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Tsirelson's problem and asymptotically commuting unitary matrices

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In this paper, we consider quantum correlations of bipartite systems having a slight interaction, and reinterpret Tsirelson's problem (and hence Kirchberg's and Connes's conjectures) in terms of finite-dimensional asymptotically commuting positive operator valued measures. We also consider the systems of asymptotically commuting unitary matrices and formulate the Stronger Kirchberg Conjecture. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4795391]

I. INTRODUCTION

A POVM (positive operator valued measure) with m outputs is an m-tuple $(A_i)_{i=1}^m$ of positive semi-definite operators on a Hilbert space $\mathcal{H}$ such that $\sum A_i = 1$. We write the convex sets of quantum correlation matrices of two independent systems of d POVMs with m outputs by

$$\mathcal{Q}_c = \left\{ \left[\langle \xi, A_i^k B_j^l \xi\rangle\right]_{\substack{k,l\\ i,j}} : \begin{array}{c} \mathcal{H} \text{ a Hilbert space}, \xi \in \mathcal{H} \text{ a unit vector} \\ (A_i^k)_{i=1}^m, k = 1, \dots, d, \text{ POVMs on } \mathcal{H}, \\ (B_j^l)_{j=1}^m, l = 1, \dots, d, \text{ POVMs on } \mathcal{H}, \\ [A_i^k, B_j^l] = 0 \text{ for all } i, j \text{ and } k, l \end{array} \right\}$$

and

$$\mathcal{Q}_s = \text{closure} \left\{ \left[\langle \xi, A_i^k B_j^l \xi\rangle\right]_{\substack{k,l\\ i,j}} : \begin{array}{c} \dim \mathcal{H} < +\infty, \xi \in \mathcal{H} \text{ a unit vector} \\ (A_i^k)_{i=1}^m, k = 1, \dots, d, \text{ POVMs on } \mathcal{H}, \\ (B_j^l)_{j=1}^m, l = 1, \dots, d, \text{ POVMs on } \mathcal{H}, \\ [A_i^k, B_j^l] = 0 \text{ for all } i, j \text{ and } k, l \end{array} \right\}.$$

Here i, j, k, l are indices and A_i^k does not mean the kth power of A_i. The sets $\mathcal{Q}_c$ and $\mathcal{Q}_s$ are closed convex subsets of $\mathbb{M}_{md}(\mathbb{R}_{\geq 0})$ such that $\mathcal{Q}_s \subset \mathcal{Q}_c$. Whether they coincide (for some/all $m, d \geq 2$, $(m, d) \neq (2, 2)$) is the well-known Tsirelson problem, and the matricial version of it is known to be equivalent to Kirchberg's and Connes's conjectures. We refer the reader to Refs. 4, 5, 8, and 11 for the literature and the proof of the equivalence. The matricial version of Tsirelson's problem asks whether $\mathcal{Q}_c^n = \mathcal{Q}_s^n$ for all n, where $\mathcal{Q}_c^n$ and $\mathcal{Q}_s^n$ are defined as follows:

$$\mathcal{Q}_c^n = \left\{ \left[V^* A_i^k B_j^l V\right]_{\substack{k,l\\ i,j}} : \begin{array}{c} \mathcal{H} \text{ a Hilbert space}, V : \ell_2^n \to \mathcal{H} \text{ an isometry} \\ (A_i^k)_{i=1}^m, k = 1, \dots, d, \text{ POVMs on } \mathcal{H}, \\ (B_j^l)_{j=1}^m, l = 1, \dots, d, \text{ POVMs on } \mathcal{H}, \\ [A_i^k, B_j^l] = 0 \text{ for all } i, j \text{ and } k, l \end{array} \right\}$$

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and

$$\mathcal{Q}_s^n = \text{closure}\left\{ \left[V^* A_i^k B_j^l V\right]_{\substack{k,l\\ i,j}} : \begin{array}{c} \dim\mathcal{H} < +\infty,\ V : \ell_2^n \to \mathcal{H} \text{ an isometry} \\ (A_i^k)_{i=1}^m,\ k = 1, \dots, d, \text{ POVMs on } \mathcal{H}, \\ (B_j^l)_{j=1}^m,\ l = 1, \dots, d, \text{ POVMs on } \mathcal{H}, \\ [A_i^k, B_j^l] = 0 \text{ for all } i, j \text{ and } k, l \end{array} \right\}.$$

In this paper, we consider "slightly interacting" systems. Suppose Alice and Bob conduct measurements by systems of operators $(A_i^{1/2})_{i=1}^m$ and $(B_j^{1/2})_{j=1}^m$, respectively. If Bob conducts a measurement immediately after Alice's measurement of a state ξ, then the probability of the output (i, j) is $\|B_j^{1/2} A_i^{1/2}\xi\|^2$—and vice versa. Therefore, when they conduct measurements of a state ξ at the same time, the probability of the output (i, j) is given by $\langle \xi, (A_i \bullet B_j)\xi\rangle$, where $A \bullet B = (A^{1/2}BA^{1/2} + B^{1/2}AB^{1/2})/2$. Thus, for $\varepsilon > 0$, we define the quantum correlation matrices of slightly interacting systems to be

$$\mathcal{Q}_\varepsilon^n = \text{closure}\left\{ \left[V^* (A_i^k \bullet B_j^l) V\right]_{\substack{k,l\\ i,j}} : \begin{array}{c} \dim\mathcal{H} < +\infty,\ V : \ell_2^n \to \mathcal{H} \text{ an isometry} \\ (A_i^k)_{i=1}^m,\ k = 1, \dots, d, \text{ POVMs on } \mathcal{H}, \\ (B_j^l)_{j=1}^m,\ l = 1, \dots, d, \text{ POVMs on } \mathcal{H}, \\ \|[A_i^k, B_j^l]\| \le \varepsilon \text{ for all } i, j \text{ and } k, l \end{array} \right\},$$

where $\|[A, B]\|$ denotes the operator norm of the commutator $[A, B] = AB - BA$. We note that $\mathcal{Q}_\varepsilon^n$ is a closed convex subset of $\mathbb{M}_{md}(\mathbb{M}_n(\mathbb{C})_+)$. Recall that a POVM $(A_i)_{i=1}^m$ is said to be *projective* if all A_i's are orthogonal projections. We also introduce the projective analogue of $\mathcal{Q}_\varepsilon^n$:

$$\mathcal{P}_\varepsilon^n = \text{closure}\left\{ \left[V^* (P_i^k \bullet Q_j^l) V\right]_{\substack{k,l\\ i,j}} : \begin{array}{c} \dim\mathcal{H} < +\infty,\ V : \ell_2^n \to \mathcal{H} \text{ an isometry} \\ (P_i^k)_{i=1}^m \text{ projective POVMs on } \mathcal{H}, \\ (Q_j^l)_{j=1}^m \text{ projective POVMs on } \mathcal{H}, \\ \|[P_i^k, Q_j^l]\| \le \varepsilon \text{ for all } i, j \text{ and } k, l \end{array} \right\}.$$

We simply write $\mathcal{P}_\varepsilon$ for $\mathcal{P}_\varepsilon^1$. The following is the main result of this paper. It probably suggests that $\mathcal{Q}_c$ is more natural than $\mathcal{Q}_s$ (cf. Introduction of Ref. 4).

Theorem. *For every* m, d, and n, *one has* $\mathcal{Q}_c^n = \bigcap_{\varepsilon>0} \mathcal{Q}_\varepsilon^n = \bigcap_{\varepsilon>0} \mathcal{P}_\varepsilon^n$. *In particular, an affirmative answer to Tsirelson's problem is equivalent to that* $\bigcap_{\varepsilon>0} \mathcal{P}_\varepsilon \subset \mathcal{Q}_s$.

Hence, the matricial version of Tsirelson's problem would have an affirmative answer if the following assertion holds for some/all (m, d).

Strong Kirchberg Conjecture (I). Let $m, d \ge 2$ be such that $(m, d) \ne (2, 2)$. For every $\kappa > 0$, there is $\varepsilon > 0$ with the following property. If $\dim\mathcal{H} < +\infty$, and $(P_i^k)_{i=1}^m$ and $(Q_j^l)_{j=1}^m$ is a pair of d projective POVMs on $\mathcal{H}$ such that $\|[P_i^k, Q_j^l]\| \le \varepsilon$, then there are a finite-dimensional Hilbert space $\tilde{\mathcal{H}}$ containing $\mathcal{H}$ and projective POVMs $(\tilde{P}_i^k)_{i=1}^m$ and $(\tilde{Q}_j^l)_{j=1}^m$ on $\tilde{\mathcal{H}}$ such that $\|[\tilde{P}_i^k, \tilde{Q}_j^l]\| = 0$ and $\|\Phi_{\mathcal{H}}(\tilde{P}_i^k) - P_i^k\| \le \kappa$ and $\|\Phi_{\mathcal{H}}(\tilde{Q}_j^l) - Q_j^l\| \le \kappa$, where $\Phi_{\mathcal{H}}$ is the compression to $\mathcal{H}$.

We will deal in Sec. IV with a parallel and equivalent conjecture in the unitary setting.

II. PRELIMINARY FROM C*-ALGEBRA THEORY

As it is observed in Refs. 4, 5, and 11, the study of quantum correlation matrices is essentially about the algebraic tensor product $\mathfrak{F}_m^d \otimes \mathfrak{F}_m^d$ of the C*-algebra

$$\mathfrak{F}_m^d = \ell_\infty^m * \cdots * \ell_\infty^m,$$

the unital full free product of d-copies of ℓ_∞^m. We note that $\mathfrak{F}_m^d$ is *-isomorphic to the full group C*-algebra $C^*\Gamma_{m,d}$ of the group $\Gamma_{m,d} = (\mathbb{Z}/m\mathbb{Z})^{*d}$. The condition $m, d \geq 2$ and $(m, d) \neq (2, 2)$ is equivalent to that $\Gamma_{m,d}$ contains the free groups $\mathbb{F}_r$. We denote by $(e_i)_{i=1}^m$ the standard basis of minimal projections in ℓ_∞^m, and by $(e_i^k)_{i=1}^m$ the kth copy of it in the free product $\mathfrak{F}_m^d$. We also write e_i^k for the elements $e_i^k \otimes 1$ in $\mathfrak{F}_m^d \otimes \mathfrak{F}_m^d$ and f_j^l for $1 \otimes e_j^l$. Thus, the maximal tensor product $\mathfrak{F}_m^d \otimes_{\max} \mathfrak{F}_m^d$ is the universal C*-algebra generated by projective POVMs $(e_i^k)_{i=1}^m$ and $(f_j^l)_{j=1}^m$ under the commutation relations $[e_i^k, f_j^l] = 0$. In passing, we note that $C^*\Gamma \otimes_{\max} C^*\Gamma$ is canonically *-isomorphic to $C^*(\Gamma \times \Gamma)$ for any group Γ. By Stinespring's dilation theorem (Theorem 1.5.3 in Ref. 3), one has

$$\mathcal{Q}_c^n = \{\left[\varphi(e_i^k f_j^l)\right]_{\substack{k,l\\ i,j}} : \varphi : \mathfrak{F}_m^d \otimes_{\max} \mathfrak{F}_m^d \to \mathbb{M}_n(\mathbb{C}) \text{ u.c.p.}\} \subset \mathbb{M}_{md}(\mathbb{M}_n(\mathbb{C})_+).$$

See Refs. 4 and 5 for the proof. Here u.c.p. stands for "unital completely positive."

We recall the notion of quasi-diagonality. We say a subset $\mathfrak{C}$ of $\mathbb{B}(\mathcal{H})$ is *quasi-diagonal* if there is an increasing net (P_r) of finite-rank orthogonal projections on $\mathcal{H}$ such that $P_r \nearrow 1$ in the strong operator topology and $\|[C, P_r]\| \to 0$ for every $C \in \mathfrak{C}$. A C*-algebra $\mathfrak{C}$ is said to be *quasi-diagonal* if there is a faithful *-representation π of $\mathfrak{C}$ on a Hilbert space $\mathcal{H}$ such that $\pi(\mathfrak{C})$ is a quasi-diagonal subset. A *-representation $\pi : \mathfrak{C} \to \mathbb{B}(\mathcal{H})$ is said to be *essential* if $\pi(\mathfrak{C})$ does not contain non-zero compact operators. The following theorem of Voiculescu is the most fundamental result on quasi-diagonal C*-algebras. See Sec. 7 of Ref. 3 (Theorems 7.2.5 and 7.3.6) for the details.

Theorem 1 (Voiculescu Ref. 13). *The following statements hold.*

- *Let $\mathfrak{C} \subset \mathbb{B}(\mathcal{H})$ be a faithful essential $*$-representation of a quasi-diagonal C^*-algebra $\mathfrak{C}$. Then, $\mathfrak{C}$ is a quasi-diagonal subset of $\mathbb{B}(\mathcal{H})$.*
- *Quasi-diagonality is a homotopy invariant.*

The following is based on Brown's idea (Ref. 2 and Proposition 7.4.5 in Ref. 3).

Theorem 2. *The C^*-algebras $\mathfrak{F}_m^d \otimes_{\max} \mathfrak{F}_m^d$ and $C^*\mathbb{F}_d \otimes_{\max} C^*\mathbb{F}_d$ are quasi-diagonal.*

Proof. We consider $\mathfrak{F}_m^d$ as a C*-subalgebra of $\mathfrak{M} = \mathbb{M}_m(\mathbb{C}) * \cdots * \mathbb{M}_m(\mathbb{C})$. Since the conditional expectation Φ from $\mathbb{M}_m(\mathbb{C})$ onto ℓ_∞^m extends to a u.c.p. map $\tilde{\Phi}$ from $\mathfrak{M}$ to $\mathfrak{F}_m^d$ which restricts to Φ on each free product component,[1] the canonical embedding $\mathfrak{F}_m^d \hookrightarrow \mathfrak{M}$ is indeed faithful and $\tilde{\Phi}$ is a conditional expectation from $\mathfrak{M}$ onto $\mathfrak{F}_m^d$. It follows that $\mathfrak{F}_m^d \otimes_{\max} \mathfrak{F}_m^d \subset \mathfrak{M} \otimes_{\max} \mathfrak{M}$. We will prove that the latter is quasi-diagonal.

Let $\theta : \mathfrak{M} \otimes_{\max} \mathfrak{M} \to \mathbb{B}(\mathcal{H})$ be a faithful *-representation on a separable Hilbert space $\mathcal{H}$. We omit writing θ for a while and denote by $\mathfrak{M}''$ the von Neumann algebra generated by $\theta(\mathfrak{M} \otimes \mathbb{C}1)$. We write $\{e_{i,j}\}_{i,j=1}^m$ for the matrix units in $\mathbb{M}_m(\mathbb{C})$ and $\{e_{i,j}^k\}$ for the kth copy of it in $\mathfrak{M}$. We note that the matrix units $\{e_{i,j}^k\}$ is unitarily equivalent to the first copy $\{e_{i,j}\}$ inside $\mathfrak{M}''$. This is a well-known fact, but we include the proof for the reader's convenience. Let $z \in \mathfrak{M}''$ be the central projection such that $z\mathfrak{M}''$ is finite and $(1 - z)\mathfrak{M}''$ is properly infinite (Theorem V.1.19 in Ref. 10). Then, the projections $ze_{1,1}$ and $ze_{1,1}^k$ are equivalent since they have the same center valued trace z/n (Corollary V.2.8 in Ref. 10). The projections $(1 - z)e_{1,1}$ and $(1 - z)e_{1,1}^k$ are also equivalent, since they are properly infinite and have full central support $1 - z$ (Theorem V.1.39 in Ref. 10). Therefore, for each k, there is a partial isometry $w_k \in \mathfrak{M}''$ such that $w_k^* w_k = e_{1,1}$ and $w_k w_k^* = e_{1,1}^k$. Now, $U_k = \sum_i e_{i,1} w_k^* e_{1,i}^k$ is a unitary element in $\mathfrak{M}''$ such that $U_k e_{i,j}^k U_k^* = e_{i,j}$ for all i, j and k. Since $\mathfrak{M}''$ is a von Neumann algebra, there is a norm-continuous path $U_k(t)$ of unitary elements connecting $U_k(0) = 1$ to $U_k(1) = U_k$. It follows that the *-homomorphisms $\pi_t : \mathfrak{M} \mapsto \mathfrak{M}''$, $e_{i,j}^k \mapsto U_k(t) e_{i,j}^k U_k(t)^*$, give rise to a homotopy from $\pi_0 : \mathfrak{M} \hookrightarrow \mathfrak{M}''$ to $\pi_1 : \mathfrak{M} \to \mathbb{M}_m(\mathbb{C}) \subset \mathfrak{M}''$. Likewise, there is a homotopy $\rho_t : \mathfrak{M} \to \theta(\mathbb{C}1 \otimes \mathfrak{M})''$ between the embedding ρ_0 of $\mathfrak{M}$ as the second tensor component and ρ_1 which ranges in $\mathbb{M}_m(\mathbb{C})$. Thus, $\pi_t \times \rho_t : \mathfrak{M} \otimes_{\max} \mathfrak{M} \to \mathbb{B}(\mathcal{H})$ is a homotopy between the embedding θ and $\pi_1 \times \rho_1$. Therefore, $\mathfrak{M} \otimes_{\max} \mathfrak{M}$ is embeddable into a C*-algebra which is homotopic to $\mathbb{M}_m(\mathbb{C}) \otimes \mathbb{M}_m(\mathbb{C})$. Now quasi-diagonality of $\mathfrak{M} \otimes_{\max} \mathfrak{M}$ follows from Theorem 1. The case for $C^*\mathbb{F}_d$ is similar (Proposition 7.4.5 in Ref. 3). □

III. PROOF OF THEOREM

We start the proof of the inclusion $\bigcap_{\varepsilon>0}\mathcal{Q}^n_\varepsilon \subset \mathcal{Q}^n_c$. Take m, d, n, and $[X^{k,l}_{i,j}] \in \bigcap_{\varepsilon>0}\mathcal{Q}^n_\varepsilon$ arbitrary. Then, for every $r \in \mathbb{N}$, there are a pair of d POVMs $(A^k_i(r))^m_{i=1}$ and $(B^l_j(r))^m_{j=1}$ on $\mathcal{H}_r$ and a u.c.p. map $\varphi_r : \mathbb{B}(\mathcal{H}_r) \to \mathbb{M}_n(\mathbb{C})$ such that $\|[A^k_i(r), B^l_j(r)]\| \le r^{-1}$ and $\|\varphi_r(A^k_i(r) \bullet B^l_j(r)) - X^{k,l}_{i,j}\| \le r^{-1}$. We consider the C*-algebras

$$\mathfrak{M} = \prod_{r=1}^{\infty} \mathbb{B}(\mathcal{H}_r) = \{(C(r))^\infty_{r=1} : C(r) \in \mathbb{B}(\mathcal{H}_r),\ \sup_r \|C(r)\| < +\infty\},$$

$$\mathfrak{K} = \bigoplus_{r=1}^{\infty} \mathbb{B}(\mathcal{H}_r) = \{(C(r))^\infty_{r=1} : C(r) \in \mathbb{B}(\mathcal{H}_r),\ \lim_r \|C(r)\| = 0\}$$

and $\mathfrak{Q} = \mathfrak{M}/\mathfrak{K}$, with the quotient map $\pi : \mathfrak{M} \to \mathfrak{Q}$. Then $A^k_i = \pi((A^k_i(r))^\infty_{r=1})$ and $B^l_j = \pi((B^l_j(r))^\infty_{r=1})$ are commuting POVMs in $\mathfrak{Q}$. Fix an ultra-limit Lim and consider the u.c.p. map $\tilde{\varphi} : \mathfrak{M} \to \mathbb{M}_n(\mathbb{C})$ defined by $\tilde{\varphi}((C(r))^\infty_{r=1}) = \mathrm{Lim}_r\, \varphi_r(C(r)) \in \mathbb{M}_n(\mathbb{C})$. It factors through $\mathfrak{Q}$ and one obtains a u.c.p. map $\varphi : \mathfrak{Q} \to \mathbb{M}_n(\mathbb{C})$ such that $\tilde{\varphi} = \varphi \circ \pi$. It follows that $\varphi(A^k_i B^l_j) = \varphi(A^k_i \bullet B^l_j) = X^{k,l}_{i,j}$, and hence $[X^{k,l}_{i,j}] \in \mathcal{Q}^n_c$.

For the inclusion $\mathcal{Q}^n_c \subset \bigcap_{\varepsilon>0}\mathcal{P}^n_\varepsilon$, take m, d, n and $[X^{k,l}_{i,j}] \in \mathcal{Q}^n_c$ arbitrary. Then, there is a u.c.p. map $\varphi : \mathfrak{F}^d_m \otimes_{\max} \mathfrak{F}^d_m \to \mathbb{M}_n(\mathbb{C})$ such that $\varphi(e^k_i f^l_j) = X^{k,l}_{i,j}$. By Stinespring's dilation theorem, there are a *-representation of $\mathfrak{F}^d_m \otimes_{\max} \mathfrak{F}^d_m$ on a separable Hilbert space $\mathcal{H}$ and an isometry $V : \ell^n_2 \to \mathcal{H}$ such that $\varphi(C) = V^*CV$ for $C \in \mathfrak{F}^d_m \otimes_{\max} \mathfrak{F}^d_m$. By inflating the *-representation, we may assume it is faithful and essential. Since $\mathfrak{F}^d_m \otimes_{\max} \mathfrak{F}^d_m$ is quasi-diagonal (Theorem 2), there is an increasing sequence $(P_r)^\infty_{r=1}$ of finite-rank orthogonal projections on $\mathcal{H}$ such that $P_r \nearrow 1$ in the strong operator topology and $\|[C, P_r]\| \to 0$ for $C \in \mathfrak{F}^d_m \otimes_{\max} \mathfrak{F}^d_m$. Thus, $P_r e^k_i P_r$ and $P_r f^l_j P_r$ are close to projections (as $r \to \infty$) and one can find projective POVMs $(E^k_i(r))^m_{i=1}$ and $(F^l_j(r))^m_{j=1}$ on $P_r\mathcal{H}$ such that $\|P_r e^k_i P_r - E^k_i(r)\| \to 0$ and $\|P_r f^l_j P_r - F^l_j(r)\| \to 0$. We note that $\|P_r V - V\| \to 0$. It follows that $\|[E^k_i(r), F^l_j(r)]\| \to 0$ and

$$\lim_{r\to\infty} V^*(E^k_i(r) \bullet F^l_j(r))V = \lim_{r\to\infty} V^* E^k_i(r) F^l_j(r) V = V^* e^k_i f^l_j V = X^{k,l}_{i,j}.$$

This implies $[X^{k,l}_{i,j}] \in \bigcap_{\varepsilon>0}\mathcal{P}^n_\varepsilon$. □

IV. ASYMPTOTICALLY COMMUTING UNITARY MATRICES

Kirchberg's conjecture[6] asserts that $\mathrm{C}^*\mathbb{F}_d \otimes_{\min} \mathrm{C}^*\mathbb{F}_d = \mathrm{C}^*\mathbb{F}_d \otimes_{\max} \mathrm{C}^*\mathbb{F}_d$ for some/all $d \ge 2$. By Choi's theorem (Theorem 7.4.1 in Ref. 3), $\mathrm{C}^*\mathbb{F}_d$ is residually finite dimensional (RFD) and so is $\mathrm{C}^*\mathbb{F}_d \otimes_{\min} \mathrm{C}^*\mathbb{F}_d$. Since finite-dimensional representations factor through the minimal tensor product, Kirchberg's conjecture is equivalent to the assertion that $\mathrm{C}^*\mathbb{F}_d \otimes_{\max} \mathrm{C}^*\mathbb{F}_d$ is RFD. For the following, let $u_1, \ldots u_d$ be the standard unitary generators of $\mathrm{C}^*\mathbb{F}_d$. We also write u_i for the elements $u_i \otimes 1$ in $\mathrm{C}^*\mathbb{F}_d \otimes \mathrm{C}^*\mathbb{F}_d$ and v_j for $1 \otimes u_j$. We denote by $\mathcal{U}(\mathcal{H})$ the set of unitary operators on $\mathcal{H}$. For $\alpha \in \mathbb{M}_d(\mathbb{M}_n(\mathbb{C}))$, we consider

$$\begin{aligned}\|\alpha\|_{\min} &= \Big\|\sum_{i,j} \alpha_{i,j} \otimes u_i v_j\Big\|_{\mathbb{M}_n(\mathbb{C}) \otimes \mathrm{C}^*\mathbb{F}_d \otimes_{\min} \mathrm{C}^*\mathbb{F}_d} \\ &= \sup\{\Big\|\sum_{i,j} \alpha_{i,j} \otimes U_i V_j\Big\| : k \in \mathbb{N},\ U_i, V_j \in \mathcal{U}(\ell^k_2) \text{s.t.} [U_i, V_j] = 0\}\end{aligned}$$

and

$$\begin{aligned}\|\alpha\|_{\max} &= \Big\|\sum_{i,j} \alpha_{i,j} \otimes u_i v_j\Big\|_{\mathbb{M}_n(\mathbb{C}) \otimes \mathrm{C}^*\mathbb{F}_d \otimes_{\max} \mathrm{C}^*\mathbb{F}_d} \\ &= \sup\{\Big\|\sum_{i,j} \alpha_{i,j} \otimes U_i V_j\Big\| : U_i, V_j \in \mathcal{U}(\ell_2) \text{s.t.} [U_i, V_j] = 0\}.\end{aligned}$$

In the above expressions, one may assume $U_1 = 1$ and $V_1 = 1$ by replacing U_i and V_j with $U_1^* U_i$ and $V_j V_1^*$. It follows that $\|\alpha\|_{\min} = \|\alpha\|_{\max}$ for $d = 2$. By Pisier's linearization trick, Kirchberg's conjecture is equivalent to the assertion that $\|\alpha\|_{\min} = \|\alpha\|_{\max}$ holds for every $d \geq 3$ (or just $d = 3$) and every $\alpha \in \mathbb{M}_d(\mathbb{M}_n(\mathbb{C}))$. See Sec. 12 of Ref. 9, Chap. 13 in Ref. 3, and Ref. 7 for the proof of this fact and more information. The proof of the following lemma is omitted because it is almost the same as that of the main theorem.

Lemma 3. For every $\alpha \in \mathbb{M}_d(\mathbb{M}_n(\mathbb{C}))$, *one has*

$$\|\alpha\|_{\max} = \inf_{\varepsilon>0} \sup\{\|\sum_{i,j} \alpha_{i,j} \otimes U_i V_j\| : k \in \mathbb{N},\ U_i, V_j \in \mathcal{U}(\ell_2^k) \text{ s.t. } \|[U_i, V_j]\| \leq \varepsilon\}.$$

We observe the following fact. Suppose $\dim \mathcal{H} < \infty$ and $U, V \in \mathcal{U}(\mathcal{H})$ are such that $\|[U, V]\| \leq \varepsilon$. It is well-known that the pair (U, V) need not be close to a commuting pair of unitary matrices,[12] but after a dilation it is. Indeed, this follows from amenability of $\mathbb{Z}^2$. Let $m = \lfloor 1/\sqrt{\varepsilon} \rfloor$ and $F = \{0, \ldots, m\}^2 \subset \mathbb{Z}^2$. We define an isometry $W : \mathcal{H} \to \ell_2\mathbb{Z}^2 \otimes \mathcal{H}$ by $W\xi = |F|^{-1/2} \sum_{x\in F} \delta_x \otimes \varphi(x)\xi$, where $\varphi((p, q)) = U^p V^q \in \mathcal{U}(\mathcal{H})$ for $(p, q) \in F$. Then, for the commuting unitary operators u and v, acting on $\ell_2\mathbb{Z}^2 \otimes \mathcal{H}$ by shifting indices in $\mathbb{Z}^2$ by $(-1, 0)$ and $(0, -1)$, respectively, one has

$$\begin{aligned}\|W^* u W - U\| &= \|\frac{1}{|F|} \sum_{x\in F\cap((-1,0)+F)} \varphi(x)^*\varphi(x + (1, 0)) - U\| \\ &\leq m\varepsilon + 1/(m + 1) < 2\sqrt{\varepsilon}.\end{aligned}$$

Similarly, one has $\|W^* v W - V\| < 2\sqrt{\varepsilon}$. Since $\mathrm{C}^*\mathbb{Z}^2$ is Abelian (and RFD), one can find a finite dimensional Hilbert space $\tilde{\mathcal{H}}$ containing $\mathcal{H}$ and commuting unitary matrices $\tilde{U}$ and $\tilde{V}$ on $\tilde{\mathcal{H}}$ such that $\|\Phi_{\mathcal{H}}(\tilde{U}) - U\| < 2\sqrt{\varepsilon}$ and $\|\Phi_{\mathcal{H}}(\tilde{V}) - V\| < 2\sqrt{\varepsilon}$, where $\Phi_{\mathcal{H}} : \mathbb{B}(\tilde{\mathcal{H}}) \to \mathbb{B}(\mathcal{H})$ is the compression. We note that $\Phi_{\mathcal{H}}(\tilde{U}) \approx U$ and $\Phi_{\mathcal{H}}(\tilde{V}) \approx V$ for any unitary elements imply $\Phi_{\mathcal{H}}(\tilde{U}\tilde{V}) \approx UV$ (see, e.g., Theorem 18 in Ref. 8). Keeping these facts in mind, we formulate the Strong Kirchberg Conjecture (II).

Strong Kirchberg Conjecture (II). Let $d \geq 2$. For every $\kappa > 0$, there is $\varepsilon > 0$ with the following property. If $\dim \mathcal{H} < +\infty$ and $U_1, \ldots, U_d, V_1 \ldots, V_d \in \mathcal{U}(\mathcal{H})$ are such that $\|[U_i, V_j]\| \leq \varepsilon$, then there are a finite-dimensional Hilbert space $\tilde{\mathcal{H}}$ containing $\mathcal{H}$ and $\tilde{U}_i, \tilde{V}_j \in \mathcal{U}(\tilde{\mathcal{H}})$ such that $\|[\tilde{U}_i, \tilde{V}_j]\| = 0$ and $\|\Phi_{\mathcal{H}}(\tilde{U}_i) - U_i\| \leq \kappa$ and $\|\Phi_{\mathcal{H}}(\tilde{V}_j) - V_j\| \leq \kappa$.

We note that the analogous statement for U_1, U_2, V is true, by the proof of the following theorem plus the fact that $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{Z})$ is RFD and has the LLP (local lifting property). See Chap. 13 in Ref. 3 for the definition of the LLP and relevant results.

Theorem 4. *The following conjectures are equivalent*.

(1) *The Strong Kirchberg Conjecture (I) holds for some/all* (m, d).
(2) *The Strong Kirchberg Conjecture (II) holds for some/all* d.
(3) *Kirchberg's conjecture holds and* $\mathrm{C}^*(\mathbb{F}_d \times \mathbb{F}_d)$ *has the LLP for some/all* $d \geq 2$.
(4) *The algebraic tensor product* $\mathrm{C}^*\mathbb{F}_d \otimes \mathrm{C}^*\mathbb{F}_d \otimes \mathbb{B}(\ell_2)$ *has unique C*-norm.*

We note that it is not known whether $\mathrm{C}^*(\mathbb{F}_d \times \mathbb{F}_d)$ has the LLP, but it is independent of $d \geq 2$ and equivalent to that the LLP is closed under the maximal tensor product. Also it is equivalent to the LLP for $\mathrm{C}^*(\Gamma_{m,d} \times \Gamma_{m,d})$. This problem seems to be independent of Kirchberg's conjecture. We will only prove the equivalence (2)⇔(3), because the proof of (1)⇔(3) is very similar and (3)⇔(4) is an immediate consequence of the tensor product characterization of the LLP (see Ref. 6 and Chap. 13 in Ref. 3).

Lemma 5. The following conjectures are equivalent:

(1) *For every* $\kappa > 0$, *there is* $\varepsilon > 0$ *with the following property. If* $\dim \mathcal{H} < +\infty$ *and* $U_1, \ldots, U_d, V_1 \ldots, V_d \in \mathcal{U}(\mathcal{H})$ *are such that* $\|[U_i, V_j]\| \leq \varepsilon$, *then there are a (not necessarily*

finite-dimensional) Hilbert space $\tilde{\mathcal{H}}$ *containing* $\mathcal{H}$ *and* $\tilde{U}_i, \tilde{V}_j \in \mathcal{U}(\tilde{\mathcal{H}})$ *such that* $\|[\tilde{U}_i, \tilde{V}_j]\| = 0$ *and* $\|\Phi_{\mathcal{H}}(\tilde{U}_i) - U_i\| \le \kappa$ *and* $\|\Phi_{\mathcal{H}}(\tilde{V}_j) - V_j\| \le \kappa$.

(2) *The C*-algebra* $\mathrm{C}^*(\mathbb{F}_d \times \mathbb{F}_d)$ *has the LLP.*

Proof. (1)⇒(2): To prove the LLP of C*-algebra $\mathrm{C}^*(\mathbb{F}_d \times \mathbb{F}_d)$, it suffices to show that the surjective *-homomorphism π from $\mathrm{C}^*(\mathbb{F}_{2d}) = \mathrm{C}^*(w_1, \ldots, w_d, w_1', \ldots, w_d')$ onto $\mathrm{C}^*(\mathbb{F}_d \times \mathbb{F}_d)$, $w_i \mapsto u_i$ and $w_j' \mapsto v_j$, is locally liftable. By the Effros–Haagerup theorem (Theorem C.4 in Ref. 3), this follows once it is shown that the canonical surjection

$$\Theta : \mathbb{B}(\ell_2) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_{2d})/\mathbb{B}(\ell_2) \otimes_{\min} \ker\pi \to \mathbb{B}(\ell_2) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_d \times \mathbb{F}_d)$$

is isometric. Let $u_0 = 1 = v_0$ and $E = \mathrm{span}\{u_i, v_j : 0 \le i, j \le d\}$ be the operator subspace of $\mathrm{C}^*(\mathbb{F}_d \times \mathbb{F}_d)$. By Pisier's linearization trick, it is enough to check that Θ is (completely) isometric on $\mathbb{B}(\ell_2) \otimes E$. For this, take $\alpha \in \mathbb{M}_{d+1}(\mathbb{B}(\ell_2))$ arbitrary and let

$$\lambda = \|\sum \alpha_{i,j} \otimes u_i v_j\|_{\mathbb{B}(\ell_2)\otimes_{\min}\mathrm{C}^*(\mathbb{F}_{2d})/\mathbb{B}(\ell_2)\otimes_{\min}\ker\pi}.$$

Let $(e_n)_{n=1}^\infty$ be a quasi-central approximate unit for $\ker\pi$ in $\mathrm{C}^*(\mathbb{F}_{2d})$, and let $w_i(n) = (1 - e_n)^{1/2} w_i (1 - e_n)^{1/2} + e_n$ and $w_j'(n)$ likewise (although the proof will equally work for $w_j'(n) = w_j'$). Then, one has

$$\|\sum \alpha_{i,j} \otimes w_i(n) w_j'(n)\|_{\mathbb{B}(\ell_2)\otimes_{\min}\mathrm{C}^*(\mathbb{F}_{2d})} \ge \lambda,$$

$$\lim_n \|[w_i(n), w_j'(n)]\| = \lim_n \|(1 - e_n)^2 [w_i, w_j']\| = \|\pi([w_i, w_j'])\| = 0$$

and $\lim_n \|[w_i^*(n), w_j'(n)]\| = 0$. Since $\mathrm{C}^*(\mathbb{F}_{2d})$ is RFD, one can find a finite-dimensional *-representation σ_n such that

$$\|\sum \alpha_{i,j} \otimes \sigma_n(w_i(n) w_j'(n))\|_{\mathbb{B}(\ell_2)\otimes_{\min}\sigma_n(\mathrm{C}^*(\mathbb{F}_{2d}))} \ge \lambda - \frac{1}{n}.$$

For every contractive matrices x and y, we consider the unitary matrices defined by

$$U_x = \begin{bmatrix} x & \sqrt{1 - xx^*} & & \\ \sqrt{1 - x^*x} & -x^* & & \\ & & x & \sqrt{1 - xx^*} \\ & & \sqrt{1 - x^*x} & -x^* \end{bmatrix} \text{ and}$$

$$V_y = \begin{bmatrix} y & & \sqrt{1 - yy^*} & \\ & y & & \sqrt{1 - yy^*} \\ \sqrt{1 - y^*y} & & -y^* & \\ & \sqrt{1 - y^*y} & & -y^* \end{bmatrix}.$$

We observe that the $(1, 1)$-entry of $U_x V_y$ is xy, and if $\|[x, y]\| \approx 0$ and $\|[x^*, y]\| \approx 0$, then $\|[U_x, V_y]\| \approx 0$. Thus, applying the assumption (1) to $U_{\sigma_n(w_i(n))}$ and $V_{\sigma_n(w_j'(n))}$, one may find unitary operators $\tilde{U}_i(n)$, $\tilde{V}_j(n)$ and the compression Φ_n such that $[\tilde{U}_i(n), \tilde{V}_j(n)] = 0$, $\|\Phi_n(\tilde{U}_i(n)) - U_{\sigma_n(w_i(n))}\| \to 0$, and $\|\Phi_n(\tilde{V}_j(n)) - V_{\sigma_n(w_j'(n))}\| \to 0$. It follows that

$$\begin{aligned}
\|\sum \alpha_{i,j} \otimes u_i v_j\|_{\mathbb{B}(\ell_2)\otimes_{\min}\mathrm{C}^*(\mathbb{F}_d\times\mathbb{F}_d)} &\ge \limsup_{n\to\infty} \|\sum \alpha_{i,j} \otimes \tilde{U}_i(n)\tilde{V}_j(n)\| \\
&\ge \limsup_{n\to\infty} \|\sum \alpha_{i,j} \otimes \Phi_n(\tilde{U}_i(n)\tilde{V}_j(n))\| \\
&= \limsup_{n\to\infty} \|\sum \alpha_{i,j} \otimes U_{\sigma_n(w_i(n))} V_{\sigma_n(w_j'(n))}\| \\
&\ge \limsup_{n\to\infty} \|\sum \alpha_{i,j} \otimes \sigma_n(w_i(n) w_j'(n))\| \\
&\ge \lambda.
\end{aligned}$$

This proves that Θ is isometric on $\mathbb{B}(\ell_2) \otimes E$, and the assertion (2) follows.

(2)⇒(1): Suppose that the assertion (1) does not hold for some $\kappa > 0$. Thus, there are unitary operators $U_i(n)$ and $V_j(n)$ on $\mathcal{H}_n$ with $\|[U_i(n), V_j(n)]\| \to 0$ which witness a violation of the conclusion of (1). We consider the C*-algebras $\mathfrak{M} = \prod \mathbb{B}(\mathcal{H}_n)$ and $\mathfrak{Q} = \prod \mathbb{B}(\mathcal{H}_n)/\bigoplus \mathbb{B}(\mathcal{H}_n)$, with the quotient map $\pi : \mathfrak{M} \to \mathfrak{Q}$. Then, $U_i = \pi((U_i(n))_{n=1}^{\infty})$ and $V_j = \pi((V_j(n))_{n=1}^{\infty})$ are commuting systems of unitary elements in $\mathfrak{Q}$, and the map $u_i \mapsto U_i$, $v_j \mapsto V_j$ extends to a $*$-homomorphism on $\mathrm{C}^*(\mathbb{F}_d \times \mathbb{F}_d)$. By the assumption (2), one may find a u.c.p. map $\varphi : \mathrm{C}^*(\mathbb{F}_d \times \mathbb{F}_d) \to \mathfrak{M}$ such that $\pi(\varphi(u_i)) = U_i$ and $\pi(\varphi(v_j)) = V_j$. We expand φ as $(\varphi_n)_{n=1}^{\infty}$ and see $\|U_i(n) - \varphi_n(u_i)\| \to 0$ and $\|V_j(n) - \varphi_n(v_j)\| \to 0$. Take N such that $\|U_i(N) - \varphi_N(u_i)\| < \kappa$ and $\|V_j(N) - \varphi_N(v_j)\| < \kappa$. By Stinespring's dilation theorem, there are a $*$-representation $\sigma : \mathrm{C}^*(\mathbb{F}_d \times \mathbb{F}_d) \to \mathbb{B}(\tilde{\mathcal{H}})$ and an isometry $W : \mathcal{H}_N \to \tilde{\mathcal{H}}$ such that $\varphi_N(x) = W^*\sigma(x)W$. Thus identifying $\mathcal{H}_N$ with $W\mathcal{H}_N$, one obtains unitary operators $\tilde{U}_i = \sigma(u_i)$ and $\tilde{V}_j = \sigma(v_j)$ which satisfy the conclusion of the assertion (1) for $U_i(N)$ and $V_j(N)$. This is a contradiction to the hypothesis. □

The analogue of Lemma 5 also holds in the projective setting, and it can be proven using the following dilation lemma.

Lemma 6. Let $m \in \mathbb{N}$ be fixed and $(A_i(n))_{i=1}^m$ and $(B_j(n))_{j=1}^m$ be sequences of POVMs on $\mathcal{H}_n$ such that $\lim_n \|[A_i(n), B_j(n)]\| = 0$. Then, there are sequences of projective POVMs $(P_i(n))_{i=1}^m$ and $(Q_j(n))_{j=1}^m$ on $\ell_2^{m+1} \otimes \ell_2^{m+1} \otimes \mathcal{H}_n$ such that $\lim_n \|[P_i(n), Q_j(n)]\| = 0$ and $\Phi_n(P_i(n)) = A_i(n)$, $\Phi_n(Q_j(n)) = B_j(n)$, and $\Phi_n(P_i(n)Q_j(n)) = A_i(n)B_j(n)$. Here Φ_n denotes the compression to $\mathbb{C}\delta_1 \otimes \mathbb{C}\delta_1 \otimes \mathcal{H}_n \cong \mathcal{H}_n$.

Proof. Let $X(n) = [A_1(n)^{1/2} \ \cdots \ A_m(n)^{1/2}] \in \mathbb{M}_{1,m}(\mathbb{B}(\mathcal{H}_n))$, and consider the unitary element

$$U(n) = \begin{bmatrix} X(n) & 0 \\ \sqrt{1 - X(n)^*X(n)} & -X(n)^* \end{bmatrix} \in \mathbb{M}_{m+1}(\mathbb{B}(\mathcal{H}_n)).$$

We denote by $E_i(n)$ the orthogonal projection in $\mathbb{M}_{m+1}(\mathbb{B}(\mathcal{H}_n))$ onto the ith coordinate, and define $P_i'(n) = U(n)E_i(n)U(n)^*$ for $i = 1, \ldots, m-1$ and $P_m'(n) = U(n)(E_m(n) + E_{m+1}(n))U(n)^*$. Then, $(P_i'(n))_{i=1}^m$ is a projective POVM on $\ell_2^{m+1} \otimes \mathcal{H}_n$ whose (1, 1)-entry is $(A_i(n))_{i=1}^m$. Similarly, one obtains a projective POVM $(Q_j'(n))_{j=1}^m$. Define $\sigma_{p,3} : \mathbb{B}(\ell_2^{m+1} \otimes \mathcal{H}_n) \to \mathbb{B}(\ell_2^{m+1} \otimes \ell_2^{m+1} \otimes \mathcal{H}_n)$ by $C \otimes D \mapsto C \otimes 1 \otimes D$ if $p = 1$, and $C \otimes D \mapsto 1 \otimes C \otimes D$ if $p = 2$; and let $P_i(n) = \sigma_{1,3}(P_i'(n))$ and $Q_j(n) = \sigma_{2,3}(Q_j'(n))$. Since $\lim_n \|[A_i(n), B_j(n)]\| = 0$, the entries of $P_i'(n)$ asymptotically commute with those of $Q_j'(n)$. It follows that $\lim_n \|[P_i(n), Q_j(n)]\| = 0$. They also satisfy the other conditions. □

We are now ready for the proof of Theorem 4.

Proof. (2)⇒(3): Assume the assertion (2). Then, Lemma 3 implies that $\|\alpha\|_{\max} = \|\alpha\|_{\min}$ for every $\alpha \in \mathbb{M}_{d+1}(\mathbb{M}_n(\mathbb{C}))$ and hence Kirchberg's conjecture follows. Lemma 5 implies that $\mathrm{C}^*(\mathbb{F}_d \times \mathbb{F}_d)$ has the LLP.

(3)⇒(2): Assume the assertion (3). Then, by Lemma 5, one has the Strong Kirchberg Conjecture (II) for a possibly infinite-dimensional $\tilde{\mathcal{H}}$. Since Kirchberg's conjecture is assumed and $\mathrm{C}^*(\mathbb{F}_d \times \mathbb{F}_d) \cong \mathrm{C}^*\mathbb{F}_d \otimes_{\min} \mathrm{C}^*\mathbb{F}_d$ is RFD, one can reduce $\tilde{\mathcal{H}}$ to a finite-dimensional Hilbert space, up to a perturbation. See Theorem 1.7.8 in Ref. 3. □

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The main theorem equally holds for three or more commuting systems. Although it is stated as "the Strong Kirchberg Conjecture," the author thinks that both Kirchberg's and the LLP conjectures for $\mathrm{C}^*(\mathbb{F}_d \times \mathbb{F}_d)$ would have negative answers. This research came out from the author's lectures for "Masterclass on sofic groups and applications to operator algebras" (University of Copenhagen, 5–9 November 2012). The author gratefully acknowledges the kind hospitality provided by University of Copenhagen during his stay in Fall 2012. This work was in part supported by JSPS (23540233) and by the Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation.

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