Asymptotic Properties of Solutions to the Homogeneous Navier-Stokes Equations in R^3

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Abstract. We show as the main result of the paper that if w is a weak global solution of homogeneous Navier-Stokes equations satisfying the strong energy inequality and $\beta \in (3/4, 1)$, then there exist $t_0 \geq 0$, $C_0 \geq 0$ and $\delta_0 > 0$ such that

$$\frac{\|A^\beta w(t)\| + \|w(t)\|}{\|A^\beta w(t+\delta)\| + \|w(t+\delta)\|} \leq C_0$$

for all $t \geq t_0$ and $\delta \in [0, \delta_0]$. So, measuring w in the graph norm $\|A^\beta w\| + \|w\|$ and starting at time t_0, we exclude fast decays of w on short time intervals.

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1. Introduction

In this paper we study some asymptotic properties of weak global solutions of the Cauchy problem for the Navier-Stokes equations in the space domain $\Omega = \mathbf{R}^3$:

$$\frac{\partial w}{\partial t} - \Delta w + w \cdot \nabla w + \nabla p = 0 \quad \text{in } \mathbf{R}^3 \times (0, \infty), \tag{1}$$

$$\nabla \cdot w = 0, \quad w(x, 0) = w_0(x), \tag{2}$$

with $w_0 \in L^2(\mathbf{R}^3)^3$, $\nabla \cdot w_0 = 0$. By a weak global solution w we mean a function

$$w \in C_w([0, \infty); L^2(\mathbf{R}^3)^3) \cap L^2_{loc}((0, \infty); W^{1,2}(\mathbf{R}^3)^3) \tag{3}$$

with $\nabla \cdot w = 0$, which satisfies the integral relation

$$(w(t), \phi(t)) + \int_0^t \left[-\left(w(s), \frac{\partial \phi}{\partial s}(s) \right) + (\nabla w(s), \nabla \phi(s)) + (w(s) \cdot \nabla w(s), \phi(s)) \right] ds = (w_0, \phi(0)), \quad t > 0,$$

for all smooth vector fields ϕ with compact support and $\nabla \cdot \phi = 0$. $(\cdot, \cdot)$ denotes the scalar product and $\| \cdot \|$ denotes the norm in $L^2(\mathbf{R}^3)^3$. C_w denotes the space of weakly continuous functions. The existence of weak global solutions is well known (see [1] or [7]).

From now on we suppose that the solutions satisfy the strong energy inequality

$$\|w(t)\|^2 + 2 \int_s^t \|\nabla w(\sigma)\|^2 d\sigma \leq \|w(s)\|^2$$

for $s = 0$ and almost all $s > 0$, and all $t \geq s$.

It is known (see [4]) that the global weak solutions with the strong energy inequality become strong after a finite time:

$$\text{there is some } T_0 = T_0(\|w_0\|) \geq 0, \text{ such that } w \in C([T_0, \infty); L^p) \text{ for every } p \in [2, \infty). \tag{4}$$

The following theorem is the main result of the paper.

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Theorem 1 *Let $\beta \in (3/4, 1)$, $w_0 \in L^2(\mathbf{R}^3)^3$, $\nabla \cdot w_0 = 0$, $w_0 \neq 0$. Let w be a weak global solution of (1) and (2) satisfying the strong energy inequality and let T_0 be from (4). Then there exist $C_0 > 1$ and $\delta_0 \in (0,1)$ such that*

$$\frac{\|A^\beta w(t)\| + \|w(t)\|}{\|A^\beta w(t+\delta)\| + \|w(t+\delta)\|} \leq C_0,\ \forall t \geq T_0 + 2,\ \forall \delta \in (0, \delta_0]. \tag{5}$$

Let us present in this connection a theorem proved in [5]:

Theorem 2 *Let $w_0 \in D(A)$, $w_0 \neq 0$. Let w be a strong global solution of the Navier-Stokes equations (1) and (2) in a smooth and bounded domain $\Omega \subset \mathbf{R}^3$ endowed with the homogeneous Dirichlet boundary conditions. If $k, l, m \in N \cup \{0\}$, then there exist $C = C(k,l,m) > 1$, $t_0 = t_0(k,l,m) \geq 0$ and $\delta_0 \in (0,1)$ such that*

$$\left\| \frac{d^k w}{dt^k}(t) \right\|_{m,2} \leq C \left\| \frac{d^l w}{dt^l}(t+\delta) \right\|,\ \forall t \geq t_0,\ \forall \delta \in [0, \delta_0].$$

It is clear that the result from Theorem 2 for the case of a bounded domain is stronger than the result presented in Theorem 1. In this paper we do not have the ambition to prove an analogical version of Theorem 2 for the whole space $\mathbf{R}^3$ and Theorem 1 is only the first step in this direction. Let us also remark that unlike the case of a bounded domain, we do not have the inequality $\|B(w,w)\| \leq \|A^{1/2}w\|\ ||A^\beta w||$, which must be replaced by $\|B(w,w)\| \leq \|A^{1/2}w\|\ (||A^\beta w|| + ||w||)$ (see the second section for the notation). It leads to the form of the left hand side in (5). Therefore, Theorem 1 says that if we measure the solution w in the graph norm $\|A^\beta \cdot\| + \|\cdot\|$, then, starting at time $T_0 + 2$, fast decays of w on short time intervals are excluded. Let us remark, that the question of fast decays of solutions on short time intervals was raised and studied in [3].

2. Notations

$L^q = L^q(\mathbf{R}^3), q \geq 1$: the Lebesgue spaces with the norm $\|\cdot\|_q$. If $q = 2$, we denote $\|\cdot\| = \|\cdot\|_2$.
$W^{s,q} = W^{s,q}(\mathbf{R}^3), s \geq 0, q \geq 2$: the Sobolev spaces endowed with the norm $\|\cdot\|_{s,q}$.
L^2_σ: the closure of $\{\varphi \in C_0^\infty(\mathbf{R}^3)^3; \nabla \cdot \varphi = 0\}$ in $L^2(\mathbf{R}^3)^3$.
P_σ: orthogonal projection of $L^2(\mathbf{R}^3)^3$ onto L^2_σ.
A: the Stokes operator on L^2_σ, $\mathcal{D}(A) = \{u \in W^{2,2}; \nabla \cdot u = 0\}$, $Au = -\Delta u,\ \forall u \in \mathcal{D}(A)$.
$A^\alpha, \alpha \geq 0$: the fractional powers of the Stokes operator.
$e^{-At}, t \geq 0$: the Stokes semigroup generated by the Stokes operator $-A$.
$B(w,w) = P_\sigma(w \cdot \nabla w)$.
the graph norm $|||w|||_\beta = \|A^\beta w\| + \|w\|$.

3. Auxiliary results

At first, let us present several known properties of weak global solutions which will be used in this paper. According to [8], if w is a weak global solutions of (1) and (2) satisfying the strong energy inequality and if $w_0 \in L^2(\mathbf{R}^3)^3 \cap L^p(\mathbf{R}^3)^3$ with $p \in [1,2)$ then

$$\|w(t)\| \leq C(1+t)^{-\frac{6-3p}{4p}}, \quad t \geq 0.$$

Using the results from [2] and [8] we can disregard the assumption $p \in [1,2)$ and derive that

$$\|w(t)\| \leq C(1+t)^{-\mu}, \quad t \geq 0$$

for any $\mu \in (0, 1/2)$ where C possibly depends on μ. Applying now a result from [4], we get that for $m, k \in N$ and $\mu \in (0,1/2)$ there is $C_{m,k} = C_{m,k}(\mu, C)$, independent of T_0, such that

$$\left\| D^m \frac{d^k w}{dt^k}(t) \right\| \leq C_m (t - T_0 - 2)^{-\mu - m/2 - k}, \quad t \geq T_0 + 1. \tag{6}$$

The following inequality can be derived as a consequence of Hölder inequality and Lemma 2.4.3 form [6]: if $\gamma \in [3/4, 1)$ then there exists $c > 0$ such that

$$\|B(u,u)\| \le c\|A^{1/2}u\| \, |||u|||_\gamma, \ \forall u \in \mathcal{D}(A^\gamma). \tag{7}$$

Finally, if $\gamma \in [3/4, 1)$ then there exists $c > 0$ such that

$$\|A^{1/2}u\| \le c|||u|||_\gamma, \ \forall u \in \mathcal{D}(A^\gamma). \tag{8}$$

4. Proofs of the main results

We prove at first the following lemma. Its corollary is substantial for the proof of Theorem 1.

Lemma 3 *If* $w \in \mathcal{D}(A^\alpha)$, $w \neq 0$, $t \ge 0$ *and* $0 \le \beta \le \alpha$ *then*

$$\frac{\|A^\alpha w\|}{\|A^\beta e^{-At}w\|} \ge \frac{\|A^\alpha e^{-At}w\|}{\|A^\beta e^{-2At}w\|}.$$

Proof: Let E_λ, $\lambda \ge 0$ be the resolution of identity for the Stokes operator A. Then

$$\|A^\beta e^{-At}w\|^2 = \int_0^\infty \lambda^{2\beta} e^{-2\lambda t} d\|E_\lambda w\|^2, \quad t \ge 0. \tag{9}$$

By the Hölder inequality we get easily that

$$\|A^\beta e^{-At}w\|^2 = \int_0^\infty \lambda^{2\beta} e^{-2\lambda t} d\|E_\lambda w\|^2 \le$$
$$\left(\int_0^\infty \lambda^{2\beta} d\|E_\lambda w\|^2\right)^{1/2} \left(\int_0^\infty \lambda^{2\beta} e^{-4\lambda t} d\|E_\lambda w\|^2\right)^{1/2} = \|A^\beta w\| \|A^\beta e^{-2At}w\|$$

and immediately

$$\frac{\|A^\beta w\|}{\|A^\beta e^{-At}w\|} \ge \frac{\|A^\beta e^{-At}w\|}{\|A^\beta e^{-2At}w\|}. \tag{10}$$

We will show further that the function $t \mapsto \|A^\alpha e^{-At}w\|^2/\|A^\beta e^{-At}w\|^2$ is non-increasing. Firstly, for every $\gamma \ge 0$

$$\frac{d}{dt}\|A^\gamma e^{-At}w\|^2 = -2\|A^{\gamma+1/2}e^{-At}w\|^2, \quad t > 0$$

and therefore

$$\frac{d}{dt}\frac{\|A^\alpha e^{-At}w\|^2}{\|A^\beta e^{-At}w\|^2} = \frac{2\|A^\alpha e^{-At}w\|^2\|A^{\beta+1/2}e^{-At}w\|^2 - 2\|A^{\alpha+1/2}e^{-At}w\|^2\|A^\beta e^{-At}w\|^2}{\|A^\beta e^{-At}w\|^4}, \quad t > 0.$$

Further,

$$\|A^\alpha e^{-At}w\|^2\|A^{\beta+1/2}e^{-At}w\|^2 \le \|A^{\alpha+1/2}e^{-At}w\|^2\|A^\beta e^{-At}w\|^2,$$

as follows from the moment inequality

$$\|A^y u\| \le \|A^z u\|^{\frac{x-y}{x-z}} \|A^x u\|^{\frac{y-z}{x-z}},$$

which holds for every $0 \le z < y < x$ and $u \in D(A^x)$. So,

$$\frac{d}{dt}\frac{\|A^\alpha e^{-At}w\|^2}{\|A^\beta e^{-At}w\|^2} \le 0, \quad t > 0$$

and due to the continuity from the right at 0 we get that the above mentioned function is non-increasing. It means especially, that

$$\frac{\|A^\alpha w\|^2}{\|A^\beta w\|^2} \geq \frac{\|A^\alpha e^{-At} w\|^2}{\|A^\beta e^{-At} w\|^2}, \quad t \geq 0. \tag{11}$$

Using now (10) and (11), we get

$$\frac{\|A^\alpha w\|}{\|A^\beta e^{-At} w\|} = \frac{\|A^\alpha w\|}{\|A^\beta w\|} \frac{\|A^\beta w\|}{\|A^\beta e^{-At} w\|} \geq \frac{\|A^\alpha e^{-At} w\|}{\|A^\beta e^{-At} w\|} \frac{\|A^\beta e^{-At} w\|}{\|A^\beta e^{-2At} w\|} = \frac{\|A^\alpha e^{-At} w\|}{\|A^\beta e^{-2At} w\|},$$

which completes the proof of the lemma. ◯

Corollary 4 *If $w \in \mathcal{D}(A^\alpha)$, $w \neq 0$, $t \geq 0$ and $0 \leq \beta \leq \alpha$ then*

$$\frac{|||w|||_\alpha}{|||e^{-At} w|||_\beta} \geq \frac{|||e^{-At} w|||_\alpha}{|||e^{-2At} w|||_\beta}.$$

Proof: The proof of the corollary follows immediately from Lemma 3 and from the elementary fact that if $\frac{\alpha_1}{\beta_1} \geq \frac{\beta_1}{\gamma_1}$ and $\frac{\alpha_2}{\beta_2} \geq \frac{\beta_2}{\gamma_2}$ for some positive $\alpha_i, \beta_i, \gamma_i$, $i = 1, 2$, then $\frac{\alpha_1+\alpha_2}{\beta_1+\beta_2} \geq \frac{\beta_1+\beta_2}{\gamma_1+\gamma_2}$. ◯

Throughout the proof of Theorem 1 c denotes the generic constant which can change from line to line.

Proof of Theorem 1: Let the assumptions of Theorem 1 be fulfilled. We will use the method from [5]. We denote

$$H = \max_{t \in [T_0+2, \infty)} |||w(t)|||_\beta.$$

It follows from (6) that $H < \infty$. Since $\|A^\beta w(t)\| \neq 0$ for all $t \in [T_0 + 2, \infty)$, there exist $C_0' > 1$ and $\delta_0' \in (0, 1)$ such that

$$\frac{|||w(t)|||_\beta}{|||w(t + \delta)|||_\beta} \leq C_0', \ \forall t \in [T_0 + 2, T_0 + 4], \ \forall \delta \in (0, \delta_0']. \tag{12}$$

We set now $D_0 = 6C_0'$ and let $\delta_0 \in (0, \delta_0']$ be such a number that

$$4Hc \left(D_0 e^{\frac{5D_0}{2(D_0-1)}} \right)^3 \left(\frac{\delta_0^{1-\beta}}{1 - \beta} + \delta_0 \right) \leq 1. \tag{13}$$

We will prove now the following proposition:

Proposition P: Let $t > T_0 + 4$, $\delta \in (0, \delta_0]$. Let further

$$\frac{|||w(t)|||_\beta}{|||w(t + \delta)|||_\beta} = C \in \left(D_0, D_0 e^{\frac{5D_0}{2(D_0-1)}} \right) \tag{14}$$

and

$$|||w(t)|||_\beta \geq |||w(s)|||_\beta, \ \forall s \in [t, t + \delta]. \tag{15}$$

Then there exists $t^* \in [t - \delta, t)$ such that

$$\frac{|||w(t^*)|||_\beta}{|||w(t)|||_\beta} \geq \frac{|||w(t)|||_\beta}{|||w(t + \delta)|||_\beta} \frac{\left(1 - \frac{|||w(t)|||_\beta}{2H}\right)^2}{\left(1 + \frac{|||w(t)|||_\beta}{2H}\right)}. \tag{16}$$

Proof of Proposition P: Let (14) and (15) be fulfilled. We can suppose that

$$\max_{s \in [t-\delta, t]} |||w(s)|||_\beta < C|||w(t)|||_\beta, \tag{17}$$

because otherwise (16) would be satisfied immediately. We begin with the integral representation of w:

$$w(t + \delta) = e^{-A\delta} w(t) + \int_0^\delta e^{-A(\delta - s)} B(w(t + s), w(t + s)) \, ds, \tag{18}$$

$$w(t) = e^{-A\delta} w(t-\delta) + \int_0^\delta e^{-A(\delta - s)} B(w(t-\delta+s), w(t-\delta+s))\, ds. \tag{19}$$

Applying gradually (7), (8) and (17) we obtain that

$$|||w(t) - e^{-A\delta} w(t-\delta)|||_\beta \leq$$
$$\int_0^\delta c((\delta-s)^{-\beta}+1)\, \|B(w(t-\delta+s), w(t-\delta+s)\|\, ds \leq$$
$$\int_0^\delta c((\delta-s)^{-\beta}+1)\, \|A^{1/2} w(t-\delta+s)\|\, |||w(t-\delta+s)|||_\beta\, ds =$$
$$|||w(t)|||_\beta \int_0^\delta c((\delta-s)^{-\beta}+1)\, \frac{||A^{1/2} w(t-\delta+s)||}{|||w(t-\delta+s)|||_\beta} \times$$
$$\frac{|||w(t-\delta+s)|||_\beta}{|||w(t)|||_\beta}\, |||w(t-\delta+s)|||_\beta\, ds \leq |||w(t)|||_\beta^2 c C^2 \int_0^\delta ((\delta-s)^{-\beta}+1)\, ds.$$

So we can get from (13) and (14) that

$$|||w(t) - e^{-A\delta} w(t-\delta)|||_\beta \leq |||w(t)|||_\beta \left[2HcC^2 \left(\frac{\delta_0^{1-\beta}}{1-\beta} + \delta_0 \right) \right] \frac{|||w(t)|||_\beta}{2H} \leq$$
$$|||w(t)|||_\beta\, \frac{|||w(t)|||_\beta}{2H} \tag{20}$$

and also

$$|||w(t) - e^{-A\delta} w(t-\delta)|||_\beta \leq |||w(t+\delta)|||_\beta \left[4HcC^3 \left(\frac{\delta_0^{1-\beta}}{1-\beta} + \delta_0 \right) \right] \times$$
$$\frac{|||w(t)|||_\beta}{4H} \leq |||w(t+\delta)|||_\beta\, \frac{|||w(t)|||_\beta}{4H}. \tag{21}$$

(21) now gives immediately that

$$|||e^{-A\delta} w(t) - e^{-2A\delta} w(t-\delta)|||_\beta \leq |||w(t+\delta)|||_\beta\, \frac{|||w(t)|||_\beta}{4H}. \tag{22}$$

It follows from (18), (7), (8), (14), (15) and (13) that

$$|||w(t+\delta) - e^{-A\delta} w(t)|||_\beta \leq$$
$$\int_0^\delta (c(\delta-s)^{-\beta}+1) ||A^{1/2} w(t+s)||\, |||w(t+s)|||_\beta\, ds =$$
$$|||w(t+\delta)|||_\beta \int_0^\delta (c(\delta-s)^{-\beta}+1)\, \frac{||A^{1/2} w(t+s)||}{|||w(t+s)|||_\beta}\, \frac{|||w(t+s)|||_\beta}{|||w(t+\delta)|||_\beta} \times$$
$$|||w(t+s)|||_\beta\, ds \leq |||w(t+\delta)|||_\beta\, |||w(t)|||_\beta\, c\, C \int_0^\delta ((\delta-s)^{-\beta}+1)\, ds =$$
$$|||w(t+\delta)|||_\beta \left[4HcC \left(\frac{\delta_0^{1-\beta}}{1-\beta} + \delta_0 \right) \right] \frac{|||w(t)|||_\beta}{4H} \leq |||w(t+\delta)|||_\beta \frac{|||w(t)|||_\beta}{4H}. \tag{23}$$

(22) and (23) provide the estimate

$$||||e^{-2A\delta} w(t-\delta) - w(t+\delta)|||_\beta \leq |||e^{-2A\delta} w(t-\delta) - e^{-A\delta} w(t)|||_\beta +$$
$$|||e^{-A\delta} w(t) - w(t+\delta)|||_\beta \leq |||w(t+\delta)|||_\beta \frac{|||w(t)|||_\beta}{2H}. \tag{24}$$

It follows now from Corollary 4 and (20) and (24) that

$$|||w(t-\delta)|||_\beta \geq \frac{|||e^{-A\delta}w(t-\delta)|||_\beta^2}{|||e^{-2A\delta}w(t-\delta)|||_\beta} \geq \frac{|||w(t)|||_\beta^2\left(1-\frac{|||w(t)|||_\beta}{2H}\right)^2}{|||w(t+\delta)|||_\beta\left(1+\frac{|||w(t)|||_\beta}{2H}\right)}.$$

If we put $t^* = t-\delta$, (16) is proved. The proof of Proposition P is finished and we can continue in the proof of Theorem 1.

Let us fix $t \in [T_0+2,\infty)$, $\delta \in (0,\delta_0]$ and suppose that

$$|||w(t)|||_\beta > H/D_0 \text{ and} \tag{25}$$

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} \geq D_0\frac{1+1/2}{(1-1/2)^2} = 6D_0. \tag{26}$$

Since $D_0 > C_0'$ and $\delta_0 \leq \delta_0'$, it follows from (12) and (26) that $t > T_0+4$. We can also suppose without loss of generality that

$$|||w(t)|||_\beta = \max_{s\in[t,t+\delta]} |||w(s)|||_\beta$$

and (by possible decreasing of δ)

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} = 6D_0.$$

Let us notice that $6D_0 < D_0 e^{\frac{5D_0}{2(D_0-1)}}$ ($D_0 > 1$) and the conditions (14) and (15) are satisfied. By Proposition P there exists $t^* \in [t-\delta, t)$ so that

$$\frac{|||w(t^*)|||_\beta}{|||w(t)|||_\beta} \geq \frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta}\frac{\left(1-\frac{|||w(t)|||_\beta}{2H}\right)^2}{\left(1+\frac{|||w(t)|||_\beta}{2H}\right)} \geq 6D_0\frac{(1-1/2)^2}{1+1/2} = D_0.$$

Thus, by (25), $|||w(t^*)|||_\beta \geq D_0|||w(t)|||_\beta > D_0H/D_0 = H$ and it is the contradiction with the definition of H. Let $D_1 = 6D_0$. We proved

Proposition P_1**:** Let $t \in [T_0+2,\infty)$, $\delta \in (0,\delta_0]$ and $|||w(t)|||_\beta > H/D_0$. Then

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} < D_1.$$

We define now

$$D_n = D_{n-1}\frac{1+\frac{1}{2D_0D_1\ldots D_{n-2}}}{\left(1-\frac{1}{2D_0D_1\ldots D_{n-2}}\right)^2},\ \forall n \in N, n \geq 2. \tag{27}$$

We have

$$6 < D_0 < D_1 < \ldots < D_{n-1} < D_n,\ \forall n \in N, \tag{28}$$

$$D_n = 6D_0\prod_{j=0}^{n-2}\frac{1+\frac{1}{2D_0D_1\ldots D_j}}{\left(1-\frac{1}{2D_0D_1\ldots D_j}\right)^2} \leq D_0\prod_{j=0}^{n-1}\frac{1+\frac{1}{2D_0^j}}{\left(1-\frac{1}{2D_0^j}\right)^2},\ \forall n \geq 2$$

and

$$\ln D_n \leq \ln D_0 + \sum_{j=0}^{n-1}\ln\left(1+\frac{1}{2D_0^j}\right) - 2\ln\left(1-\frac{1}{2D_0^j}\right),\ \forall n \geq 1.$$

It follows from the elementary properties of the function $x \to \ln(1+x)$ that

$$\ln D_n < \ln D_0 + \sum_{j=0}^{n-1} \left(\frac{1}{2D_0^j} + 4\frac{1}{2D_0^j} \right) < \ln D_0 + \frac{5D_0}{2(D_0-1)}$$

and

$$D_n < D_0 e^{\frac{5D_0}{2(D_0-1)}}, \; \forall n \in N. \tag{29}$$

We will prove now that for every $n \in N$ the following proposition is valid:
Proposition P_n: Let $t \in [T_0+2, \infty)$, $\delta \in (0, \delta_0]$ and

$$|||w(t)|||_\beta > \frac{H}{D_0 D_1 \dots D_{n-1}}.$$

Then

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} < D_n.$$

We will use the mathematical induction. Proposition P_1 has already been proved. Let us suppose that P_n holds for some $n \in N$ and we will prove the validity of P_{n+1}. Thus, let $t \in [T_0+2, \infty)$, $\delta \in (0, \delta_0]$ and $|||w(t)|||_\beta > H/D_0 D_1 \dots D_n$. We can suppose that

$$|||w(t)|||_\beta \le H/D_0 D_1 \dots D_{n-1}, \tag{30}$$

since otherwise we would apply Proposition P_n, get $|||w(t)|||_\beta / |||w(t+\delta)|||_\beta < D_n < D_{n+1}$ and Proposition P_{n+1} would be proved. We suppose by contradiction that

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} \ge D_{n+1}. \tag{31}$$

It follows then from (12) and (28) that $t > T_0 + 4$. We can suppose without loss of generality that

$$|||w(t)|||_\beta \ge |||w(s)|||_\beta, \; \forall s \in [t, t+\delta] \tag{32}$$

and also

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} = D_{n+1}. \tag{33}$$

Due to (28), (29), (32) and (33) we see that (14) and (15) are satisfied. Therefore, Proposition P, (33), (30) and (27) yield that there exists $t^* \in [t-\delta, t)$ so that

$$\frac{|||w(t^*)|||_\beta}{|||w(t)|||_\beta} \ge \frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} \frac{\left(1 - \frac{|||w(t)|||_\beta}{2H}\right)^2}{\left(1 + \frac{|||w(t)|||_\beta}{2H}\right)} \ge D_{n+1} \frac{\left(1 - \frac{1}{2D_0 D_1 \dots D_{n-1}}\right)^2}{\left(1 + \frac{1}{2D_0 D_1 \dots D_{n-1}}\right)} = D_n. \tag{34}$$

If we use the assumptions of Proposition P_{n+1} we obtain that

$$|||w(t^*)|||_\beta \ge D_n |||w(t)|||_\beta > D_n \frac{H}{D_0 D_1 \dots D_n} = \frac{H}{D_0 D_1 \dots D_{n-1}}$$

and according to Proposition P_n we get that

$$\frac{|||w(t^*)|||_\beta}{|||w(t)|||_\beta} < D_n,$$

which is the contradiction to (34). Therefore, (31) does not hold, in fact

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} < D_{n+1}$$

and Proposition P_{n+1} is proved. We proved that Proposition P_n holds for every $n \in N$.

We now finish the proof of Theorem 1. Let us fix $t \in [T_0 + 2, \infty)$ and $\delta \in (0, \delta_0]$. Then there exists $n \in N$ so that $|||w(t)|||_\beta > \frac{H}{D_0 D_1 \ldots D_{n-1}}$. By Proposition P_n and by (29) we get that

$$\frac{|||w(t)|||_\beta}{|||w(t+\delta)|||_\beta} < D_n < D_0 e^{\frac{5D_0}{2(D_0-1)}}.$$

Setting $C_0 = D_0 e^{\frac{5D_0}{2(D_0-1)}}$ the proof of Theorem 1 is complete. ◯

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