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THE NAVIER-STOKES FLOW FOR GLOBALLY LIPSCHITZ CONTINUOUS INITIAL DATA

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Abstract. Consider the Cauchy problem of the incompressible Navier-Stokes equations with initial velocity $U_0$ of the form $U_0(x) := u_0(x) - f(x)$, where $f$ is a Lipschitz function and $u_0 \in L^p_\sigma(\mathbb{R}^n)$. It is shown that under these assumptions the equations of Navier-Stokes admit a unique local in time mild solution.

1. Introduction

We consider the flow of an incompressible, viscous fluid in the whole space $\mathbb{R}^n$, $n \geq 2$ described by the Cauchy problem for the system of the Navier-Stokes equations, i.e.,

\begin{equation}
\left\{
\begin{array}{ll}
U_t - \Delta U + (U, \nabla)U + \nabla P = F, & \text{in } \mathbb{R}^n \times (0, T),
\n\nabla \cdot U = 0, & \text{in } \mathbb{R}^n \times (0, T),
\nU|_{t=0} = U_0 & \text{in } \mathbb{R}^n.
\end{array}
\right.
\end{equation}

Here, $U = (U^1, \ldots, U^n)$ and $P$ represent the unknown velocity and the unknown pressure of the fluid; $U_0$ is the given initial velocity, and $F$ is a given external force term.

There is a vast literature on existence of solutions of (1.1) in $\mathbb{R}^n$, see e.g. [1, 7, 9, 12, 16, 19, 22]. All these results assume that the initial data decay as $|x| \to \infty$. In particular, when $F = 0$, it is well known that there exists a locally-in-time smooth solution to (1.1) provided the initial velocity $U_0$ belongs to $L^p_\sigma(\mathbb{R}^n)$ and $p \geq n$ (see e.g. [15, 19]).

On the other hand, there is strong interest in equation (1.1) for initial data which do not decay at infinity. For results in this direction, we refer to [5, 6, 13] and [8]. Also, H. Okamoto [24] showed that for certain concrete flow problems there exist exact solutions to (1.1) which have the property that $u$ grows linearly as $|x| \to \infty$.

In this paper, we consider initial data of the form

\begin{equation}
U_0(x) = u_0(x) - f(x), \quad x \in \mathbb{R}^n,
\end{equation}

where $u_0 \in L^p(\mathbb{R}^n)^n$ satisfies $\nabla \cdot u_0 = 0$ and $f$ fulfills the following three conditions:

(H1) $\nabla \cdot f = 0$,

(H2) $\Delta f \in L^p_\sigma$,

(H3) $\exists \Pi : \text{ scalar function s.t. } (f, \nabla)f + \nabla \Pi \in L^p_\sigma$.

The particular case where $f(x) = Mx$ was considered in [17]. Here $M$ denotes a real $n \times n$ matrix having $\text{tr } M = 0$. It was shown that this case there exists a unique, local solution to (1.1). It was also shown that this solution is analytic in the spatial variables provided $M$ is skew-symmetric. In this paper, we generalize the result of [17] to the case of Lipschitz continuous functions $f$ satisfying (H1), (H2) and (H3).

For the time being consider again the case where $f(x) = Mx$. Then it is known that (1.1) admits many exact solutions, which are studied e.g. in [10, 21, 25]. In fact, let $f$ be of the form

\[ f(x) = Mx. \]
\[ f(x) = Mx + V, \] where \( M = (m_{ij})_{i,j} \) is an \( n \times n \) real-valued constant matrix satisfying that \( \text{tr} M = 0 \) and such that \( M^2 \) is symmetric. Moreover, let \( V \) be a vector. Then \( \Delta f = 0 \) and

\[ \text{tr} M = 0 \iff (H1) \quad \text{and} \quad M^2 \text{ is symmetric} \iff (H3). \]

In fact, take \( \Pi = \frac{1}{2}(M^2x, x) + (V, M^T x) \). Then \((U, P)\) given by \( U = -f \) and \( P = -\Pi \) solves (1.1) with \( F = 0 \) provided \( \Delta f = 0 \) and (H3) holds.

The particular case, where \( M = R \) describes pure rotation, was investigated by Hishida and by Babin, Mahalov and Nicolaenko. Indeed, Hishida constructed in [18] a local solution to the equation (1.3) written below in the \( L^2 \) context and provided \( u_0 \) belongs to a certain fractional power space. Babin, Mahalov and Nicolaenko [2, 3] proved the existence of a local solution and even a global solution to (1.1)-(1.2) provided the speed of rotation is fast enough. Further, the case \( f(x) = (ax_1, ax_2, -2ax_3) \) with some constant \( a \in \mathbb{R} \), was investigated by Giga and Kambe [14]. They studied the axisymmetric irrotational flow and the stability of the vortex.

In [26], the third author proved the existence of a local solution of (1.1)-(1.2), still for \( M = R \) provided \( u_0 \) belongs to the homogeneous Besov space \( \dot{B}^{0}_{2,1} \). Although \( \dot{B}^{0}_{2,1} \) is strictly smaller than \( L^{\infty} \), this space still contains the nondecaying function \( f(x) = \sin x \). He also showed the uniqueness of the solution for general matrices \( M \); see [27].

In the following consider the the substitutions \( u := U + f \) and \( \tilde{P} := P + \Pi \). Then the pair \((U, P)\) satisfies (1.1) in the classical sense, if and only if \((u, \tilde{P})\) satisfies

\begin{equation}
\begin{cases}
u_t - \Delta u + (u, \nabla)u - (f, \nabla)u - (u, \nabla)f + \nabla \tilde{P} = \tilde{F} & \text{in } \mathbb{R}^n \times (0, T), \\
\nabla \cdot u = 0 & \text{in } \mathbb{R}^n \times (0, T), \\
u(0) = u_0 & \text{with } \nabla \cdot u_0 = 0 \text{ in } \mathbb{R}^n.
\end{cases}
\end{equation}

Here \( \tilde{F} := F + \Delta f - (f, \nabla)f - \nabla \Pi \). Of course, if \((f, \Pi)\) is a stationary solution to (1.1) with \( F = F(x) \), then \( \tilde{F} \equiv 0 \). Our approach to equation (1.1) is based on equation (1.3).

## 2. Main Results

Let \( u_0 \in L^p(\mathbb{R}^n) \) for some \( p \) satisfying \( 1 < p < \infty \). Moreover, let \( f \) be a vector-valued globally Lipschitz continuous function satisfying hypothesis (H1), (H2), (H3).

We then rewrite the first equation of (1.3) as the abstract equation

\begin{equation}
u' + Au + (u, \nabla)u - 2(u, \nabla)f + \nabla \tilde{P} = \tilde{F}.
\end{equation}

with \( A \) being an operator in \( L^p(\mathbb{R}^n) \) defined by

\begin{equation}
Au := -\Delta u - (f, \nabla)u + (u, \nabla)f.
\end{equation}

Equipped with the domain \( D(A) := \{ u \in W^{2,p}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n); (f, \nabla)u \in L^p(\mathbb{R}^n) \} \), \(-A\) generates a \( C_0\)-semigroup \( \{ e^{-tA} \}_{t \geq 0} \) on \( L^p \) for \( 1 < p < \infty \). This follows from the results in [20] and standard perturbation theory.

Applying the Helmholtz projection \( \mathbb{P} \) to (2.1), we may rewrite (1.3) as

\begin{equation}
\begin{cases}
u' + Au + \mathbb{P}(u, \nabla)u - 2\mathbb{P}(u, \nabla)f = \tilde{F} \\
u(0) = u_0.
\end{cases}
\end{equation}

Note that in our case the Helmholtz projection \( \mathbb{P} \) can be expressed explicitly by \( \mathbb{P} := (\delta_{ij} + R_i R_j)_{i,j} \), where \( \delta_{ij} \) stands for Kronecker’s delta, and \( R_i \) is the Riesz transform defined by \( R_i := \partial_i (\Delta)^{-1/2} \) for \( i = 1, \ldots, n \). Observe that \( A \) and \( \mathbb{P} \) commute in our case, since \( \nabla \cdot Au = 0 \) if \( \nabla \cdot u = 0 \). Since \( u, F \) and \( f \) are divergence-free, \( \mathbb{P}u = u \) as well as \( \mathbb{P}F = F \).
For $T > 0$ we call a function $u \in C([0, T); L^p_\sigma(\mathbb{R}^n))$ a **mild solution** of (2.3) if $u$ satisfies the integral equation

\[(2.4)\]
\[
  u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}\mathcal{P}(u(s), \nabla)u(s)ds + 2\int_0^t e^{-(t-s)A}\hat{F}(s)ds + \int_0^t e^{-(t-s)A}F(s)ds
\]

for $t \in (0, T)$, and $u(0) = u_0$.

We now state the our existence and uniqueness results for mild solutions of (2.3) in $L^p$ spaces.

**2.1. Theorem.** Let $n \geq 2$, $T > 0$, $p \in [n, \infty)$ and $q \in [p, \infty)$. Let $f$ be a vector-valued globally Lipschitz continuous function satisfying (H1), (H2) and (H3). Assume that $u_0 \in L^p_\sigma(\mathbb{R}^n)$, and that $F \in C(0, T; L^p_\sigma(\mathbb{R}^n))$. Then there exist $T_0 \in (0, T)$ and a unique mild solution $u$ of (2.3) such that

\[(2.5)\]
\[
  [t \mapsto t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}u(t)] \in C([0, T_0); L^q_\sigma(\mathbb{R}^n))
\]

\[(2.6)\]
\[
  [t \mapsto t^\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}\nabla u(t)] \in C([0, T_0); L^q_\sigma(\mathbb{R}^n))
\]

**2.2. Remark.** (i) The semigroup \( \{e^{-tA}\}_{t \geq 0} \) is not analytic.

(ii) Consider the case $p = \infty$ and $u_0 \in L^\infty_\sigma(\mathbb{R}^n)$ or $u_0 \in BUC_\sigma$, i.e., $u_0$ do not decay at space infinity. In this case, one might expect to obtain the existence result for the mild solutions $u \in C([0, T_0); B_{\infty,1}^p(\mathbb{R}^n))$ satisfying (2.3) provided that $u_0 \in B_{\infty,1}^p(\mathbb{R}^n)$ and $\nabla \cdot u_0 = 0$. In [27], this is discussed for the case $f(x) = Mx$.

The proof of Theorem 2.1 is based on Kato’s iteration procedure. The key is to derive appropriate smoothing estimates for the semigroup and its gradient; see Proposition 3.3. Uniqueness follows by Gronwall’s inequality.

3. **Estimates for the semigroup**

In this section we prepare the linear estimates needed for the iteration scheme. Let $f$ be a vector-valued globally Lipschitz continuous function satisfying (H1), (H2) and (H3).

We define the realization of the operator

\[(3.1)\]
\[
  \mathcal{L}u := -\Delta u - (f, \nabla)u, \quad x \in \mathbb{R}^n,
\]

in $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$ as follows. Set

\[
  Lu := \mathcal{L}u, \quad D(L) := \{u \in W^{2,p}(\mathbb{R}^n); (f, \nabla)u \in L^p(\mathbb{R}^n)\}.
\]

Then the following result was proved by Lunardi and Metafune [20].

**3.1. Proposition.** Let $1 < p < \infty$. Then the operator $-L$ generates a $C_0$-semigroup \( \{e^{-tL}\}_{t \geq 0} \) on $L^p(\mathbb{R}^n)$.

**3.2. Remark.** (i) The semigroup \( \{e^{-tL}\}_{t \geq 0} \) is not analytic; see [20].

(ii) The family \( \{e^{-tL}\}_{t \geq 0} \) is also a semigroup on $L^1(\mathbb{R}^n)$ and on $L^\infty(\mathbb{R}^n)$, which in the latter case is not strongly continuous.

(iii) If $f(x) = Mx$ where $M$ is a constant matrix, the semigroup \( \{e^{-tL}\}_{t \geq 0} \) has an explicit representation given by

\[
  e^{-tL}\varphi(x) := \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^n} \varphi(e^{tM}x - y)e^{-\frac{1}{4}(Q_t(x-y),y)}dy, \quad x \in \mathbb{R}^n, \quad t > 0,
\]

where $Q_t$ for $t > 0$ is given by $Q_t := \int_0^te^{sM}e^{sM^T}ds$. 


For the iteration scheme described in the next section it is essential that the associated semigroup maps an $L^p$-function $u$ with $\nabla \cdot u = 0$ into the space of $L^p$-functions which are divergence free. We therefore introduce the operator $A$ by

$$Au := Lu + (u, \nabla)f,$$

where $u = (u^1, \ldots, u^n)$. Thus $A$ is an $n \times n$ operator matrix given by

$$A = L \text{Id} + (\nabla f)$$

where $\text{Id}$ denotes the identity matrix. Observe that

$$\nabla \cdot \{(f, \nabla)u - (u, \nabla)f\} = 0, \quad \text{provided } \nabla \cdot u = 0 \text{ and } \nabla \cdot f = 0.$$

Hence, we define the realization $A$ of $A$ in $L^p_0(\mathbb{R}^n)$ as

$$A_0 := Au, \quad D(A) := D(L)^n \cap L^p_0(\mathbb{R}^n).$$

By standard perturbation theory, $-A$ generates a $C_0$-semigroup $\{e^{-tA}\}_{t \geq 0}$ on $L^p$ for all $p \in (1, \infty)$. In the case where $f(x) = Mx$, the semigroup $\{e^{-tA}\}_{t \geq 0}$ is given by

$$(e^{-tA}u)(x) := \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}}e^{-tM} \int_{\mathbb{R}^n} u(e^{tM}x-y)e^{-\frac{1}{4}(Q_t^{-1}y, y)}dy.$$

We cannot expect to have such a formula for the semigroup $\{e^{-tA}\}_{t \geq 0}$, in general. We are now state $L^p - L^q$ smoothing properties for the semigroup $e^{-tA}$ as well as gradient estimates for $e^{-tA}$. Note that due to the non-analyticity of $\{e^{-tA}\}_{t \geq 0}$, gradient estimates for $e^{-tA}$ do not follow from the general theory of semigroups (like the Stokes semigroup). Notice also that in the special case where $f(x) = x$, $L^p - L^q$ smoothing estimates as well as gradient estimates for $e^{-tA}$ were obtained by Gallay and Wayne [11]. For $f(x) = Mx$, these estimates were obtained in [17]. For the general case, we rely on the recent results of Lunardi and Metafune [20] and Bertholdi and Lorenzi [4].

3.3. Proposition. [[20], Prop. 5.4], [[4], Thm. 4.7, Cor, 4.8]. Let $n \geq 2$, $1 < p < \infty$ and $p \leq q \leq \infty$.

a) Then there exist constants $C > 0$ and $\omega \in \mathbb{R}$ such that

$$\|e^{-tA}\varphi\|_q \leq Ce^{\omega t}t^{-\frac{3}{2}n}t^{\frac{n}{q}}\|\varphi\|_p, \quad t \geq 0, \quad \varphi \in L^p(\mathbb{R}^n),$$

$$\|\nabla e^{-tA}\varphi\|_p \leq Ce^{\omega t}t^{-\frac{n}{q}}\|\varphi\|_p, \quad t \geq 0, \quad \varphi \in L^p(\mathbb{R}^n).$$

b) There exist constants $C' > 0$ and $\omega' \in \mathbb{R}$ such that

$$\|\nabla^2 e^{-tA}\varphi\|_p \leq C'e^{\omega' t}t^{-1} \|\varphi\|_p, \quad t \geq 0, \quad \varphi \in L^p(\mathbb{R}^n).$$

c) Moreover, let $1 < p < q \leq \infty$ and $\varphi \in L^p(\mathbb{R}^n)$. Then

$$t^{\frac{n}{2}(1-\frac{1}{p})}\|e^{-tA}\varphi\|_q \rightarrow 0 \text{ as } t \rightarrow 0,$$

$$t^{\frac{n}{2}}\|\nabla e^{-tA}\varphi\|_p \rightarrow 0 \text{ as } t \rightarrow 0,$$

$$t^{\frac{n}{2}}\|\nabla^2 e^{-tA}\varphi\|_p \rightarrow 0 \text{ as } t \rightarrow 0.$$

4. Proof of the Main Result

For a given globally Lipschitz continuous function $f$ satisfying (H1), (H2), (H3), consider the substitution $u(x, t) := U(x, t) + f(x)$ and $\mathcal{P}(x, t) := P(x, t) + \Pi(x)$. Then $(U, P)$ is a solution of (1.1) in the classical sense if and only if $(u, \mathcal{P})$ satisfies (1.3). We thus consider in the following (1.3) and its abstract formulation in (2.3), or (2.4). We only show the proof for the case $p = n$; the case $p > n$ is similar.
Proof of Theorem 2.1. Let $n \geq 2$ and $u_0 \in L^\sigma_n(\mathbb{R}^n)$. Assume that $F \in C(0, \infty; L^\sigma_n(\mathbb{R}^n))$. Recall that $\tilde{F} = F + \Delta f - (f, \nabla)f - \nabla \cdot \tilde{F} = 0$. For $j \geq 1$ and $t > 0$ we define functions $u_j$ successively by

(4.1) \[ u_1(t) := e^{-tA}u_0 + \int_0^t e^{-(t-s)A} \tilde{F}(s)ds, \]

(4.2) \[ u_{j+1}(t) := u_1(t) - \int_0^t e^{-(t-s)A} \mathbb{P}(u_j(s), \nabla)u_j(s)ds + 2 \int_0^t e^{-(t-s)A} \mathbb{P}(u_j(s), \nabla)f ds. \]

Since $\{e^{-tA}\}_{t \geq 0}$ acts on $L^p_n(\mathbb{R}^n)$ for $p \in (1, \infty)$, it follows from the definition of the Helmholtz projection that the functions $u_j$ are divergence-free for all $t > 0$ and all $j$.

For $T \in (0,1]$ and $\delta \in (0,1)$ we define

\[ K_0 := \sup_{0<t\leq T} t\frac{1-\delta}{2} \Vert e^{-tA}u_0 \Vert_{n/\delta} \quad \text{and} \quad K_0^0 := \sup_{0<t\leq T} t^{1/2} \Vert \nabla e^{-tA}u_0 \Vert_n. \]

By (3.7) and (3.8) in Proposition 3.3-(c), $K_0 \to 0$ and $K_0^0 \to 0$ as $T \to 0$. Similarly, we define $K_j := K_j(T)$ and $K_j^0 := K_j^0(T)$ for $j \geq 1$ by

\[ K_j(T) := \sup_{0<t\leq T} t\frac{1-\delta}{2} \Vert u_j(t) \Vert_{n/\delta} \quad \text{and} \quad K_j^0(T) := \sup_{0<t\leq T} t^{1/2} \Vert \nabla u_j(t) \Vert_n. \]

Let us estimate $K_1$ and $K_1'$; by definition and the $L^p - L^n$ smoothing property (3.4), we have

\[ K_1 = \sup_{0<t\leq T} t^{1/2} \Vert u_1(t) \Vert_{n/\delta} \]

\[ \leq K_0 + C \sup_{0<t\leq T} t^{1/2} \int_0^t \Vert e^{-(t-s)A} \tilde{F}(s) \Vert_{n/\delta} ds \]

\[ \leq K_0 + C \sup_{0<t\leq T} t^{1/2} \int_0^t (t-s)^{-\frac{1-\delta}{2}} \Vert \tilde{F}(s) \Vert_n ds \]

\[ \leq K_0 + CT \left( \Vert \Delta f + (f, \nabla)f + \Pi \Vert_n + \Vert F \Vert_{L^\infty(0,T;L^n(\mathbb{R}^n))} \right). \]

Similarly,

\[ K_1' \leq K_0' + CT \left( \Vert \Delta f + (f, \nabla)f + \Pi \Vert_n + \Vert F \Vert_{L^\infty(0,T;L^n(\mathbb{R}^n))} \right). \]

We thus have

(4.3) \[ K_1, K_1' \to 0 \quad \text{as} \quad T \to 0. \]

Next, it follows from (4.2), the $L^p - L^n$ smoothing of the semigroup and from the boundedness of $\mathbb{P}$ from $L^p(\mathbb{R}^n)$ into $L^\sigma_p(\mathbb{R}^n)$ that

\[ \Vert u_{j+1}(t) \Vert_{n/\delta} \]

\[ \leq \Vert u_1 \Vert_{n/\delta} + \int_0^t \Vert e^{-(t-s)A} \mathbb{P}(u_j(s), \nabla)u_j(s) \Vert_{n/\delta} ds + 2 \int_0^t \Vert e^{-(t-s)A} \mathbb{P}(u_j(s), \nabla)f \Vert_{n/\delta} ds \]

\[ \leq t^{-\frac{1-\delta}{2}} K_1 + C \int_0^t (t-s)^{-\frac{1}{2}} \Vert (u_j(s), \nabla)u_j(s) \Vert_r ds + C \int_0^t \Vert u_j(s) \Vert_{n/\delta} ds, \]

where $r = \frac{n}{1+\delta}$. In order to estimate the second term on the right hand side of last inequality, we apply Hölder’s inequality to conclude that

\[ \Vert (u_j(s), \nabla)u_j(s) \Vert_r \leq \Vert u_j(s) \Vert_{n/\delta} \Vert \nabla u_j(s) \Vert_n \leq K_j K_j' s^{-\frac{1-\delta}{2} - \frac{1}{2}}. \]

This implies

\[ \Vert u_{j+1}(t) \Vert_{n/\delta} \leq t^{-\frac{1-\delta}{2}} K_1 + CK_j K_j' \int_0^t (t-s)^{-\frac{1}{2}} s^{1+\frac{\delta}{2}} ds + CK_j \int_0^t s^{-\frac{1-\delta}{2}} ds. \]

Multiplying with $t^{\frac{1-\delta}{2}}$ and taking $\sup_{0<t\leq T}$ on both sides, we obtain

\[ K_{j+1} \leq K_1 + C_1 K_j K_j' + C_2 TK_j \]
with some constants $C_1, C_2$, independent of $j$ and $T$.

Similarly, applying $\nabla$ to (4.2) and estimating it with respect to the $L^n$-norm, it follows from (3.4) and (3.5) that

$$K'_j( T) \leq \min(\frac{1}{8C_6}, \frac{1}{8C_8})$$

for some constants $C_3$ and $C_4$. By (4.3), for any $\lambda > 0$ there exists $\tilde{T}_0 > 0$ such that $K_1, K'_1 \leq \lambda$ for all $T \leq \tilde{T}_0$. So, we fix $T_0 \leq \min(\frac{1}{8C_6}, \frac{1}{8C_8})$ provided $\lambda \leq \min(\frac{1}{8C_6}, \frac{1}{8C_8})$. We thus obtain bounds for $K_j(T)$ and $K'_j(T)$ for any $T \leq T_0$ uniformly in $j$ provided that $T_0$ is small enough. Indeed, $\sup_j K_j, K'_j \leq 3\lambda$ for $T \leq \tilde{T}_0$.

The uniform bounds of $K_j$ and $K'_j$ imply that $t^{1-\frac{n}{2q}}\left\|u_j(t)\right\|_q$ as well as $t^{1-\frac{n}{2q}}\left\|\nabla u_j(t)\right\|_q$ are bounded for $q \in [n, \infty)$, $t \leq \tilde{T}_0$ and all $j \in \mathbb{N}$. The continuity of the above functions follows from similar calculations and (3.7).

We finally derive estimates for the differences $u_{j+1} - u_j$. Indeed, for all $j \geq 1$ put

$$L_j(T) := \sup_{0 \leq t \leq T} t^{1-\frac{n}{2q}}\left\|u_{j+1}(t) - u_j(t)\right\|_{n/8} \text{ and } L'_j(T) := \sup_{0 < t \leq T} t^{1/2}\left\|\nabla u_{j+1}(t) - \nabla u_j(t)\right\|_n.$$

Similarly as before, we have for all $j \geq 1$

$$L_j(T) \leq C_6\lambda(L_{j-1} + L'_{j-1}) + C_6TL_{j-1},$$

$$L'_j(T) \leq C_7\lambda(L_{j-1} + L'_{j-1}) + C_8TL_{j-1}.$$

with some positive constants $C_5, C_6, C_7$ and $C_8$. We now choose $T_0 \leq \tilde{T}_0$ small enough so that $T_0 \leq \min(\frac{1}{8C_6}, \frac{1}{8C_8})$ provided $4(C_6 + C_7)\lambda \leq 1$. Hence we have $(L_{j+1} + L'_{j+1})/(L_j + L'_{j}) \leq 1/2$ for all $j \geq 1$ and $T \leq T_0$. This implies that $L_j$ and $L'_j$ tend to zero as $j \to \infty$. It thus follows that the above sequences are Cauchy sequences and we conclude that there are unique limit functions

$$\{t \to t^{1-\frac{n}{2q}}u_j(t)\}_{j \geq 1} \subset C([0, T_0]; L^n_q), \quad \{t \to t^{1-\frac{n}{2q}}v_j(t)\}_{j \geq 1} \subset C([0, T_0]; L^q)$$

of the sequences $\{t^{1-\frac{n}{2q}}u_j(t)\}_{j \geq 1}$ (if necessary, we shall take its subsequence) and $\{t^{1-\frac{n}{2q}}\nabla u_j(t)\}_{j \geq 1}$. Finally, note that $v(t) = t^{1/2}\nabla u(t)$ and that $u$ is a mild solution of (2.3) on $[0, T_0]$.

Uniqueness of mild solutions follows from standard Gronwall's inequality. This completes the proof of the first assertion of Theorem 2.1. \hfill $\Box$

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