NAVIER-STOKES DYNAMICS ON A DIFFERENTIAL ONE-FORM

TROY L. STORY

Abstract. After transforming the Navier-Stokes dynamic equation into a differential one-form on an odd-dimensional differentiable manifold, exterior calculus is used to construct a pair of differential equations and tangent vector(vortex vector) characteristic of Hamiltonian geometry. A solution to the Navier-Stokes dynamic equation is then obtained by solving this pair of equations for the position x^k and the conjugate $\mathbf{b}_k$ to the position as functions of time. The solution $\mathbf{b}_k$ is shown to be divergence-free by contracting the differential 3-form corresponding to the divergence of the gradient of the velocity with a triple of tangent vectors, implying constraints on two of the tangent vectors for the system. Analysis of the solution $\mathbf{b}_k$ shows it is bounded since it remains finite as $\left|x^k\right| \to \infty$, and is physically reasonable since the square of the gradient of the principal function is bounded. By contracting the principal differential one-form with the vortex vector, the Lagrangian is obtained.

1. Introduction

In fluid dynamics, the Euler and Navier-Stokes equations model the dynamics of a fluid in $\mathbb{R}^n (n = 2 \; or \; 3)$ for times $t \geq 0$. For incompressible fluids filling all of $\mathbb{R}^n$, the Navier-Stokes equations are given by the three equations

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{v} + \left[-\nabla P + \nu \sum_{j=1}^{n} \frac{\partial}{\partial x^j} \left(\frac{\partial \mathbf{v}}{\partial x^j} \right) + \mathbf{f} \right] \tag{1.1}$$

$$div \; \mathbf{v} = 0 \tag{1.2}$$

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$$\mathbf{v}^0\left(x^1,...,x^n\right)=\mathbf{v}\left(x^1,...,x^n,\ t\right)|_{t=0} \tag{1.3}$$

where $((x^1,...,x^n)\in\mathbb{R}^n,\ t\geq 0)$. For the case of zero viscosity ν, these equations are the Euler equations. Eqn.(1.3) is the initial condition for position x^k and time t, eqn.(1.2) is is the divergence-free condition, and eqn.(1.1) is the equation describing the dynamics, with externally applied force $\mathbf{f}(x^1,\ ...,\ x^n,\ t)\in\mathbb{R}^n$, velocity $\mathbf{v}(x^1,\ ...,\ x^n,\ t)\in\mathbb{R}^n$, pressure $P(x^1,\ ...,\ x^n,\ t)\in\mathbb{R}$, and with forces due to pressure gradient ∇P and viscous friction $\nu\sum_{j=1}^{n}\frac{\partial}{\partial x^j}\left(\frac{\partial\mathbf{v}}{\partial x^j}\right)$.

Many investigations have focused on finding solutions $\mathbf{v}$ and P to the Navier-Stokes equations satisfying the first three equations or on proving or disproving the global existence, smoothness and breakdown of solutions on $\mathbb{R}^3$ or on $\mathbb{R}^3/\mathbb{Z}^3$, e.g., the work of Ladyzhenskaya [1] and later the work of Bertozzi and Majda [2], and Constantin [3]. Examples of the development of weak and strong solutions are given in the works of Leray [4], Scheffer [5], Caffarelli, Kohn and Nirenberg [6], Shnirelman [7], Lin [8], and Amann [9]. A critical analysis on many analytic and numerical solutions to the Navier-Stokes equations led Fefferman [10] to doubt whether standard methods of solving these equations are adequate.

In the present investigation a different approach is employed; namely, the dynamic Navier-Stokes equation is transformed into a differential one-form on an odd-dimensional differentiable manifold. It is then shown that the use of exterior calculus predicts a set of differential equations and tangent vector characteristic of Hamiltonian geometry [11, 12]. This pair of equations is solved for the position x^k as a function of time and for the conjugate $\mathbf{b}_k$ to the position as a function of time. The solution $\mathbf{b}_k$ is shown to be divergence-free by contracting the differential 3-form corresponding to the divergence of the gradient of the velocity with a triple of tangent vectors, implying constraints on two of the tangent vectors for the system. Analysis of the solution $\mathbf{b}_k$ shows it is bounded since it remains finite as $\left|x^k\right|\rightarrow\infty$, and is physically reasonable since the square of the gradient of the principal function is bounded. By contracting this differential one-form with the characteristic tangent vector, the Lagrangian is obtained.

2. Differential one-form for the Navier-Stokes dynamic equation

Multiplying the first equation by $-\,dt$ gives

$$d\,\mathbf{S} \;=\; \mathbf{B}_j\,dx^j \;-\; \mathbf{\Omega}\,dt \tag{2.1}$$

where

$$\mathbf{B}_j \equiv \left(\frac{\partial \mathbf{v}}{\partial x^j}\right) \tag{2.2}$$

$$\mathbf{B}_j dx^j = \;(\mathbf{v}\,\cdot\nabla)\,\mathbf{v}\,dt \tag{2.3}$$

$$\Omega \equiv -\nabla P \,+\, \nu\sum_{j\,=\,1}^{n}\left(\frac{\partial \mathbf{B}_j}{\partial x^j}\right) \;+\; \mathbf{f} \tag{2.4}$$

$$d\,\mathbf{S} \equiv -\left(\frac{\partial \mathbf{v}}{\partial t}\right) dt \tag{2.5}$$

where $\mathbf{S}$ will be referred to as the principal function.

To develop $\mathbf{\Omega}$ as a function of $(\mathbf{B}_j\,,\,x^j,\,t)$ and further characterize the equation for $d\,\mathbf{S}\left(x^k,\,t\right)$, the quantity $\partial_{x^j}\mathbf{B}_j$ in $\mathbf{\Omega}$ is analyzed in the following manner: first Taylor's expansion of $\mathbf{B}_j$ is taken in the neighborhood of initial position $\left(\mathbf{B}_j\,(0)\,,\,x_0^j,\,t_0\right)$, then $\partial_{x^j}\mathbf{B}_j$ is taken, then $\partial_{x^j}\mathbf{B}_j(0)$ from Taylor's expansion of $\mathbf{B}_j$ is substituted into the expression for $\partial_{x^j}\mathbf{B}_j$, giving

$$\begin{aligned}\partial_{x^j}\mathbf{B}_j = \;& [\mathbf{B}_j - \mathbf{B}_j(0) - \,(t\,-\,t_0)\,\partial_t\,\mathbf{B}_j(0)]\;\,(x^j\,-\,x_0^j)^{-1} \\ & +\;\sum_{N\,=\,2}^{\infty}\sum_{r\,=\,0}^{N}\left[\frac{N\,-\,r\,-\,1}{r!(N-r)!}\;\partial_{x^j}^{N\,-\,r}\partial_t^r\mathbf{B}_j(0)\right]\;(x^j\,-\,x_0^j)^{N\,-\,r\,-\,1}(t-t_0)^r\end{aligned} \tag{2.6}$$

with $\mathbf{B}_j(0) = \mathbf{B}_j(x_0^j,\,t_0)$. Substituting this $\partial_{x^j}\mathbf{B}_j$ into $\mathbf{\Omega}$ gives

$$\Omega \;=\; -\nabla P \;+\; \mathbf{f} \;+\; \nu \sum_{j=1}^{n} \left[\mathbf{B}_j \;-\; \mathbf{B}_j(0) - \;\partial_t \mathbf{B}_j(0)(t \;-\; t_0)\right]\;(x^j \;-\; x_0^j)^{-1}$$

$$+\;\nu \sum_{j=1}^{n} \sum_{N=2}^{\infty} \sum_{r=0}^{N} \left[\frac{(N \;-\; r \;-\; 1)}{r!(N-r)!}\;\partial_{x^j}^{N-r}\partial_t^r \mathbf{B}_j(0)\right]\;(x^j - \;x_0^j)^{N-r-1}(t - \;t_0)^r \tag{2.7}$$

The differential one-form corresponding to $d\,\mathbf{S}$ is

$$\mathbf{d}\,\mathbf{S} \;=\; \mathbf{B}_j\,\mathbf{d}x^j \;\;-\;\; \Omega\,\mathbf{d}t \tag{2.8}$$

where boldface symbol $\mathbf{d}$ is the exterior derivative operator and $\mathbf{dS}$ is the exterior derivative of vector field $\mathbf{S}$. Let the set of x^j now represent a configuration space. In order for $\mathbf{d}\,\mathbf{S}$ to satisfy Hamiltonian geometry, three conditions must be satisfied; namely, (1) $\mathbf{B}_j$ must be the gradient of the function $\mathbf{S}$, (2) x^j and $\mathbf{B}_j$ must be functions of temporal coordinate t alone and (3) $\Omega \;=\; \Omega\,(\mathbf{B}_j\;,\, x^j,\, t)$. The first condition is automatically satisfied by reference to the equation for $\mathbf{dS}$, i.e., $\mathbf{B}_j$ is the gradient of $\mathbf{S}$. Since the existence of $\mathbf{v}$ implies $x^j \;=\; x^j\,(t)$ and since $\mathbf{B}_j \;=\; \mathbf{B}_j\;(x^j, t) \;=\; \mathbf{B}_j\;(x^j(t), t) \;=\; \mathbf{b}_j(t)$, then the second condition is satisfied. Condition three is satisfied by the definition of Ω. Hence $\mathbf{dS}$ becomes,

$$\mathbf{d}\,\mathbf{S} \;=\; \mathbf{b}_j\,\mathbf{d}x^j \;\;-\;\; \Omega\,\mathbf{d}t \tag{2.9}$$

which is analogous to the expression for the differential one-form for the action in Hamiltonian mechanics. The geometric object $\mathbf{dS}$ is called a vector-valued differential one-form on extended cotangent space $T^*M_{x^j}$(coordinates $(\mathbf{b}_j\;,\, x^j,\, t)$), with basic differential one-forms $\mathbf{db}_j\;$, $\mathbf{d}x^j$, $\mathbf{d}t$ and function $\Omega\,(\mathbf{b}_j\;,\, x^j,\, t)$(analgous to the Hamiltonian). With this development, the Navier-Stokes equation is expressed as a differential form useful for applying exterior calculus to analyze Navier-Stokes dynamics.

3. Navier-Stokes dynamics on a differential one-form

Using the symbol $\omega \equiv \mathbf{dS}$, the exterior derivative of $\mathbf{dS}$ is

$$\mathbf{d}\omega = \mathbf{db}_j \wedge \mathbf{d}x^j - \left[\left(\frac{\partial \mathbf{\Omega}}{\partial x^j} \right) \mathbf{d}x^j + \left(\frac{\partial \mathbf{\Omega}}{\partial \mathbf{b}_j} \right) \mathbf{db}_j + \left(\frac{\partial \mathbf{\Omega}}{\partial t} \right) \mathbf{d}t \right] \wedge \mathbf{d}t \tag{3.1}$$

Following the procedure of Story [11], consider the vectors ξ , $\eta \in T\,(T^* M_x)$, where

$$\xi = \left(\frac{d\,\mathbf{b}_j}{dt} \right) \partial_{\mathbf{b}_j} + \left(\frac{dx^j}{dt} \right) \partial_{x^j} + \partial_t \,, \tag{3.2}$$

and

$$\eta = \beta_{\mathbf{b}_j} \partial_{\mathbf{b}_j} + \beta_{x^j} \partial_{x^j} + \partial_t \tag{3.3}$$

and the mapping $\mathbf{d}\omega : \ (\xi, \eta) \rightarrow \mathbf{d}\omega\,(\xi, \eta)$. This mapping and the contraction

$$\mathbf{d}\omega\,(\xi \,, \eta\,) = 0 \tag{3.4}$$

are defined only if the coordinates $\frac{dx^j}{dt}$ and $\frac{d\,\mathbf{b}_j}{dt}$ of ξ have the values

$$\frac{dx^j}{dt} = \frac{\partial \mathbf{\Omega}}{\partial \mathbf{b}_j} \quad \text{and} \quad \frac{d\,\mathbf{b}_j}{dt} = - \frac{\partial \mathbf{\Omega}}{\partial x^j} \tag{3.5}$$

Using the definition of $\mathbf{\Omega}$, with $\mathbf{b}_j$ replacing $\mathbf{B}_j$, the above equations become

$$\frac{dx^k}{dt} = \frac{\nu}{x^k - x_0^k} \tag{3.6}$$

and

$$\begin{aligned} &\frac{d\mathbf{b}_k}{dt} = \partial_{x^k} \left(\nabla P \right) - \partial_{x^k} \mathbf{f} \\ &-\nu\, \partial_{x^k} \sum_{j=1}^{n} \left[\mathbf{b}_j - \mathbf{B}_j\,(0) - (t - t_0)\, \partial_t \mathbf{B}_j\,(0) \right] \left(x^j - x_0^j \right)^{-1} \\ &-\nu\, \partial_{x^k} \sum_{j=1}^{n} \sum_{N=2}^{\infty} \sum_{r=0}^{N} \left[\frac{N - r - 1}{r!\,(N-r)!}\, \partial_{x^j}^{N-r} \partial_t^r \mathbf{B}_j\,(0) \right] \left(x^j - x_0^j \right)^{N-r-1} (t - t_0)^r \end{aligned} \tag{3.7}$$

This is an equation whose solution is $\mathbf{b}_j$, with constants of the type $\partial_{x^j}^{N\,-\,r}\, \partial_t^r\, \mathbf{B}_j(0)$ appearing, not $\mathbf{B}_j\,(x^j, \, t)$.

4. The solution

The solution to the differential equation for $\frac{dx^k}{dt}$ is

$$x^k = x_0^k \pm \sqrt{2\nu(t - t_0)} \tag{4.1}$$

To change the equation for $\frac{d\mathbf{b}_k}{dt}$ so that a series expansion method can be used for its solution, first P and $\mathbf{f}$ are approximated by a Taylor's series to second order and ∇P is taken, then partial derivatives $\partial_{x^k}\mathbf{f}$ and $\partial_{x^k}\nabla P$ are taken. When comparing the terms $\partial_{x^k}\partial_{x^{k+1}}\mathbf{f}(0)\left(x^{k+1} - x_0^{k+1}\right)$ and $\partial_{x^{k+2}}\partial_{x^k}\mathbf{f}(0)\left(x^{k+2} - x_0^{k+2}\right)$ with $\partial^2_{x^k}\mathbf{f}(0)\left(x^k - x_0^k\right)$, all being terms from $\partial_{x^k}\mathbf{f}$, it is assumed $\partial_{x^{k+1}}\partial_{x^k}\mathbf{f}(0) \ll \partial^2_{x^k}\mathbf{f}(0)$ and $\partial_{x^{k+2}}\partial_{x^k}\mathbf{f}(0) \ll \partial^2_{x^k}\mathbf{f}(0)$; these terms are excluded as an approximation. The notation k, $k+1$, $k+2$ is intended to imply cyclic order in x, y, z. Following the above indicated procedure and noting once again that $\mathbf{b}_j = \mathbf{b}_j(t)$, the differential equation for $\frac{d\mathbf{b}_k}{dt}$ becomes

$$\begin{aligned}
\frac{d\mathbf{b}_k}{dt} &= -\sum_{N=2}^{\infty}\sum_{r=0}^{N}\left[\frac{\nu\,(N-r-1)^2}{r!(N-r)!}\partial_{x^k}^{N-r}\partial_t^r\mathbf{B}_k(0)\right](x^k - x_0^k)^{N-r-2}(t-t_0)^r \\
&- \partial_{x^k}\partial_t\mathbf{f}(0)\,(t-t_0) - \partial^2_{x^k}\mathbf{f}(0)\,(x^k - x_0^k) \\
&+ \left[\left(\mathbf{e}_k\partial^2_{x^k} + \mathbf{e}_{x^{k+1}}\partial_{x^k}\partial_{x^{k+1}} + \mathbf{e}_{x^{k+2}}\partial_{x^{k+2}}\partial_{x^k}\right)P(0) - \partial_{x^k}\mathbf{f}(0)\right] \\
&+ \nu\left[\mathbf{b}_k - \mathbf{B}_k(0)\right](x^k - x_0^k)^{-2} \\
&- \nu\,\partial_t\mathbf{B}_k(0)\,(x^k - x_0^k)^{-2}(t-t_0)
\end{aligned} \tag{4.2}$$

where $\mathbf{e}_k$ are unit vectors arising from use of the gradient. After multiplying the foregoing equation by $(x^k - x_0^k)^2$ and using $x^k = x_0^k \pm \sqrt{2\nu(t - t_0)}$ to remove the remaining $(t - t_0)$ dependence, the equation for $\frac{d\mathbf{b}_k}{dt}$ becomes

$$
\begin{aligned}
(x^k - x_0^k)^2 \frac{d\,\mathbf{b}_k}{dt} \;&=\; \nu\,(\mathbf{b}_k - \mathbf{B}_k(0)) \\
&- \sum_{N=2}^{\infty}\sum_{r=0}^{N}\left[\frac{(N - r - 1)^2}{2\,(2\nu)^{r-1}\,r!(N-r)!}\partial_{x^k}^{N-r}\,\partial_t^r\,\mathbf{B}_k(0)\right](x^k - x_0^k)^{N+r} \\
&- (2\nu)^{-1}\partial_{x^k}\partial_t\,\mathbf{f}(0)\,(x^k - x_0^k)^4 - \partial_{x^k}^2\mathbf{f}(0)\,(x^k - x_0^k)^3 \qquad (4.3)\\
&+ \begin{bmatrix} -\frac{1}{2}\partial_t\,\mathbf{B}_k(0) - \partial_{x^k}\mathbf{f}(0) \\ \\ +\left(\mathbf{e}_k\partial_{x^k}^2 + \mathbf{e}_{x^{k+1}}\partial_{x^k}\partial_{x^{k+1}} + \mathbf{e}_{x^{k+2}}\partial_{x^{k+2}}\partial_{x^k}\right)P(0)\end{bmatrix}(x^k - x_0^k)^2
\end{aligned}
$$

The series solution to the foregoing equation proceeds by assuming $\mathbf{b}_k$ is given by

$$\mathbf{b}_k\,(t) \;=\; \sum_{N=1}^{\infty}\mathbf{C}_N\,(t - t_0)^{N/2}\exp\left(-Na(t - t_0)^{1/2}\right) \qquad (4.4)$$

Using $x^k = x_0^k \pm \sqrt{2\nu(t - t_0)}$,

$$\mathbf{b}_k\left(x^k\right) \;=\; \sum_{N=1}^{\infty}\frac{\mathbf{C}_N}{(2\nu)^{\frac{N}{2}}}(x^k - x_0^k)^N\exp\left(-\frac{Na}{\sqrt{2\nu}}(x^k - x_0^k)\right) \qquad (4.5)$$

where the $\mathbf{C}_N$ and the a are constants. Computing $\frac{d\,\mathbf{b}_k}{dt}$ with the use of the equation for $\mathbf{b}_k\,(t)$, followed by use of $x^k = x_0^k \pm \sqrt{2\nu(t - t_0)}$ to express $\frac{d\,\mathbf{b}_k}{dt}$ as a function of $\left(x^k - x_0^k\right)$, use of the equation for $\mathbf{b}_k\left(x^k\right)$, expanding the exponential function by a Taylor's series to second-order, rearranging and combining terms, changes $\left(x^k - x_0^k\right)^2\,\frac{d\,\mathbf{b}_k}{dt}$ into the following result:

$$
\begin{aligned}
&\sum_{N=2}^{\infty}\sum_{r=0}^{N}\left[\frac{(N\,-\,r\,-\,1)^2}{2\,(2\nu)^{r-1}\,r!(N-r)!}\partial_{x^k}^{N-r}\,\partial_t^r\,\mathbf{B}_k(0)\right]\;(x^k-\;x_0^k)^{N+r}\\
&+\;\sum_{N=4}^{\infty}\left[\begin{matrix}\dfrac{-a^3\,(N-3)^3\,\mathbf{C}_{N-3}+a^2\,(N-\,2)^2\,(N-\,1)\,\mathbf{C}_{N-2}}{4(2\nu)^{\frac{N}{2}-1}}\\ \\ \dfrac{-\,2\,a\;(N-1)^2\;\mathbf{C}_{N-1}\;+\;2\,(N\;-\;1)\;\mathbf{C}_N}{4(2\nu)^{\frac{N}{2}-1}}\end{matrix}\right]\;\left(x^k-\;x_0^k\right)^N\\
&+\;(2\nu)^{-1}\partial_{x^k}\partial_t\,\mathbf{f}(0)\;(x^k-\;x_0^k)^4\\
&+\;\left[\partial_{x^k}^2\mathbf{f}(0)\;+\;\frac{1}{\sqrt{2\nu}}\;\left(\frac{1}{2}\,a^2\mathbf{C}_1-\,2a\mathbf{C}_2\;+\;\mathbf{C}_3\right)\right]\;(x^k-\;x_0^k)^3\\
&+\;\left[\begin{matrix}-\;\left(\dfrac{a}{2}\right)\mathbf{C}_1\;+\;\left(\dfrac{1}{2}\right)\mathbf{C}_2\;\;+\;\dfrac{1}{2}\partial_t\,\mathbf{B}_k(0)\;+\;\partial_{x^k}\mathbf{f}(0)\\ \\ -\;\left(\mathbf{e}_k\partial_{x^k}^2\;+\;\mathbf{e}_{x^{k+1}}\partial_{x^k}\partial_{x^{k+1}}\;+\;\mathbf{e}_{x^{k+2}}\partial_{x^{k+2}}\partial_{x^k}\right)\,P(0)\end{matrix}\right]\;(x^k-\;x_0^k)^2\\
&+\;\nu\,\mathbf{B}_k(0)\;=\;0
\end{aligned}
\tag{4.6}
$$

Changing $\left(x^k-\;x_0^k\right)^{N+r}$ to $\left(x^k-\;x_0^k\right)^N$ gives

$$\sum_{N=2}^{\infty} \sum_{r=0}^{r \leq N/2} \mathbf{A}_{N-r,r} \left(x^k - x_0^k\right)^N - \left[\frac{\partial_t \mathbf{B}_k(0)}{2}\right] \left(x^k - x_0^k\right)^2$$
$$= \sum_{N=2}^{\infty} \sum_{r=0}^{N} \left[\frac{(N - r - 1)^2}{2\,(2\nu)^{r-1}\; r!\,(N - r)!} \partial_{x^k}^{N-r}\, \partial_t^r\, \mathbf{B}_k(0)\right] (x^k - x_0^k)^{N+r} \tag{4.7}$$

where

$$\mathbf{A}_{N-r,\,r} = \left[\frac{(N - 2r - 1)^2}{2\,(2\nu)^{r-1}\; r!\,(N - 2r)!} \partial_{x^k}^{N-2r}\, \partial_t^r\, \mathbf{B}_k(0)\right] \text{ for } N - 2r \geq 0 \tag{4.8}$$

and where the term $\left[\frac{\partial_t \mathbf{B}_k(0)}{2}\right] \left(x^k - x_0^k\right)^2$ is generated by the sums $\sum_{N=2}^{\infty} \sum_{r=0}^{r \leq N/2}$ but is not generated by the sums $\sum_{N=2}^{\infty} \sum_{r=0}^{N}$, hence it is subtracted; there are no other terms of this type. The meaning of the limit $r_{\max} \leq \frac{N}{2}$ is illustrated as follows: if $N =$ odd number, e.g., $N = 3$ then $r_{\max} = 1$ since $r_{\max} = 2$ would contradict $r_{\max} \leq \frac{N}{2}$. If $N =$ even number, e.g., $N = 4$, then $r_{\max} = 2$ since $r_{\max} = 3$ would contradict $r_{\max} \leq \frac{N}{2}$. This new sum is expanded to start at $N = 5$ and the $A_{N-r,r}$ are evaluated, giving

$$\sum_{N=5}^{\infty}\left[\begin{array}{l}\displaystyle\sum_{r=0}^{r\le N/2}\frac{2\,(N-2r-1)^2}{(2\nu)^{r-\frac{N}{2}}\;r!\,(N-2r)!}\partial_{x^k}^{N-2r}\,\partial_t^r\,\mathbf{B}_k(0)\\ \\ -a^3\,(N-3)^3\,\mathbf{C}_{N-3}+\;a^2\,(N-2)^2\,(N-1)\mathbf{C}_{N-2}\\ \\ -\,2\,a\;(N-1)^2\;\mathbf{C}_{N-1}\,+\,2\,(N-1)\;\mathbf{C}_N\end{array}\right]\frac{\left(x^k-x_0^k\right)^N}{4(2\nu)^{\frac{N}{2}-1}}$$

$$+\left[\begin{array}{l}\left(\dfrac{-a^3\nu^{-1}}{8}\right)\mathbf{C}_1+\left(\dfrac{3a^2\nu^{-1}}{2}\right)\mathbf{C}_2+\left(\dfrac{-9a\,\nu^{-1}}{4}\right)\mathbf{C}_3+\left(\dfrac{3\,\nu^{-1}}{4}\right)\mathbf{C}_4\\ \\ +\,(2\nu)^{-1}\partial_{x^k}\partial_t\,\mathbf{f}(0)\\ \\ +\,\left(\dfrac{3}{8}\,\nu\,\partial_{x^k}^4\,+\,\dfrac{1}{4}\partial_{x^k}^2\partial_t\,+\,\dfrac{1}{4}(2\nu)^{-1}\partial_t^2\right)\mathbf{B}_k(0)\end{array}\right](x^k-x_0^k)^4$$

$$+\left[\begin{array}{l}\left(\dfrac{a^2}{2\sqrt{2\nu}}\right)\mathbf{C}_1\;-\;\left(\dfrac{2a}{\sqrt{2\nu}}\right)C_2\;+\;\left(\dfrac{1}{\sqrt{2\nu}}\right)\mathbf{C}_3\\ \\ +\,\dfrac{2}{3}\,\nu\,\partial_{x^k}^3\,\mathbf{B}_k(0)\;+\;0\;+\;\partial_{x^k}^2\mathbf{f}(0)\end{array}\right](x^k-x_0^k)^3$$

$$+\left[\begin{array}{l}-\;\left(\dfrac{a}{2}\right)\mathbf{C}_1\;+\;\left(\dfrac{1}{2}\right)\mathbf{C}_2\;+\;\partial_{x^k}\mathbf{f}(0)+\dfrac{1}{2}\left(\nu\,\partial_{x^k}^2\;+\;\partial_t\right)\mathbf{B}_k(0)\\ \\ -\,\left(\mathbf{e}_k\partial_{x^k}^2\;+\;\mathbf{e}_{x^{k+1}}\partial_{x^k}\partial_{x^{k+1}}\;+\;\mathbf{e}_{x^{k+2}}\partial_{x^{k+2}}\partial_{x^k}\right)P(0)\end{array}\right](x^k-x_0^k)^2$$

$$+\;\nu\,\mathbf{B}_k(0)\;=\;0 \tag{4.9}$$

The right side of the above equation is zero only if the coefficients of the individual powers of $(x^k - x_0^k)$ are zero; hence,

$$\mathbf{B}_k(0) \; = \; 0 \tag{4.10}$$

$$\mathbf{C}_2 \; = \; \begin{bmatrix} a\,\mathbf{C}_1 \; - \; \left(\nu\,\partial^2_{x^k} \; + \; \partial_t\right)\,\mathbf{B}_k(0) \; - \; 2\,\partial_{x^k}\mathbf{f}(0) \\ \\ +2\,\left(\mathbf{e}_k\partial^2_{x^k} \; + \; \mathbf{e}_{x^{k+1}}\partial_{x^k}\partial_{x^{k+1}} \; + \; \mathbf{e}_{x^{k+2}}\partial_{x^{k+2}}\partial_{x^k}\right)P(0) \end{bmatrix} \tag{4.11}$$

$$\mathbf{C}_3 = \begin{bmatrix} \left(\dfrac{3a^2}{2}\right)\mathbf{C}_1 \; - \; \left(2a\,\nu\,\partial^2_{x^k} \; + \dfrac{(2\nu)^{\frac{3}{2}}}{3}\,\partial^3_{x^k} \; + \; 2a\,\partial_t\right)\mathbf{B}(\mathbf{0}) \\ \\ - \; \left(4a\,\partial_{x^k} \; + \; \sqrt{2\nu}\,\partial^2_{x^k}\right)\,\mathbf{f}(0) \\ \\ + \; 4a\,\left(\mathbf{e}_k\partial^2_{x^k} \; + \; \mathbf{e}_{x^{k+1}}\partial_{x^k}\partial_{x^{k+1}} \; + \; \mathbf{e}_{x^{k+2}}\partial_{x^{k+2}}\partial_{x^k}\right)P(0) \end{bmatrix} \tag{4.12}$$

$$\mathbf{C}_4 = \begin{bmatrix} \left(\dfrac{8a^3}{3}\right)\,\mathbf{C}_1 \; - \; \begin{pmatrix} 2a^2\;(2\nu)\;\partial^2_{x^k} \; + a\,(2\nu)^{\frac{3}{2}}\;\partial^3_{x^k} \\ \\ +\dfrac{1}{8}\,(2\nu)^2\;\partial^4_{x^k} + \; 4a^2\,\partial_t \\ \\ + \; \dfrac{1}{6}\,\partial^2_t \; + \; \dfrac{1}{3}\nu\,\partial^2_{x^k}\partial_t \end{pmatrix}\mathbf{B}_k(0) \\ \\ - \; \left(8\,a^2\,\partial_{x^k} + \; 3a\,(2\nu)^{\frac{1}{2}}\;\partial^2_{x^k} + \; \dfrac{2}{3}\,\partial_{x^k}\partial_t\right)\mathbf{f}\,(0) \\ \\ + \; 8a^2\,\left(\mathbf{e}_k\partial^2_{x^k} \; + \; \mathbf{e}_{x^{k+1}}\partial_{x^k}\partial_{x^{k+1}} \; + \; \mathbf{e}_{x^{k+2}}\partial_{x^{k+2}}\partial_{x^k}\right)P(0) \\ \end{bmatrix} \tag{4.13}$$

$$\mathbf{C}_N = a^3\frac{(N-3)^3}{2\,(N-1)}\mathbf{C}_{N-3} - a^2\frac{(N-2)^2}{2}\mathbf{C}_{N-2} + a\,(N-1)\,\mathbf{C}_{N-1}$$

$$(4.14)$$

$$-\sum_{r=0}^{r\le N/2}\frac{(N-2r-1)^2}{(N-1)\,(2\nu)^{r-\frac{N}{2}}\,r!\,(N-2r)!}\,\partial_{x^k}^{N-2r}\,\partial_t^r\,\mathbf{B}_k(0)\quad ;N\ge 5$$

The equations for $\mathbf{B}_k\,(0)$ and the coefficients can be used to compute all constants relative to the value of $\mathbf{C}_1$, but do not provide an explicit calculation of $\mathbf{C}_1$. These constants are functions of constant coefficients of the type $\partial_{x^k}^{\alpha}\,\partial_t^{\beta}\,\mathbf{B}_k(0)$, $\partial_{x^k}^{\alpha}\,\partial_t^{\beta}\,\mathbf{f}(0)$ and $\partial_{x^k}^{\alpha}\,\partial_t^{\beta}\,P(0)$. Recall that $\mathbf{B}_k = \mathbf{B}_k\,(x^k,t) = \mathbf{B}_k\,(x^k(t),t) = \mathbf{b}_k(t)$; hence, the equation $\mathbf{B}_k(0) = 0$ states that the gradient of the velocity is zero at the initial conditions defined by $\mathbf{b}_k(t_0) = 0$, $x^k = x_0^k$ and $t = t_0$.

5. Analysis of the solution

5.1. **The solution (initial conditions, bounds on $\mathbf{b}_k$, bounded energy, graphs, zero external forces).** The solution to eqn.(1.1) depends on the existence of smooth functions $\mathbf{B}_k$, $\mathbf{f}$, and P such that Taylor's expansion theorem can be used; hence the solution depends on these functions being C^∞, although $\mathbf{f}$ and P are expanded only to second - order. These three functions are real and belong to $\mathbb{R}^n \times [0,\infty)$.

5.1.1. *Initial conditions.* The solution, $x^k = x_0^k \pm \sqrt{2\nu(t-t_0)}$, shows that if $t = t_0$ then $x^k = x_0^k$; hence, the equations for $\mathbf{b}_k(t)$ and $\mathbf{b}_k(x^k)$ give $\mathbf{b}_k(t_0) = 0$. The initial conditions are given in the appropriate variables for extended cotangent space $\big(\mathbf{b}_k,\,x^k,\,t)\big)$ rather than extended tangent space$\big(\mathbf{v},\,x^k,\,t)\big)$.

5.1.2. *Bounds on* $\mathbf{b_k}$. Using $\mathbf{b}_k = \mathbf{b}_k\left(x^k\right)$, the α-th derivative of $\mathbf{b}_k$ is

$$\frac{d^\alpha\mathbf{b}_k}{dx^{k\,\alpha}} = \sum_{N=1}^{\infty}\sum_{m=0}^{\alpha}\beta_{N\,m\,\alpha}\,\mathbf{C}_N\left(x^k - x_0^k\right)^{N-(\alpha-m)}\exp\left[\frac{-a\,N}{\sqrt{2\nu}}\left(x^k - x_0^k\right)\right] \tag{5.1}$$

where

$$\beta_{N\,m\,\alpha} \;=\; \frac{1}{(2\nu)^{N/2}} \left[\frac{-a\,N}{\sqrt{2\,\nu}}\right]^m \begin{pmatrix} \alpha \\ m \end{pmatrix} \left[\frac{N!}{[N-\,(\alpha-\,m)]!}\right] \tag{5.2}$$

and where $N \;\geq\; \alpha \;-\; m$. At $t = t_0$, note that $\frac{d^\alpha \mathbf{b}_k}{dx^{k\,\alpha}} \;=\; 0$, since $x^k = x_0^k$ at $t = t_0$. This analysis shows that at $x^k = x_0^k$, $\lim_{|x^k| \to \infty} \left|\frac{d^\alpha \mathbf{b}_k}{dx^{k\,\alpha}}\right| \;=\; 0$ for any α . Hence $\mathbf{b}_k$ will not grow large as $\left|x^k\right| \to \infty$. The behavior of $\mathbf{b}_k$ as $\left|x^k\right| \to \infty$ can also be examined directly from $\mathbf{b}_k(x^k)$. As $\left|x^k\right| \to \infty$, $\exp\,\left(-Na\,(2\nu)^{-1/2}\,(x^k \,-\, x_0^k)\right) \;\to\; 0$ faster than $(x^k \,-\, x_0^k)^N \to \infty$, hence $\mathbf{b}_k \to\; 0$; blow-up does not occur.

5.1.3. *Bounded energy.* Since the motion of the system occurs in extended cotangent space rather than extended tangent space, evaluation of the following integral will show that the energy is bounded:

$$\int\limits_{\mathbb{R}^n} \left|\mathbf{b}_k\right|^2 dx^k \;<\; C \text{ for all } t \;\geq\; t_0 \text{ and } C \,<\, \infty \tag{5.3}$$

Evaluation of this integral gives

$$\begin{aligned} \textstyle\int_0^\infty \left|\mathbf{b}_k\right|^2 dx^k \;&=\; \sum_{M=1}^{\infty}\sum_{N=1}^{\infty} \left|\tfrac{\sqrt{2\nu}\,\mathbf{C}_M \mathbf{C}_N}{a^{M+N+1}}\right| \left[\tfrac{(M+N)!}{(M+N)^{M+N+1}}\right] \\ &=\; \text{constant}, \end{aligned} \tag{5.4}$$

which is finite for finite $\mathbf{C}_M$, $\mathbf{C}_N$ and where $\left[\frac{(M+N)!}{(M+N)^{M+N+1}}\right] \;\to 0$ for large M, N. Hence the above results imply the function $\left|\mathbf{b}_k\right|^2$ is bounded.

A physically reasonable solution has a bounded energy in field-free space when

$$\int\limits_{\mathbb{R}^{\mathrm{n}}} \left|\mathbf{p}\right|^2 dx \;<\; \text{constant, for all } t \;\geq\; 0 \tag{5.5}$$

since in this case, the energy is proportional to the square of the momentum $\left|\mathbf{p}\right|^2$. The solution $\mathbf{b}_k$ (the gradient of $\mathbf{S}$) can be used as the integrand in $\int_{\mathbb{R}^n} \left|\mathbf{p}\right|^2 dx$ in place of the momentum (the gradient of the action) for proof of a physically reasonable solution. This is based on the fact that both principal functions ($\mathbf{S}$ and the action) can be represented by a family of surfaces with the gradient of the principal function always normal to a surface at a point; the larger the gradient, the slower the fronts representing the surfaces. When the square of the gradient of the principal function is a function of time ($|\mathbf{b}_k(t)|^2$ or $|\mathbf{p}(t)|^2$) it characterizes the motion in field-free space; hence, the square of the gradient of the principle function is proportional to the kinetic energy. Therefore, the fact that $\int_{\mathbb{R}^n} |\mathbf{b}_k|^2 \, dx^k < C$ for all $t \geq 0$ and $C < \infty$, shows the solution is physically reasonable.

5.1.4. *Graphing the solution.* The solution $\mathbf{b}_k$ contains constants $\mathbf{C}_N$ and a, which cannot be graphed without knowledge of these constants. Quantity a is merely a unit constant present to make the argument of the exponential dimensionless; hence, its value 1 sec $^{-1/2}$. Constants $\mathbf{C}_N$ are functions of the constant coefficients $\partial^{\alpha}_{x^k} \partial^{\beta}_t \mathbf{B}_k(0)$, $\partial^{\alpha}_{x^k} \partial^{\beta}_t \mathbf{f}(0)$ and $\partial^{\alpha}_{x^k} \partial^{\beta}_t P(0)$. The procedure to obtain the expansion coefficients is to treat them as parameters and determine them experimentally. This involves fitting the experimental data with the use of these parameters, then designating these evaluated parameters as the characteristic constants for the system. This is a commonly used technique for precise quantum mechanical measurements, for example the older frequency standard work on cesium by means of atomic beam magnetic resonance spectroscopy, where hyperfine structure constants are treated as parameters.

5.1.5. *Solution for* $\mathbf{b}_k$ *when* $\mathbf{f} = 0$. By setting the external force $\mathbf{f} = 0$, $\mathbf{b}_k$ then depends on the expansion coefficients $\partial^{\alpha}_{x^k} \partial^{\beta}_t \mathbf{B}_k(0)$ and $\partial^{\alpha}_{x^k} \partial^{\beta}_t P(0)$. By this procedure it is possible to eliminate some of parameters required to fit experimental data and hence allow a first approximation for determination of some of the required coefficients.

5.2. **Incompressibility.** The divergence equation is the condition for the velocity vector field $\mathbf{v}$ to be divergence-free. If ∂_{x^k} is taken on each side of this equation, the result becomes $div\, \mathbf{B}_k = div\, \mathbf{b}_k = 0$. In differential geometry the divergence of a vector field belonging to $\mathbb{R}^3$ is the source density in the expression for the 3-form on $\mathbb{R}^3$. In the present case the divergence of vector field $\mathbf{b}_k$ belonging to $\mathbb{R}^3$ is the source density in the expression for the 3-form ω^3 on $\mathbb{R}^3$, given by

$$\omega^3 \;=\; (div\;\; \mathbf{b}_k)\;\; \mathbf{d}x^{(1)} \;\wedge\; \mathbf{d}x^{(2)} \;\wedge\; \mathbf{d}x^{(3)} \tag{5.6}$$

where ω^3 characterizes the sources in an elementary parallelepiped with edges $(\varepsilon\,\xi_\alpha,\ \varepsilon\,\xi_\beta,\ \varepsilon\,\xi_\kappa)$ and tangent vectors ξ, where $\mathbf{d}x^{(1)}$, $\mathbf{d}x^{(2)}$, and $\mathbf{d}x^{(3)}$ are basis one-forms on $\mathbb{R}^3$ at point $\left(x^{(1)},\ x^{(2)}, x^{(3)}\right)$, where $\mathbf{d}x^{(1)} \wedge \mathbf{d}x^{(2)} \wedge \mathbf{d}x^{(3)}$ is the volume form, and ε is an arbitrarily small number. In order for $div\ \mathbf{b}_k \;=\; 0$, then the contraction $\mathbf{d}x^{(1)} \wedge \mathbf{d}x^{(2)} \wedge \mathbf{d}x^{(3)}\ (\xi_\alpha,\ \xi_\beta\,,\ \xi_\kappa) \;=\; 0$. For tangent vector ξ_α

$$\xi_\alpha = \left(\frac{dx^{(1)}}{dt}\right)\partial_{x^{(1)}} + \left(\frac{dx^{(2)}}{dt}\right)\partial_{x^{(2)}} + \left(\frac{dx^{(3)}}{dt}\right)\partial_{x^{(3)}} \tag{5.7}$$

and arbitrary tangent vectors ξ_β, ξ_κ , namely

$$\xi_\beta = \beta_{x^{(1)}}\left(\frac{dx^{(1)}}{dt}\right)\partial_{x^{(1)}} + \beta_{x^{(2)}}\left(\frac{dx^{(2)}}{dt}\right)\partial_{x^{(2)}} + \beta_{x^{(3)}}\left(\frac{dx^{(3)}}{dt}\right)\partial_{x^{(3)}} \tag{5.8}$$

and

$$\xi_\kappa = \kappa_{x^{(1)}}\left(\frac{dx^{(1)}}{dt}\right)\partial_{x^{(1)}} + \kappa_{x^{(2)}}\left(\frac{dx^{(2)}}{dt}\right)\partial_{x^{(2)}} + \kappa_{x^{(3)}}\left(\frac{dx^{(3)}}{dt}\right)\partial_{x^{(3)}} \tag{5.9}$$

it results that the contraction $\mathbf{d}x^{(1)} \wedge \mathbf{d}x^{(2)} \wedge \mathbf{d}x^{(3)}\ (\xi_\alpha,\ \xi_\beta\,,\ \xi_\kappa) \;=\; 0$ implies

$$\beta_{x^{(1)}}\left(\kappa_{x^{(2)}} \;-\; \kappa_{x^{(3)}}\right) \;+\; \beta_{x^{(2)}}\left(\kappa_{x^{(3)}} \;-\; \kappa_{x^{(1)}}\right) \;+\; \beta_{x^{(3)}}\left(\kappa_{x^{(1)}} \;-\; \kappa_{x^{(2)}}\right) \;=\; 0 \tag{5.10}$$

where $(\partial_{x^{(1)}},\ \partial_{x^{(2)}},\ \partial_{x^{(3)}})$ are basis tangent vectors belonging to $\mathbb{R}^3$. The condition on the β_{x^k} and κ_{x^k} implies that the vectors ξ_β and ξ_κ are not entirely arbitrary; the condition distorts the parallelepiped $(\xi_\alpha,\ \xi_\beta,\ \xi_\kappa)$ to allow the gradient of $\mathbf{v}$ to be divergence-free.

This condition is strictly true from a mathematical point of view, but involves assumptions which have not been adequately studied in terms of physical reasonableness. However, if the volume of this parallelepiped is in the same region of space in which the motion of the system occurs, then the requirements of the divergence equation are fulfilled.

5.3. **The Euler equation.** Referring back to the introduction, the Euler equations are the first three equations when the viscosity is zero. Hence, it would seem that the solution to the Euler equations can be obtained by setting the viscosity to zero in the solution to the Navier-Stokes dynamic equation. However, this procedure depends on the use of the solution $x^k = x_0^k \pm \sqrt{2\nu(t - t_0)}$ to change the functional dependence from $t - t_0$ to $x^j - x_0^j$. If the viscosity is set to zero, $\frac{dx^k}{dt} = \frac{\nu}{x^k - x_0^k}$ predicts x^j to be independent of time; therefore, this procedure cannot be used. The present solution to the Navier-Stokes equations can be used as an approximation to solutions for the Euler equations only in the case of exceptionally small, but non-vanishing viscosity.

6. Vortex vector, Lagrangian

The vortex vector $\mathbf{R}$, the vector which gives the direction of the system change, is obtained by substituting the coordinate values for the coordinates of the tangent vector ξ; hence the vortex vector is

$$\begin{aligned} \mathbf{R} &= -\ (\partial_{x^k}\boldsymbol{\Omega})\ \partial_{\mathbf{b}_k}\ +\ (\partial_{\mathbf{b}^k}\boldsymbol{\Omega})\ \partial_{x^k}\ +\ \partial_t \\ &= -\ (\partial_{x^k}\boldsymbol{\Omega})\ \partial_{\mathbf{b}_k}\ +\ \nu\left(x^k - x_0^k\right)^{-1}\partial_{x^k}\ +\ \partial_t \end{aligned} \tag{6.1}$$

To obtain the Lagrangian for the system, the principal differential one-form $\mathbf{dS}$ is contracted with the vortex vector giving

$$\begin{aligned} \mathbf{dS}\,(\mathbf{R}) &= \mathbf{b}_k\,\partial_{\mathbf{b}_k}\boldsymbol{\Omega}\ -\ \boldsymbol{\Omega} \\ &= \mathbf{b}_k\,\nu\,\left(x^k - x_0^k\right)^{-1} -\ \boldsymbol{\Omega} \end{aligned} \tag{6.2}$$

This equation can be made more detailed by substitution for $\mathbf{b}_k(t)$ and $\boldsymbol{\Omega}$. Note that the same technique for obtaining the Lagrangian has been demonstrated for Hamiltonian mechanics, geometric optics, irreversible thermodynamics, black hole mechanics, and electromagnetic and classical string field theory [11].

7. Conclusion

The technique employed in this paper for solving the Navier-Stokes model for fluid dynamics in the case of incompressible fluids was to transform the dynamic equation into a differential one-form, and then use methods from exterior calculus to generate a pair of differential equations and a vortex vector satisfying Hamiltonian geometry. This pair of equations was solved for the position x^k as a function of time and for the conjugate $\mathbf{b}_k$ to the position as a function of time.

The value of the solution $\mathbf{b}_k$ as $\left|x^k\right| \to \infty$ was shown to be finite, hence the solution is bounded; blow-up does not occur. The solution was shown to be physically reasonable since the square of the gradient of the principal function is bounded. It is not possible to plot the solution without knowledge of some of the constants contained in the solution, but these constants can be treated as parameters and evaluated experimentally. One example of this procedure is the older frequency standard work on cesium atom using atomic beam magnetic resonance spectroscopy, where hyperfine interaction constants are treated as parameters and determined with experimental data.

The gradient was taken on each side of the equation for the divergence of the velocity, $div\ \mathbf{v} = 0$, resulting in an equation for the divergence of the gradient of the velocity. Then the 3-form corresponding to the divergence of the gradient of the velocity was contracted with a triple of tangent vectors and set to zero. As a result, a condition was placed on arbitrary tangent vectors in $\mathbb{R}^3$, distorting the volume where the motion of the system occurs in a manner that restricts the gradient of the velocity to be divergence-free.

The vortex vector (characteristic tangent vector) giving the direction of the system change was constructed by substituting coordinate values for coordinates of a basic tangent vector in $T(T^*M_x)$. By contracting the principal differential one-form defining the system with the vortex vector, the Lagrangian was obtained.

The present solution to the Navier-Stokes equations is based on several assumptions, namely, (1) assuming the gradient $\mathbf{B}_k$ of the velocity, the pressure P, the force $\mathbf{f}$, and the exponential part of the series solution for $\mathbf{b}_k$ are all smooth functions, with all but $\mathbf{B}_k$ (infinite order expansion) expanded to second order in a Taylor's series, (2) assuming the cross terms in $\partial_{x^k}\mathbf{f}$ can be neglected and (3) assuming a certain condition on the coordinates of two otherwise arbitrary tangent vectors in $\mathbb{R}^3$.

References

[1] O. Ladyzhenskaya, *The mathematical theory of viscous incompressible flows*, 2nd edition, Gordon and Breach, 1969.

[2] A. Bertozzi and A. Majda, *Vorticity and incompressible flows*, Cambridge U. Press, 2000.

[3] P. Constantin, *Some open problems and research directions in the mathematical study of fluid dynamics*, Univ. Chicago Preprint. (2000), 1–11.

[4] J. Leray, *Sur le mouvement d'un liquide visquez emplissent l'espace*, Acta. Math. J. **63** (1934), 193–248.

[5] V. Scheffer, em An inviscid flow with compact support in spacetime, J. Geom. Analysis **3** (1993), 343–401.

[6] L. Caffarelli, R. Kohn, and L. Nirenberg, *Partial regularity of suitable weak solutions*, Pure and Appl. Math. **35**, (1982), 771–831.

[7] A. Shnirelman, *On the nonuniqueness of weak solutions of the Euler equation*, Comm. Pure and Appl. Math. **50**, (1997), 1260–1286.

[8] F. H. Lin, *A new proof of the Caffarelli-Kohn-Nirenberg theorem*, Comm. Pure and Appl. Math. **51**, (1998), 241–257.

[9] H. Amann, *On the strong solvability of the Navier-Stokes equations*, J. Math. Fluid Mech. **2**, (2000), 16–98.

[10] C. L. Fefferman, *Existence and smoothness of the Navier-Stokes equation*, www.claymath.org (2000), 1–5.

[11] Troy L. Story, *Dynamics on differential one-forms*, J. Math. Chem. **29** (2001), 85–96.

[12] Troy L. Story, *Navier-Stokes dynamics on a differential one-form*, MSRI Preprint **2003-015** (2003), 1–33.

Morehouse College, 830 Westview Drive, Atlanta, GA 30314-3773 USA
E-mail address: `tstory@morehouse.edu`