consider the systems of asymptotically commuting unitary matrices and formulate the Stronger Kirchberg Conjecture. © 2013 American Institute of Physics. [http://dx.doi.org/10.1063/1.4795391]

I. INTRODUCTION

A POVM (positive operator valued measure) with $m$ outputs is an $m$-tuple $(A_i)_{i=1}^m$ of positive semi-definite operators on a Hilbert space $\mathcal{H}$ such that $\sum A_i = 1$. We write the convex sets of quantum correlation matrices of two independent systems of $d$ POVMs with $m$ outputs by

$$Q_c = \left\{ (\xi, B_{1}^{k}j_{j})_{i,j}^{k}, \xi \in \mathcal{H}, \text{a unit vector} \right\}$$

$$Q_s = \left\{ (\xi, B_{1}^{k}j_{j})_{i,j}^{k}, \xi \in \mathcal{H}, \text{a unit vector} \right\}$$

Here $i, j, k, l$ are indices and $A_{i}^{k}$ does not mean the $k$th power of $A_{i}$. The sets $Q_c$ and $Q_s$ are closed convex subsets of $\mathbb{M}_{md}(\mathbb{R}_{\geq 0})$ such that $Q_s \subset Q_c$. Whether they coincide (for some/all $m, d \geq 2, (m, d) \neq (2, 2)$) is the well-known Tsirelson problem, and the matricial version of it is known to be equivalent to Kirchberg’s and Connes’s conjectures. We refer the reader to Refs. 4, 5, 8, and 11 for the literature and the proof of the equivalence. The matricial version of Tsirelson’s problem asks whether $Q^n_c = Q^n_s$ for all $n$, where $Q^n_c$ and $Q^n_s$ are defined as follows:

$$Q^n_c = \left\{ (\xi, B_{1}^{k}j_{j})_{i,j}^{k}, \xi \in \mathcal{H}, \text{a unit vector} \right\}$$

$$Q^n_s = \left\{ (\xi, B_{1}^{k}j_{j})_{i,j}^{k}, \xi \in \mathcal{H}, \text{a unit vector} \right\}$$

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and

\[ Q^\mu_\varepsilon = \text{closure} \left\{ V^* A_i^k B_j^l V \right\}_{k,l}^{i,j} : \begin{array}{l}
\dim \mathcal{H} < +\infty, V : \ell_2^n \to \mathcal{H} \text{ an isometry} \\
(A_i^k)_{i=1}^m, k = 1, \ldots, d, \text{POVMs on } \mathcal{H}, \\
(B_j^l)_{j=1}^m, l = 1, \ldots, d, \text{POVMs on } \mathcal{H}, \\
[A_i^k, B_j^l] = 0 \text{ for all } i, j, k, \ell \end{array} \].

In this paper, we consider “slightly interacting” systems. Suppose Alice and Bob conduct measurements by systems of operators \((A_i^{1/2})_{i=1}^m\) and \((B_j^{1/2})_{j=1}^m\), respectively. If Bob conducts a measurement immediately after Alice’s measurement of a state \(\xi\), then the probability of the output \((i, j)\) is \(\|B_j^{1/2} A_i^{1/2} \xi\|^2\)—and vice versa. Therefore, when they conduct measurements of a state \(\xi\) at the same time, the probability of the output \((i, j)\) is given by \((\xi, (A_i \cdot B_j) \xi)\), where \(A \cdot B = (A^{1/2} B A^{1/2} + B^{1/2} A B^{1/2})/2\). Thus, for \(\varepsilon > 0\), we define the quantum correlation matrices of slightly interacting systems to be

\[ Q^\mu_\varepsilon = \text{closure} \left\{ V^* (A_i^k \cdot B_j^l) V \right\}_{k,l}^{i,j} : \begin{array}{l}
\dim \mathcal{H} < +\infty, V : \ell_2^n \to \mathcal{H} \text{ an isometry} \\
(A_i^k)_{i=1}^m, k = 1, \ldots, d, \text{POVMs on } \mathcal{H}, \\
(B_j^l)_{j=1}^m, l = 1, \ldots, d, \text{POVMs on } \mathcal{H}, \\
\| [A_i^k, B_j^l] \| \leq \varepsilon \text{ for all } i, j, k, \ell \end{array} \],

where \(\|[A, B]\|\) denotes the operator norm of the commutator \([A, B] = AB - BA\). We note that \(Q^\mu_\varepsilon\) is a closed convex subset of \(\mathcal{M}_{md}(\mathbb{M}_n(\mathbb{C}))_+\). Recall that a POVM \((A_i)_{i=1}^m\) is said to be projective if all \(A_i’s\) are orthogonal projections. We also introduce the projective analogue of \(Q^\mu_\varepsilon\):

\[ P^\mu_\varepsilon = \text{closure} \left\{ V^* (P_i^k \cdot Q_j^l) V \right\}_{k,l}^{i,j} : \begin{array}{l}
\dim \mathcal{H} < +\infty, V : \ell_2^n \to \mathcal{H} \text{ an isometry} \\
(P_i^k)_{i=1}^m \text{ projective POVMs on } \mathcal{H}, \\
(Q_j^l)_{j=1}^m \text{ projective POVMs on } \mathcal{H}, \\
\| [P_i^k, Q_j^l] \| \leq \varepsilon \text{ for all } i, j, k, \ell \end{array} \].

We simply write \(P_\varepsilon\) for \(P^1_\varepsilon\). The following is the main result of this paper. It probably suggests that \(Q^\mu_\varepsilon\) is more natural than \(Q_\varepsilon\) (cf. Introduction of Ref. 4).

**Theorem.** For every \(m, d,\) and \(n\), one has \(Q^\mu_\varepsilon = \bigcap_{\varepsilon > 0} Q^\mu_\varepsilon = \bigcap_{\varepsilon > 0} P^\mu_\varepsilon\). In particular, an affirmative answer to Tsirelson’s problem is equivalent to that \(\bigcap_{\varepsilon > 0} P_\varepsilon \subseteq Q_\varepsilon\).

Hence, the matricial version of Tsirelson’s problem would have an affirmative answer if the following assertion holds for some/all \((m, d)\).

**Strong Kirchberg Conjecture (I).** Let \(m, d \geq 2\) be such that \((m, d) \neq (2, 2)\). For every \(\kappa > 0\), there is \(\varepsilon > 0\) with the following property. If \(\dim \mathcal{H} < +\infty\), and \((P_i^k)_{i=1}^m\) and \((Q_j^l)_{j=1}^m\) is a pair of \(d\) projective POVMs on \(\mathcal{H}\) such that \(\|[P_i^k, Q_j^l]\| \leq \varepsilon\), then there are a finite-dimensional Hilbert space \(\tilde{\mathcal{H}}\) containing \(\mathcal{H}\) and projective POVMs \((\tilde{P}_i^k)_{i=1}^m\) and \((\tilde{Q}_j^l)_{j=1}^m\) on \(\tilde{\mathcal{H}}\) such that \(\|[\tilde{P}_i^k, \tilde{Q}_j^l]\| = 0\) and \(\|\Phi_{\mathcal{H}}(\tilde{P}_i^k) - P_i^k\| \leq \kappa\) and \(\|\Phi_{\mathcal{H}}(\tilde{Q}_j^l) - Q_j^l\| \leq \kappa\), where \(\Phi_{\mathcal{H}}\) is the compression to \(\mathcal{H}\).

We will deal in Sec. IV with a parallel and equivalent conjecture in the unitary setting.

### II. PRELIMINARY FROM C*-ALGEBRA THEORY

As it is observed in Refs. 4, 5, and 11, the study of quantum correlation matrices is essentially about the algebraic tensor product \(\mathfrak{S}_m^d \otimes \mathfrak{S}_m^d\) of the C*-algebra

\[ \mathfrak{S}_m^d = \ell_2^m \otimes \cdots \otimes \ell_2^m. \]
the unital full free product of $d$-copies of $\ell^m_{\infty}$. We note that $\mathfrak{S}^d_m$ is $\ast$-isomorphic to the full group 
$C^*$-algebra $C^*(\Gamma_{m,d})$ of the group $\Gamma_{m,d} = (\mathbb{Z}/m\mathbb{Z})^d$. The condition $m, d \geq 2$ and $(m, d) \neq (2, 2)$ is equivalent to that $\Gamma_{m,d}$ contains the free groups $\mathcal{F}_r$. We denote by $(e_i^m)_{i=1}^m$ the standard basis of minimal projections in $\ell^m_{\infty}$, and by $(e_i^m)_{i=1}^m$ the $k$th copy of it in the free product $\mathfrak{S}^d_m$. We also write $e_i^d$ for the elements $e_i^m \otimes 1$ in $\mathfrak{S}^d_m \otimes \mathfrak{S}^d_m$ and $f_j^d$ for $1 \otimes e_j^d$. Thus, the maximal tensor product $\mathfrak{S}^d_m \otimes_{\text{max}} \mathfrak{S}^d_m$ is the universal $C^*$-algebra generated by projective POVMs $(e_i^m)_{i=1}^m$ and $(f_j^d)_{j=1}^d$, under the commutation relations $[e_i^k, f_j^d] = 0$. In passing, we note that $C^*(\Gamma_{m,d})$ is canonically $\ast$-isomorphic to $C^*(\Gamma \times \Gamma)$ for any group $\Gamma$. By Stinespring's dilation theorem (Theorem 1.5.3 in Ref. 3), one has

$$Q_c = \left\{ \left[ (\varphi(e_i^k f_j^d))_{i,j} \right] \in M_n(\mathbb{C}) : \varphi : \mathfrak{S}^d_m \otimes_{\text{max}} \mathfrak{S}^d_m \rightarrow M_n(\mathbb{C}) \text{ u.c.p.} \right\} \subset M_{md}(M_n(\mathbb{C})_{u.c.p.})$$

See Refs. 4 and 5 for the proof. Here u.c.p. stands for “unital completely positive.”

We recall the notion of quasi-diagonality. We say a subset $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ is quasi-diagonal if there is an increasing net $(P_r)$ of finite-rank orthogonal projections on $\mathcal{H}$ such that $P_r \mathcal{C}^1 \mathcal{H}$ in the strong operator topology and $\|P_r \mathcal{C} \mathcal{H}\| \rightarrow 0$ for every $C \in \mathcal{C}$. A $C^*$-algebra $\mathcal{C}$ is said to be quasi-diagonal if there is a faithful $\ast$-representation $\pi$ of $\mathcal{C}$ on a Hilbert space $\mathcal{H}$ such that $\pi(\mathcal{C})$ is a quasi-diagonal subset. A $\ast$-representation $\pi : \mathcal{C} \rightarrow \mathcal{B}(\mathcal{H})$ is said to be essential if $\pi(\mathcal{C})$ does not contain non-zero compact operators. The following theorem of Voiculescu is the most fundamental result on quasi-diagonal $C^*$-algebras. See Sec. 7 of Ref. 3 (Theorems 7.2.5 and 7.3.6) for the details.

**Theorem 1 (Voiculescu Ref. 13).** The following statements hold.

- Let $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a faithful essential $\ast$-representation of a quasi-diagonal $C^*$-algebra $\mathcal{E}$. Then, $\mathcal{E}$ is a quasi-diagonal subset of $\mathcal{B}(\mathcal{H})$.
- Quasi-diagonality is a homotopy invariant.

The following is based on Brown’s idea (Ref. 2 and Proposition 7.4.5 in Ref. 3).

**Theorem 2.** The $C^*$-algebras $\mathfrak{S}^d_m \otimes_{\text{max}} \mathfrak{S}^d_m$ and $C^*\mathfrak{F}_d \otimes_{\text{max}} C^*\mathfrak{F}_d$ are quasi-diagonal.

**Proof.** We consider $\mathfrak{S}^d_m$ as a $C^*$-subalgebra of $\mathfrak{M} = M_{m_n}(\mathbb{C}) \ast \cdots \ast M_{m_1}(\mathbb{C})$. Since the conditional expectation $\Phi$ from $M_{m_n}(\mathbb{C})$ onto $\ell^\infty_{\infty}$ extends to a u.c.p. map $\Phi$ from $\mathfrak{M}$ to $\mathfrak{S}^d_m$, which restricts to $\Phi$ on each free product component, the canonical embedding $\mathfrak{S}^d_m \hookrightarrow \mathfrak{M}$ is indeed faithful and $\Phi$ is a conditional expectation from $\mathfrak{M}$ onto $\mathfrak{S}^d_m$. It follows that $\mathfrak{S}^d_m \otimes_{\text{max}} \mathfrak{S}^d_m \subset M_{m_n}(\mathbb{C})_{\text{max}}$. We will prove that the latter is quasi-diagonal.

Let $\theta : \mathfrak{M} \otimes_{\text{max}} \mathfrak{M} \rightarrow \mathcal{B}(\mathcal{H})$ be a faithful $\ast$-representation on a separable Hilbert space $\mathcal{H}$. We omit writing $\theta$ for a while and denote by $\mathfrak{M}^\theta$ the von Neumann algebra generated by $\theta(\mathfrak{M} \otimes 1)$. We write $\{e_{i,j}^d\}_{i,j=1}^{m_1 \cdots m_n}$ for the matrix units in $M_{m_1}(\mathbb{C})$ and $\{e_{i,j}^d\}_{i,j=1}^{m_1 \cdots m_n}$ for the $k$th copy of it in $\mathfrak{M}$. We note that the matrix units $\{e_{i,j}^d\}$ are unitarily equivalent to the first copy $\{e_{i,j}^d\}$ inside $\mathfrak{M}$. This is a well-known fact, but we include the proof for the reader’s convenience. Let $z \in \mathfrak{M}^\theta$ be the central projection such that $z \mathfrak{M}^\theta$ is finite and $(1 - z) \mathfrak{M}^\theta$ is properly infinite (Theorem V.1.19 in Ref. 10). Then, the projections $ze_{1,1}^d$ and $ze_{1,1}^d$ are equivalent since they have the same center valued trace $zn$ (Corollary V.2.8 in Ref. 10). The projections $(1 - z)e_{1,1}^d$ and $(1 - z)e_{1,1}^d$ are also equivalent, since they are properly infinite and have full central support $1 - z$ (Theorem V.1.39 in Ref. 10). Therefore, for each $k$, there is a partial isometry $w_k \in \mathfrak{M}^\theta$ such that $w_k^* w_k = e_{1,1}$ and $w_k w_k^* = e_{1,1}$. Now, $U_k = \sum_{i,j} e_{i,j}^d w_k^* e_{i,j}^d$ is a unitary element in $\mathfrak{M}^\theta$ such that $U_k e_{i,j}^d U_k^* = e_{i,j}^d$ for all $i, j$. Since $\mathfrak{M}^\theta$ is a von Neumann algebra, there is a norm-continuous path $U_k(t)$ of unitary elements connecting $U_k(0) = 1$ to $U_k(1) = U_k$. It follows that the $\ast$-homomorphisms $\pi_k : \mathfrak{M} \rightarrow \mathfrak{M}^\theta$, $e_{i,j}^d \mapsto U_k(t) e_{i,j}^d U_k^*$, give rise to a homotopy from $\pi_0 : \mathfrak{M} \otimes \mathfrak{M} \rightarrow \mathfrak{M}^\theta$ to $\pi_1 : \mathfrak{M} \otimes \mathfrak{M} \rightarrow M_{m_n}(\mathbb{C}) \subset \mathfrak{M}^\theta$. Likewise, there is a homotopy $\rho_\mathfrak{M} : \mathfrak{M} \rightarrow \theta(\mathfrak{C} \otimes \mathfrak{M})^\theta$ between the embedding $\rho_\mathfrak{M}$ of $\mathfrak{M}$ as the second tensor component and $\rho_\mathfrak{M}$ which ranges in $M_{m_n}(\mathbb{C})$. Thus, $\pi_1 \times \rho_\mathfrak{M} : \mathfrak{M} \otimes_{\text{max}} \mathfrak{M} \rightarrow \mathcal{B}(\mathcal{H})$ is a homotopy between the embedding $\theta$ and $\pi_1 \times \rho_\mathfrak{M}$. Therefore, $\mathfrak{M} \otimes_{\text{max}} \mathfrak{M}$ is embeddable into a $C^*$-algebra which is homotopic to $M_{m_n}(\mathbb{C}) \otimes M_{m_n}(\mathbb{C})$. Now quasi-diagonality of $\mathfrak{M} \otimes_{\text{max}} \mathfrak{M}$ follows from Theorem 1. The case for $C^*\mathfrak{F}_d$ is similar (Proposition 7.4.5 in Ref. 3).
III. PROOF OF THEOREM

We start the proof of the inclusion $\bigcap_{k>0} Q^n_k \subseteq Q^n$. Take $m, d, n$, and $[X_{i,j}^k] \in \bigcap_{k>0} Q^n_k$ arbitrary. Then, for every $r \in \mathbb{N}$, there are a pair of $d$ POVMs $(A_i^k(r))_{i=1}^m$ and $(B_j^k(r))_{j=1}^m$ on $\mathcal{H}_r$ and a u.c.p. map $\varphi_r : \mathcal{B}(\mathcal{H}_r) \to \mathcal{M}_n(\mathbb{C})$ such that $\|A_i^k(r) B_j^k(r)\| \leq r^{-1}$ and $\|\varphi_r(A_i^k(r) \otimes B_j^k(r) - X_{i,j}^k)\| \leq r^{-1}$. We consider the C*-algebras

$$\mathcal{M} = \prod_{r=1}^{\infty} \mathcal{B}(\mathcal{H}_r) = \{(C(r))_{r=1}^{\infty} : C(r) \in \mathcal{B}(\mathcal{H}_r), \sup_r \|C(r)\| < +\infty\},$$

and $\mathcal{R} = \mathcal{M}/\mathbb{R}$ with the quotient map $\pi : \mathcal{M} \to \mathcal{R}$. Then $A_i^k = \pi((A_i^k(r))_{r=1}^{\infty})$ and $B_j^k = \pi((B_j^k(r))_{r=1}^{\infty})$ are commuting POVMs in $\mathcal{R}$. Fix an ultra-limit Lim and consider the u.c.p. map $\varphi : \mathcal{M} \to \mathcal{M}_n(\mathbb{C})$ defined by $\varphi((C(r))_{r=1}^{\infty}) = \text{Lim} \varphi_r(C(r)) \in \mathcal{M}_n(\mathbb{C})$. It factors through $\mathcal{R}$ and one obtains a u.c.p. map $\varphi : \mathcal{R} \to \mathcal{M}_n(\mathbb{C})$ such that $\varphi = \varphi \circ \pi$. It follows that $\varphi(A_i^k B_j^k) = \varphi(A_i^k) \otimes \varphi(B_j^k) = X_{i,j}^k$, and hence $[X_{i,j}^k] \subseteq Q^n$.

For the inclusion $Q^n_c \subseteq \bigcap_{k \geq 0} P^n_k$, take $m, d, n$, and $[X_{i,j}^k] \in Q^n_c$ arbitrary. Then, there is a u.c.p. map $\varphi : \mathcal{R}^d \otimes \text{max} \mathcal{R}^d \to \mathcal{M}_n(\mathbb{C})$ such that $\varphi(e_i^k f_j^k) = X_{i,j}^k$. By Stinespring’s dilation theorem, there are a *-representation of $\mathcal{R}^d \otimes \text{max} \mathcal{R}^d$ on a separable Hilbert space $\mathcal{H}$ and an isometry $\mathcal{V} : l_2^d \to \mathcal{H}$ such that $\varphi(C) = \mathcal{V}^* CV$ for $C \in \mathcal{R}^d \otimes \text{max} \mathcal{R}^d$. By inflicting the *-representation, we may assume it is faithful and essential. Since $\mathcal{R}^d \otimes \text{max} \mathcal{R}^d$ is quasi-diagonal (Theorem 2), there is an increasing sequence $(P_r)_{r=1}^{\infty}$ of finite-rank orthogonal projections on $\mathcal{H}$ such that $P_r \to 1$ in the strong operator topology and $\|\mathcal{V}_r(C P_r)\| \to 0$ for $C \in \mathcal{R}^d \otimes \text{max} \mathcal{R}^d$. Thus, $P_r e_i^k P_r$ and $P_r f_j^k P_r$ are close to projections (as $r \to \infty$) and one can find projective POVMs $(E_i^k(r))_{r=1}^{m}$ and $(F_j^k(r))_{j=1}^{m}$ on $P_r \mathcal{H}$ such that $\|P_r e_i^k P_r - E_i^k(r)\| \to 0$ and $\|P_r f_j^k P_r - F_j^k(r)\| \to 0$. We note that $\|P_r V - V\| \to 0$. It follows that $\|E_i^k(r), F_j^k(r)\| \to 0$ and

$$\lim_{r \to \infty} \mathcal{V}^* (E_i^k(r) \otimes F_j^k(r)) V = \lim_{r \to \infty} \mathcal{V}^* (E_i^k(r) F_j^k(r)) V = \mathcal{V}^* e_i^k f_j^k V = X_{i,j}^k,$$

This implies $[X_{i,j}^k] \subseteq \bigcap_{k \geq 0} P^n_k$. \hfill \square

IV. ASYMPTOTICALLY COMMUTING UNITARY MATRICES

Kirchberg’s conjecture\(^6\) asserts that $C^* \mathcal{F}_d \otimes \text{min} C^* \mathcal{F}_d = C^* \mathcal{F}_d \otimes \text{max} C^* \mathcal{F}_d$ for some/all $d \geq 2$. By Choi’s theorem (Theorem 7.4.1 in Ref. 3), $C^* \mathcal{F}_d$ is residually finite dimensional (RFD) and so is $C^* \mathcal{F}_d \otimes \text{min} C^* \mathcal{F}_d$. Since finite-dimensional representations factor through the minimal tensor product, Kirchberg’s conjecture is equivalent to the assertion that $C^* \mathcal{F}_d \otimes \text{max} C^* \mathcal{F}_d$ is RFD. For the following, let $u_1, \ldots, u_d$ be the standard unitary generators of $C^* \mathcal{F}_d$. We also write $u_i$ for the elements $u_i \otimes 1$ in $C^* \mathcal{F}_d \otimes C^* \mathcal{F}_d$ and $v_j$ for $1 \otimes u_j$. We denote by $\mathcal{U}(\mathcal{H})$ the set of unitary operators on $\mathcal{H}$. For $a \in \mathcal{M}_d(\mathcal{M}_n(\mathbb{C}))$, we consider

$$\|a\|_\text{min} = \|\sum_{i,j} a_{i,j} \otimes u_i v_j \mathcal{M}_n(\mathbb{C}) \otimes C^* \mathcal{F}_d \| = \sup\{\|\sum_{i,j} a_{i,j} \otimes U_i V_j\| : k \in \mathbb{N}, U_i, V_j \in \mathcal{U}(l_2^k) \text{ s.t. } \{U_i, V_j\} = 0\}$$

and

$$\|a\|_\text{max} = \|\sum_{i,j} a_{i,j} \otimes u_i v_j \mathcal{M}_n(\mathbb{C}) \otimes C^* \mathcal{F}_d \| = \sup\{\|\sum_{i,j} a_{i,j} \otimes U_i V_j\| : U_i, V_j \in \mathcal{U}(l_2) \text{ s.t. } \{U_i, V_j\} = 0\}.$$
In the above expressions, one may assume $U_1 = 1$ and $V_1 = 1$ by replacing $U_i$ and $V_j$ with $U_i^*U_1$ and $V_jV_j^*$. It follows that $\|\alpha\|_{\text{min}} = \|\alpha\|_{\text{max}}$ for $d = 2$. By Pisier’s linearization trick, Kirchberg’s conjecture is equivalent to the assertion that $\|\alpha\|_{\text{min}} = \|\alpha\|_{\text{max}}$ holds for every $d \geq 3$ (or just $d = 3$) and every $\alpha \in M_d(M_n(\mathbb{C}))$. See Sec. 12 of Ref. 9, Chap. 13 in Ref. 3, and Ref. 7 for the proof of this fact and more information. The proof of the following lemma is omitted because it is almost the same as that of the main theorem.

**Lemma 3.** For every $\alpha \in M_d(M_n(\mathbb{C}))$, one has

$$\|\alpha\|_{\text{max}} = \inf_{\varepsilon > 0} \sup_{i,j} \|\sum_{\ell} \alpha_{ij,\ell} \otimes V_j\| : k \in \mathbb{N}, U_i, V_j \in U(\ell_2^k) \text{ s.t. } \|[U_i, V_j]\| \leq \varepsilon.$$

We observe the following fact. Suppose dim $\mathcal{H} < \infty$ and $U, V \in U(\mathcal{H})$ are such that $\|U, V\| < \varepsilon$. It is well-known that the pair $(U, V)$ need not be close to a commuting pair of unitary matrices, but after a dilation it is. Indeed, this follows from amenability of $\mathbb{Z}^2$. Let $m = \lfloor 1/\sqrt{\varepsilon} \rfloor$ and $F = \{0, \ldots, m\}^2 \subset \mathbb{Z}^2$. We define an isometry $W : \mathcal{H} \to \ell_2 \mathbb{Z}^2 \otimes \mathcal{H}$ by $W \xi = |F|^{-1/2} \sum_{y \in F} \delta_y \otimes \varphi(x)\xi$, where $\varphi((p, q)) = U^p V^q \in U(H)$ for $(p, q) \in F$. Then, for the commuting unitary operators $u$ and $v$, acting on $\ell_2 \mathbb{Z}^2 \otimes \mathcal{H}$ by $(\mathbb{Z}^\infty \oplus 1, 0)$ and $(0, -1)$, respectively, one has

$$\|W^*uW - U\| = \frac{1}{|F|} \sum_{x \in F \cap (\ell_2 \mathbb{Z}^2)} \varphi(x)^\ast \varphi(x + (1, 0)) - \|U\| \leq \frac{m \varepsilon}{1 + (m - 1) < 2\sqrt{\varepsilon}}.$$

Similarly, one has $\|W^*vW - V\| < 2\sqrt{\varepsilon}$. Since $C^\ast \mathbb{H} \mathbb{Z}^2$ is Abelian (and RFD), one can find a finite dimensional Hilbert space $\tilde{\mathcal{H}}$ containing $\mathcal{H}$ and commuting unitary matrices $\tilde{U}$ and $\tilde{V}$ on $\tilde{\mathcal{H}}$ such that $\|\Phi_{\tilde{\mathcal{H}}} (\tilde{U}) - U\| < 2\sqrt{\varepsilon}$ and $\|\Phi_{\tilde{\mathcal{H}}} (\tilde{V}) - V\| < 2\sqrt{\varepsilon}$, where $\Phi_{\tilde{\mathcal{H}}} : B(\tilde{\mathcal{H}}) \to B(\mathcal{H})$ is the compression. We note that $\Phi_{\tilde{\mathcal{H}}} (\tilde{U}) \approx U$ and $\Phi_{\tilde{\mathcal{H}}} (\tilde{V}) \approx V$ for any unitary elements imply $\Phi_{\tilde{\mathcal{H}}} (U \tilde{V}) \approx UV$ (see, e.g., Theorem 18 in Ref. 8). Keeping these facts in mind, we formulate the Strong Kirchberg Conjecture (II).

**Strong Kirchberg Conjecture (II).** Let $d \geq 2$. For every $\kappa > 0$, there is $\varepsilon > 0$ with the following property. If dim $\mathcal{H} < +\infty$ and $U_1, \ldots, U_d, V_1, \ldots, V_d \in U(\mathcal{H})$ are such that $\|[U_i, V_j]\| \leq \varepsilon$, then there are a finite-dimensional Hilbert space $\tilde{\mathcal{H}}$ containing $\mathcal{H}$ and commuting unitary matrices $\tilde{U}_i, \tilde{V}_j \in U(\tilde{\mathcal{H}})$ such that $\|[\tilde{U}_i, \tilde{V}_j]\| = 0$ and $\|\Phi_{\tilde{\mathcal{H}}} (\tilde{U}_i) - U_i\| \leq \kappa$ and $\|\Phi_{\tilde{\mathcal{H}}} (\tilde{V}_j) - V_j\| \leq \kappa$.

We note that the analogous statement for $U_1, U_2, V$ is true, by the proof of the following theorem plus the fact that $C^\ast (\ell_2 \times \mathbb{Z})$ is RFD and has the LLP (local lifting property). See Chap. 13 in Ref. 3 for the definition of the LLP and relevant results.

**Theorem 4.** The following conjectures are equivalent.

1. The Strong Kirchberg Conjecture (I) holds for some/all $(m, d)$.
2. The Strong Kirchberg Conjecture (II) holds for some/all $d$.
3. Kirchberg’s conjecture holds and $C^\ast (\mathbb{F}_d \times \mathbb{F}_d)$ has the LLP for some/all $d \geq 2$.
4. The algebraic tensor product $C^\ast \mathbb{F}_d \otimes C^\ast \mathbb{F}_d \otimes B(\ell_2)$ has unique $C^\ast$-norm.

We note that it is not known whether $C^\ast (\mathbb{F}_d \times \mathbb{F}_d)$ has the LLP, but it is independent of $d \geq 2$ and equivalent to that the LLP is closed under the maximal tensor product. Also it is equivalent to the LLP for $C^\ast (\Gamma^\ast_m \times \Gamma^\ast_m, d)$. This problem seems to be independent of Kirchberg’s conjecture. We will only prove the equivalence (2)$\iff$(3), because the proof of (1)$\iff$(3) is very similar and (3)$\iff$(4) is an immediate consequence of the tensor product characterization of the LLP (see Ref. 6 and Chap. 13 in Ref. 3).

**Lemma 5.** The following conjectures are equivalent:

1. For every $\kappa > 0$, there is $\varepsilon > 0$ with the following property. If dim $\mathcal{H} < +\infty$ and $U_1, \ldots, U_d, V_1, \ldots, V_d \in U(\mathcal{H})$ are such that $\|[U_i, V_j]\| \leq \varepsilon$, then there are $a$ (not necessarily
finite-dimensional) Hilbert space \( \tilde{H} \) containing \( H \) and \( \tilde{U}_i, \tilde{V}_j \in \mathcal{U}(\tilde{H}) \) such that \( \| \tilde{U}_i, \tilde{V}_j \| = 0 \) and \( \| \Phi_H(\tilde{U}_i) - U_i \| \leq \kappa \) and \( \| \Phi_H(\tilde{V}_j) - V_j \| \leq \kappa \).

(2) The \( C^* \)-algebra \( C^*(F_d \times F_d) \) has the LLP.

**Proof.** (1)\( \Rightarrow \) (2): To prove the LLP of \( C^* \)-algebra \( C^*(F_d \times F_d) \), it suffices to show that the surjective \( * \)-homomorphism \( \pi \) from \( C^*(F_{2d}) = C^*(w_1, \ldots, w_d, w'_1, \ldots, w'_d) \) onto \( C^*(F_d \times F_d) \), \( w_i \mapsto u_i \) and \( w'_i \mapsto v_j \), is locally liftable. By the Effros–Haagerup theorem (Theorem C.4 in Ref. 3), this follows once it is shown that the canonical surjection

\[
\Theta : \mathbb{B}(\ell_2) \otimes_{\text{min}} C^*(F_{2d}) \to \mathbb{B}(\ell_2) \otimes_{\text{min}} C^*(F_d \times F_d)
\]

is isometric. Let \( u_0 = 1 = v_0 \) and \( E = \text{span}\{u_i, v_j : 0 \leq i, j \leq d\} \) be the operator subspace of \( C^*(F_d \times F_d) \). By Pisier’s linearization trick, it is enough to check that \( \Theta \) is (completely) isometric on \( \mathbb{B}(\ell_2) \otimes E \). For this, take \( \lambda \in \mathbb{M}_{d+1}(\mathbb{B}(\ell_2)) \) arbitrary and let

\[
\lambda = \| \sum \alpha_{i,j} \otimes u_i v_j \|_{\mathbb{B}(\ell_2) \otimes_{\text{max}} C^*(F_d \times F_d) / \text{ker} \pi}.
\]

Let \( (e_n)_{n=1}^\infty \) be a quasi-central approximate unit for \( \ker \pi \) in \( C^*(F_{2d}) \), and let \( w_i(n) = (1 - e_n)^{1/2} w_i (1 - e_n)^{1/2} + e_n \) and \( w'_i(n) \) likewise (although the proof will equally work for \( w'_i(n) = w'_i \)). Then, one has

\[
\lim_n \| [w_i(n), w'_i(n)] \| = \lim_n \| (1 - e_n)^2 [w_i, w'_i] \| = \| \pi([w_i, w'_i]) \| = 0
\]

and \( \lim_n \| w_i(n), w'_i(n) \| = 0 \). Since \( C^*(F_{2d}) \) is RFD, one can find a finite-dimensional \( * \)-representation \( \sigma_n \) such that

\[
\| \sum \alpha_{i,j} \otimes \sigma_n(w_i(n)w'_i(n)) \|_{\mathbb{B}(\ell_2) \otimes_{\text{max}} C^*(F_{2d})} \geq \lambda - \frac{1}{n}.
\]

For every contractive matrices \( x \) and \( y \), we consider the unitary matrices defined by

\[
U_x = \begin{bmatrix}
\sqrt{1 - x^*x} & x \\
-\sqrt{1 - x^*x} & -x^*
\end{bmatrix}
\quad \text{and} \quad
V_y = \begin{bmatrix}
\sqrt{1 - y^*y} & y \\
-\sqrt{1 - y^*y} & -y^*
\end{bmatrix}.
\]

We observe that the \( (1, 1) \)-entry of \( U, V \) is \( xy \), and if \( \| [x, y] \| \approx 0 \) and \( \| [x^*, y] \| \approx 0 \), then \( \| [U_x, V_y] \| \approx 0 \). Thus, applying the assumption (1) to \( U_{\sigma_n(w_i(n))} \) and \( V_{\sigma_n(w'_i(n))} \), one may find unitary operators \( \tilde{U}_i(n), \tilde{V}_j(n) \) and the compression \( \Phi_n \) such that \( [\tilde{U}_i(n), \tilde{V}_j(n)] = 0 \), \( \| \Phi_n(\tilde{U}_i(n)) - U_{\sigma_n(w_i(n))} \| \to 0 \), and \( \| \Phi_n(\tilde{V}_j(n)) - V_{\sigma_n(w'_i(n))} \| \to 0 \). It follows that

\[
\| \sum \alpha_{i,j} \otimes \tilde{U}_i(n) \tilde{V}_j(n) \| \geq \limsup_{n \to \infty} \| \sum \alpha_{i,j} \otimes \tilde{U}_i(n) \tilde{V}_j(n) \|
\geq \limsup_{n \to \infty} \| \sum \alpha_{i,j} \otimes \Phi_n(\tilde{U}_i(n) \tilde{V}_j(n)) \|
\geq \limsup_{n \to \infty} \| \sum \alpha_{i,j} \otimes U_{\sigma_n(w_i(n))} V_{\sigma_n(w'_i(n))} \|
\geq \limsup_{n \to \infty} \| \sum \alpha_{i,j} \otimes \sigma_n(w_i(n)w'_i(n)) \|
\geq \lambda.
\]

This proves that \( \Theta \) is isometric on \( \mathbb{B}(\ell_2) \otimes E \), and the assertion (2) follows.
(2)⇒(1): Suppose that the assertion (1) does not hold for some $\kappa > 0$. Thus, there are unitary operators $U_i(n)$ and $V_j(n)$ on $H_n$ with $\|U_i(n), V_j(n)\| \to 0$ which witness a violation of the conclusion of (1). We consider the C*-algebras $B = \prod B(H_n)$ and $\Omega = \bigoplus B(H_n)$, with the quotient map $\pi : B \to \Omega$. Then, $U_i = \pi(U_i(n))_{n=1}^{\infty}$ and $V_j = \pi(V_j(n))_{n=1}^{\infty}$ are commuting systems of unitary elements in $\Omega$, and the map $u_j \mapsto U_i, v_j \mapsto V_j$ extends to a *-homomorphism on $C^*(F_d \times F_d)$. By the assumption (2), one may find a u.c.p. map $\varphi : C^*(F_d \times F_d) \to B$ such that $\pi(\varphi(U_i)) = U_i$ and $\pi(\varphi(V_j)) = V_j$. We expand $\varphi$ as $\varphi_n = (\varphi_n)_{n=1}^{\infty}$ and see $\|U_i(n) - \varphi_n(U_i)\| \to 0$ and $\|V_j(n) - \varphi_n(V_j)\| \to 0$. Take $N$ such that $\|U_i(N) - \varphi(U_i)\| < \kappa$ and $\|V_j(N) - \varphi(V_j)\| < \kappa$. By Stinespring’s dilation theorem, there are a *-representation $\sigma : C^*(F_d \times F_d) \to B(\tilde{H})$ and an isometry $W : H_N \to \tilde{H}$ such that $\varphi_n(x) = W^* \sigma(x) W$. Thus identifying $H_N$ with $W H_N$, one obtains unitary operators $\tilde{U}_i = \sigma(u_i)$ and $\tilde{V}_j = \sigma(v_j)$ which satisfy the conclusion of the assertion (1) for $U_i(N)$ and $V_j(N)$. This is a contradiction to the hypothesis. $\square$

The analogue of Lemma 5 also holds in the projective setting, and it can be proven using the following dilation lemma.

**Lemma 6.** Let $m \in \mathbb{N}$ be fixed and $(A_i(n))_{i=1}^{m}$ and $(B_j(n))_{j=1}^{m}$ be sequences of POVMs on $H_n$ such that $\lim_{n} \|A_i(n), B_j(n)\| = 0$. Then, there are sequences of projective POVMs $(P_i(n))_{i=1}^{m}$ and $(Q_j(n))_{j=1}^{m}$ on $\ell^2_n \otimes H_n$ such that $\lim_{n} \|P_i(n), Q_j(n)\| = 0$ and $\Phi_i(P_i(n)) = A_i(n)$, $\Phi_j(Q_j(n)) = B_j(n)$, and $\Phi_n(P_i(n)Q_j(n)) = A_i(n)B_j(n)$.

**Proof.** Let $X(n) = [A_1(n)/2 \cdots A_m(n)/2] \in M_{1, m}(B(H_n))$, and consider the unitary element

$$U(n) = \left[ \begin{array}{cc} X(n) & 0 \\ \sqrt{1 - X(n)^*X(n)} & -X(n)^* \end{array} \right] \in M_{m+1}(B(H_n)).$$

We denote by $E_i(n)$ the orthogonal projection in $M_{m+1}(B(H_n))$ onto the $i$th coordinate, and define $P_i(n) = U(n)E_i(n)U(n)^*$ for $i = 1, \ldots, m-1$ and $P_m(n) = U(n)(E_m(n) + E_{m+1}(n))U(n)^*$. Then, $(P_i(n))_{i=1}^{m}$ is a projective POVM on $\ell^2_n \otimes H_n$ whose $(1, 1)$-entry is $(A_1(n))_{i=1}^{m}$. Similarly, one obtains a projective POVM $(Q_j(n))_{j=1}^{m}$ on $\ell^2_n \otimes H_n$ whose $(1, 1)$-entry is $(A_1(n))_{j=1}^{m}$. Define $\sigma_{p, 3} : B(\ell^2_n \otimes H_n) \to B(\ell^2_n \otimes H_n)$ by $C \otimes D \mapsto C \otimes 1 \otimes D$ if $p = 1$, and $C \otimes D \mapsto 1 \otimes C \otimes D$ if $p = 2$; and let $P_i(n) = \sigma_{3, i}(P_i(n))$ and $Q_j(n) = \sigma_{1, j}(Q_j(n))$. Since $\lim_{n} \|A_i(n), B_j(n)\| = 0$, the entries of $P_i(n)$ asymptotically commute with those of $Q_j(n)$. It follows that $\lim_{n} \|P_i(n), Q_j(n)\| = 0$. They also satisfy the other conditions. $\square$

We are now ready for the proof of Theorem 4.

**Proof.** (2)⇒(3): Assume the assertion (2). Then, Lemma 3 implies that $\|e\|_{\max} = \|e\|_{\min}$ for every $e \in M_{d+1}(M_{d}(\mathbb{C}))$ and hence Kirchberg’s conjecture follows. Lemma 5 implies that $C^*(F_d \times F_d)$ has the LLP.

(3)⇒(2): Assume the assertion (3). Then, by Lemma 5, one has the Strong Kirchberg Conjecture (II) for a possibly infinite-dimensional $\tilde{H}$. Since Kirchberg’s conjecture is assumed and $C^*(F_d \times F_d) \cong C^*F_d \otimes_{\min} C^*F_d$ is RFD, one can reduce $\tilde{H}$ to a finite-dimensional Hilbert space, up to a perturbation. See Theorem 1.7.8 in Ref. 3. $\square$

**ACKNOWLEDGMENTS**

The main theorem equally holds for three or more commuting systems. Although it is stated as “the Strong Kirchberg Conjecture,” the author thanks that both Kirchberg’s and the LLP conjectures for $C^*(F_d \times F_d)$ would have negative answers. This research came out from the author’s lectures for “Masterclass on sofic groups and applications to operator algebras” (University of Copenhagen, 5–9 November 2012). The author gratefully acknowledges the kind hospitality provided by University of Copenhagen during his stay in Fall 2012. This work was in part supported by JSPS (23540233) and by the Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation.