Navier-Stokes equations with absorption under slip boundary conditions: existence, uniqueness and extinction in time*

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To the memory of Alexandre Vasilievich Kazhikhov.

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Abstract

The Navier-Stokes equations modified with absorption terms under slip boundary conditions are investigated in a cylinder $\Omega \times (0,T) \subset \mathbb{R}^N \times \mathbb{R}^+$. On the boundary $\partial\Omega$, we assume there is no-penetration of fluid flow and the slip is considered with or without friction. We present several results concerning a localization in time effect for different absorption terms. This property is established by using a suitable energy method and is independent if the slip on boundary occurs with friction or not. We prove, also, existence and uniqueness results for the related mathematical model.

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Contents

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1 Introduction

During the recent past the most studied boundary condition was the no-slip condition, *i.e.*,

$$\mathbf{u}(\mathbf{x},t)=\mathbf{0} \quad \text{on} \quad \partial\Omega\times(0,T). \tag{1.1}$$

This condition is a mathematical expression for the adherence of the fluid to the boundary of the flow domain. In Section 3 we will precise the notation we are using. For the time being, it suffices to know that $\partial\Omega$ denotes the boundary of a spatial domain Ω and $(0,T)$ is a time interval. The acceptance of the no-slip boundary condition as the correct physical model in fluid flows, goes back to the work of G.G. Stokes in 1845. However, the experiments by J.C. Maxwell in 1879, in the kinetic theory of gases, very soon pointed out that this condition does not explain well all the physical phenomena (see Serrin [29, pp. 240-241] and the references therein). Nowadays, the acceptance of the no-slip boundary condition, for fluid flows with moderate velocities and pressures, is justified by direct observations and comparisons between numerical simulations and experimental results (see John and Liakos [21, p. 713] and the references therein).

In the last years there have been an increase of interest in studying fluid problems with slip boundary conditions. It is known the slip condition applies mainly to free surfaces in free boundary problems such as the coating problem (see, *e.g.*, Friedman and Velázquez [19]), which are modeled as being stress free, *i.e.*

$$\mathbf{t}\cdot\tau=0 \quad \text{on} \quad \partial\Omega\times(0,T); \tag{1.2}$$

where $\mathbf{t}=\mathbf{n}\cdot\mathbf{S}$ is the stress vector and τ is a tangential vector to the boundary $\partial\Omega$. But, the new interest in the slip condition came essentially from the large eddy simulation, one of the most promising approaches for modeling turbulence. To describe many phenomena which can be observed in nature, the slip boundary conditions are more appropriated. For instance, hurricanes and tornadoes, do slip along the ground, lose energy as they slip and do not penetrate the ground (see John and Liakos [21, p. 714]). In spite of the mathematical convenience to treat boundary value problems with no-slip boundary conditions, there are also some mathematical aspects which show the inadequacy of the no-slip condition. For instance, in Le Roux [25, pp. 310-311], is addressed one of these aspects, when one considers the problem with nonhomogeneous Dirichlet boundary conditions. There, is pointed out that the uniqueness for that problem is guaranteed only in the case of an impermeable boundary. This implies that additional boundary conditions are necessary to ensure the well-posedness of the problem.

Although its recent interest, Navier-Stokes equations with slip boundary conditions have already been studied analytically by many authors. Solonnikov and Ščadilov have proved in [30] the existence of a generalized solution, as well its smoothness, for a linearized stationary system which on a part of the boundary satisfies a slip boundary condition. The time dependent incompressible Navier-Stokes equations was investigated, respectively, in Sobolev-Sobolevsky and Hölder function spaces by Tani *et al.* [31, 20]. With respect to the 2-D case, there are some works concerning the inviscid limit of the Navier-Stokes equations. Clopeau *et al.* have proved in [14] the existence of regular solutions with bounded vorticity for the 2D evolutive system with a slip boundary condition on a part of the boundary. Coron has proved earlier in [15] the same, but with smooth compatible data. Lopes Filho *et al.* [18] and Kelliher [22] have extended the results of Clopeau *et al.* [14], and Berselli and Romito [12], by using rather elementary tools, have proved existence and uniqueness of weak solutions.

Historically slip boundary conditions were proposed by C.L. Navier in 1827 (see Serrin [29, p.240]) in the following form:

$$\mathbf{u}\cdot\tau = k\,\mathbf{t}\cdot\tau \quad \text{on} \quad \partial\Omega\times(0,T); \tag{1.3}$$

where k is a given negative-valued function. Note that this condition alone is a mathematical expression of slip with friction ($k<0$). Specifically, slip occurs in the opposite direction as the resistive force the wall exerts on the fluid. However, Navier condition is absent to what happens through the boundary. For instance, there can exist cross of fluid through the boundary or, simply, it may happens that there is no penetration. That is the reason why in the literature (1.3) is considered as a partial slip boundary condition. Since the works of J.C. Maxwell in 1879, various slip conditions have been proposed in place of the no-slip boundary condition, the most important being

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{and} \quad \mathbf{u}\cdot\tau = \beta^{-1}\,\mathbf{t}\cdot\tau \quad \text{on} \quad \partial\Omega\times(0,T); \tag{1.4}$$

where $\mathbf{n}$ and τ denote, respectively, unit normal and tangential vectors to the boundary $\partial\Omega$, and here β is a coefficient with no defined sign (as in Serrin [29, p.240]). The case $\beta^{-1}<0$ corresponds to the most studied case in the literature, slip with friction. But, in this work, we will consider also $\beta^{-1}>0$, the case which the boundary walls accelerate the fluid. The limit $\beta\to 0$ leads to free slip boundary conditions (1.2), while the limit $\beta\to\infty$ recovers the no-slip boundary conditions (1.1). In some situations the parameter β can be explicitly calculated in terms of the Reynolds number and of a spatial scale length (see John and Liakos [21] and the references therein). Mathematically, $(1.4)_1$ is an expression for the no-penetration of fluid on the boundary and $(1.4)_2$ for flow with resistance, friction or not.

2 Statement of the problem

We consider the mathematical problem of an incompressible viscous fluid in a cylinder $Q_T := \Omega\times(0,T)\subset\mathbb{R}^N\times\mathbb{R}^+$, where Ω is a bounded domain with a locally Lipschitz boundary $\partial\Omega$. Here, we will consider the dimensions of physical interest $N = 2$ and $N = 3$. However, the main results of this article, established in Section 5, are valid for any dimension $N \geq 2$. From the principle of conservation of mass for an incompressible fluid and from the principle of conservation of momentum, we obtain the Navier-Stokes equations:

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad Q_T\,; \tag{2.5}$$

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f} + \operatorname{div} \mathbf{S} \quad \text{in} \quad Q_T\,; \tag{2.6}$$

where the stress tensor $\mathbf{S}$ obeys the Stokes law:

$$\mathbf{S} = -p\mathbf{I} + 2\nu\mathbf{D}, \quad \mathbf{D} = \frac{1}{2}\left(\nabla\,\mathbf{u} + \nabla\,\mathbf{u}^T\right)\,. \tag{2.7}$$

In these equations, $\mathbf{u}$ is the velocity, $\mathbf{D}$ is the rate of the strain tensor, $\mathbf{I}$ is the unit tensor, p is the pressure divided by the constant density of the fluid, ν is the constant kinematics viscosity, and $\mathbf{f}$ is a forces field. Solutions of (2.5)-(2.7) are assumed to satisfy the initial condition:

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in} \quad \Omega\,. \tag{2.8}$$

We assume the problem is supplemented with the following slip boundary conditions:

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{and} \quad \mathbf{u}\cdot\tau = \beta^{-1}\mathbf{t}\cdot\tau \quad \text{on} \quad \partial\Omega\times(0,T); \tag{2.9}$$

where $\mathbf{t} = \mathbf{n}\cdot\mathbf{S}$ is the stress vector and β^{-1} is a proportional factor (as in Serrin [29, p. 240]). Note that $\mathbf{u}\cdot\tau = \beta^{-1}\,\mathbf{t}\cdot\tau$ can be written as $\mathbf{u}\cdot\tau = \beta^{-1}\,\mathbf{n}\cdot\,\mathbf{S}\cdot\tau$, or $\mathbf{u}\cdot\tau = \beta^{-1}\,(\mathbf{S}\,\mathbf{n})\cdot\tau$. Mathematically, $(2.9)_2$ expresses the fact that, on the boundary, tangential velocities are proportional to the tangential stresses.

The new contribution of this work is the consideration, in the momentum equation (2.6), of a forces field $\mathbf{f}$ such that

$$-\mathbf{f}(\mathbf{x},t,\mathbf{u})\cdot\mathbf{u} \geq C_{\mathbf{f}}|\mathbf{u}|^{\sigma} \quad \forall\,\mathbf{u}\in\mathbb{R}^2, \qquad \sigma\in(1,2), \tag{2.10}$$

for some positive constant $C_{\mathbf{f}}$. Note that such forces field depends, in a sublinear way, on the own velocity $\mathbf{u}$ and, physically, maybe considered as a feedback field. The motivation for this forces field is purely mathematical and goes back to the works of Benilan *et al.* [8], Díaz and Herrero [11], and Bernis [9, 10]. There, was studied the importance of the absorption term $|\mathbf{u}|^{\sigma-2}\mathbf{u}$ to prove qualitative properties related with compact supported solutions, or solutions which exhibit finite speed of propagations, or which extinct in time. Theses properties were there proved with the equivalent assumption of $\sigma\in(1,2)$. Therefore, in a certain sense, we are doing nothing but to introduce, in the left-hand of momentum equation (2.6), the absorption term $|\mathbf{u}|^{\sigma-2}\mathbf{u}$. We notice that we already have considered similar forces field in a variety of fluid mechanics problems with no-slip boundary conditions in [2]-[7].

3 Mathematical framework

Notation. The notation used throughout this text is largely standard in Mathematical Fluid Mechanics - see, *e.g.*, Galdi [16, 17], Layzhenskaya [23], or Temam [33]. We distinguish vectors from scalars by using boldface letters. For functions and function spaces we will use this distinction as well. The symbol C will denote a generic constant - almost the time a positive constant, whose value will not be specified; it can change from one inequality to another. The dependence of C on other constants or parameters will always be clear from the exposition. Sometimes we will use letter subscripts to relate a constant with the result from where it derives. In this article, the notation Ω stands always for a domain, *i.e.*, a connected open subset of $\mathbb{R}^N$, whose compact boundary is denoted by $\partial\Omega$. The letters $\mathbf{n}$ and τ denote unit normal and tangent vectors to the boundary $\partial\Omega$. The boundary $\partial\Omega$ is assumed to be smooth enough such that $\mathbf{n}$ and τ exist a.e. on $\partial\Omega$ - for instance, C^1.

Function spaces. Let $1 \leq p \leq \infty$. We shall use the classical Lebesgue spaces $L^p(\Omega)$, whose norm is denoted by $\|\cdot\|_{L^p(\Omega)}$. For any nonnegative k, $H^k(\Omega)$ denotes the Sobolev space $W^{k,2}(\Omega)$, and its norm we simbolize by $\|\cdot\|_{H^k(\Omega)}$. For $m \geq 1$ the associated trace spaces are denoted by $W^{q,m-1/q}(\partial\Omega)$, with $1 \leq q < \infty$, and $H^{m-1/2}(\partial\Omega)$. Given $T > 0$ and a Banach space X, $L^p(0,T;X)$ and $H^k(0,T;X)$, k is any nonnegative number, denote the usual Lebesgue and Sobolev spaces used in evolutive problems, with norm denoted by $\|\cdot\|_{L^p(0,T;X)}$ and $\|\cdot\|_{H^k(0,T;X)}$. The corresponding spaces of vector-valued functions are denoted by boldface letters. All these spaces are Banach spaces and the Hilbert framework corresponds to $p = 2$. The H^k Sobolev spaces already correspond to $p = 2$ and therefore are Hilbert spaces. For a detailed exposition of these spaces, we address the reader, for instance, to the monograph by Adams [1].

For the mathematical setting of our the problem, we define the following function spaces:

$$\mathbf{H} = \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \operatorname{div} \mathbf{v} = 0 \ \text{ and } \ \mathbf{v}\cdot\mathbf{n} = 0 \ \text{ on } \ \partial\Omega\},$$

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \operatorname{div} \mathbf{v} = 0 \ \text{ and } \ \mathbf{v}\cdot\mathbf{n} = 0 \ \text{ on } \ \partial\Omega\},$$

$$\mathcal{W} = \{\mathbf{v} \in \mathbf{V} \cap \mathbf{H}^2(\Omega) : \mathbf{v} \text{ satisfies to (2.9)}\}.$$

The space $\mathcal{W}$ is endowed with the $\mathbf{H}^1(\Omega)$ norm, $\mathbf{H}$ is endowed with the $\mathbf{L}^2(\Omega)$ inner product and norm, and $\mathbf{V}$ is endowed with the inner product $(u,v)_{\mathbf{V}} = \sum_{i=1}^N (\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i})$ and with the associated norm. From (3.12) below, we see that this norm is equivalent to the $\mathbf{H}^1(\Omega)$ norm.

Auxiliary results. Throughout this text we will make reference, at least once, to the following inequalities (see Antontsev *et al.* [7, Appendix]):
(1) Algebraic inequality - for every α, $\beta \in \mathbb{R}$ and every A, $B \geq 0$,

$$A^\alpha B^\beta \leq (A+B)^{\alpha+\beta}; \tag{3.11}$$

(2) Young's inequality - for every $a,\ b \geq 0$, $\varepsilon > 0$ and $1 < p,\ q < \infty$ such that $1/p + 1/q = 1$,

$$ab \leq \varepsilon a^p + C(\varepsilon) b^q \,.$$

If $p = q = 2$, this is known as Cauchy's inequality.
(3) Hölder's inequality - for every $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, with $1 \leq p,\ q \leq \infty$ such that $1/p + 1/q = 1$,

$$\int_\Omega u\, v\, d\mathbf{x} \leq \|u\|_{p,\Omega} \|v\|_{q,\Omega}.$$

It is worth recalling the following result of Temam [33, Theorem 1.2]: there exists a continuous linear operator $\gamma_{\mathbf{n}}$ mapping the space $\mathbf{E}(\Omega) := \{\mathbf{v} \in \mathbf{L}^2 : \operatorname{div} \mathbf{v} \in L^2(\Omega)\}$ into $\mathbf{H}^{-1/2}(\partial\Omega)$, the dual space $\mathbf{H}^{1/2}(\partial\Omega)$, such that $\gamma_{\mathbf{n}}(\mathbf{v})$ is the restriction to $\partial\Omega$ of every compact supported function $\mathbf{v} \in \mathbf{C}^\infty(\Omega)$. Also, the following divergence theorem holds

$$\int_\Omega \mathbf{v} \cdot \nabla\, h\, d\mathbf{x} = \int_{\partial\Omega} \gamma_{\mathbf{n}}(\mathbf{v}) \cdot \gamma_0(h)\, dS - \int_\Omega \operatorname{div} \mathbf{v}\, h\, d\mathbf{x}\,,$$

for every $\mathbf{v} \in \mathbf{E}(\Omega)$ and $h \in H^1(\Omega)$. In the sequel we always suppress the trace function γ_0 and write $\mathbf{v} \cdot \mathbf{n}$ in place of $\gamma_{\mathbf{n}}(\mathbf{v})$.

For the main properties we will prove in this article, play important roles two known results. The first is related with the famous Gagliardo-Nirenberg inequality and the second with the trace theorem.

Lemma 3.1 *Let Ω be a domain of $\mathbb{R}^N$, $N \geq 1$, with a compact boundary $\partial\Omega$. Assume that $u \in W^{1,p}(\Omega)$, $\partial\Omega$ is locally Lipschitz and $\int_\Omega u\, dx = 0$. For every fixed number $r \geq 1$ there exists a constant C_{GN} depending only on N, p, r and Ω such that*

$$\|u\|_{q,\Omega} \leq C_{GN} \|\nabla u\|^\theta_{L^p(\Omega)} \|u\|^{1-\theta}_{L^r(\Omega)}, \tag{3.12}$$

where $\theta \in [0,1]$, $p,\ q \geq 1$, are linked by $\theta = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{N} - \frac{1}{p} + \frac{1}{r}\right)^{-1}$, and their admissible range is:
(1) If $N = 1$, $q \in [r, \infty]$, $\theta \in \left[0, \frac{p}{p+r(p-1)}\right]$;
(2) If $p < N$, $q \in \left[\frac{Np}{N-p}, r\right]$ if $r \geq \frac{Np}{N-p}$ and $q \in \left[r, \frac{Np}{N-p}\right]$ if $r \leq \frac{Np}{N-p}$;
(3) If $p \geq N > 1$, $q \in [r, \infty)$ and $\theta \in \left[0, \frac{Np}{Np+r(p-N)}\right]$.

See the proof of this result in Ladyzhenskaya *et al.* [24, p. 62] (see also Nirenberg [27, p. 125]). A precise definition of locally Lipschitz boundary is given in Galdi [16, p. 36], which turns out to be equivalent to the definition of piecewise-smooth boundary (with nonzero interior angles) given in [24, p. 9]. If Ω is unbounded, or if Ω is bounded and $u \in W^{1,p}_0(\Omega)$, than the assumptions on the boundary are not needed, as well the zero average of u in Ω, and the constant C_{GN} does not depend on Ω (see [27]).

Lemma 3.2 *Let Ω be a domain of $\mathbb{R}^N$, $N \geq 2$, with a compact boundary $\partial\Omega$. Assume that $u \in W^{1,p}(\Omega)$, $\partial\Omega$ is locally Lipschitz and $\int_\Omega u\,dx = 0$. There exists a constant C_{tr} depending only on N, q and Ω such that*

$$\|u\|_{L^q(\partial\Omega)} \leq C_{tr}\|\nabla u\|^{\alpha}_{L^2(\Omega)}\|u\|^{1-\alpha}_{L^2(\Omega)}, \tag{3.13}$$

where $\alpha = \frac{N}{2} - \frac{N-1}{q}$ and $q \in \left[\frac{2(N-1)}{N}, \frac{2(N-1)}{N-2}\right]$ if $N \geq 3$, or $q \in [1, \infty)$ for $N = 2$.

This inequality is established in [24, p. 69] (see also [16, p. 43]). In the two previous results the dependence of constants C_{GN} and C_{tr} on Ω is understood in the sense that it depends on the structure of $\partial\Omega$. However, it does not depend on the size of Ω, *i.e.*, it does not change under dilatations of Ω. Sometimes, in the sequel, we will denote this situation by writing $C = C(\partial\Omega)$.

Under the assumption that $\partial\Omega$ is locally Lipschitz, we obtain, as a straightforward consequence of (2.5) and (2.9), that every component u_i, $i = 1, \ldots, N$, of a velocity field $\mathbf{u} \in \mathbf{E}(\Omega)$ satisfies

$$\int_\Omega u_i\,d\mathbf{x} = \int_\Omega \operatorname{div}(\mathbf{u}\,x_i)d\mathbf{x} = \int_{\partial\Omega} x_i(\mathbf{u}\cdot\mathbf{n})\,dS = 0\,. \tag{3.14}$$

We will also make use of an important inequality often used in Continuum Mechanics.

Lemma 3.3 *Let Ω be a domain of $\mathbb{R}^N$, $N \geq 2$, with a locally Lipschitz compact boundary $\partial\Omega$. If $\mathbf{u}$ is in $\mathbf{H}^1(\Omega)$ and satisfies to (2.9)$_1$, then*

$$\|\nabla\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} \leq C_K\|\mathbf{D}(\mathbf{u})\|^2_{\mathbf{L}^2(\Omega)}\,, \tag{3.15}$$

where C_K is a positive constant depending on Ω.

This is the so-called second Korn's inequality and it extends for suitable unbounded domains. See Oleinik and Yosifian [30] for the proof and related questions (see also Solonnikov and Ščadilov [28, Lemma 2]).

4 On the existence and uniqueness

In this section, we assume the forces field is given by

$$\mathbf{f}(\mathbf{x}, t, \mathbf{u}) = -\alpha|\mathbf{u}|^{\sigma-2}\mathbf{u} + \mathbf{g}(\mathbf{x}, t)\,, \tag{4.16}$$

where α is a non-negative constant and $\mathbf{g}$ is a prescribed function. Note that such forces field satisfies to (2.10) with $\alpha = C_{\mathbf{f}}$ and only if $\mathbf{g} \equiv \mathbf{0}$. As a consequence, we obtain the following modified Navier-Stokes problem

$$\operatorname{div}\mathbf{u} = 0\,, \quad \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \alpha|\mathbf{u}|^{\sigma-2}\mathbf{u} = \mathbf{g} + \operatorname{div}\mathbf{S} \quad \text{in} \quad Q_T\,, \tag{4.17}$$

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}) \quad \text{in} \quad \Omega\,, \tag{4.18}$$

$$\mathbf{u}\cdot\mathbf{n}=0 \quad \text{and} \quad \mathbf{u}\cdot\tau=\beta^{-1}\,\mathbf{t}\cdot\tau \quad \text{on} \quad \partial\Omega\times(0,T). \tag{4.19}$$

If $\alpha=0$, then we fall in the usual Navier-Stokes problem. For the weak formulation of (4.17)-(4.19), we start by noting that the no-penetration condition $\mathbf{v}\cdot\mathbf{n}=0$ on the boundary allows us to write $\mathbf{v}=\sum_{i=1}^{N-1}(\mathbf{v}\cdot\tau_{\mathbf{i}})\tau_{\mathbf{i}}$ for every $N\geq 2$, where $\{\tau_{\mathbf{1}},\ldots,\tau_{\mathbf{N-1}},\mathbf{n}\}$ forms an orthonormal system of vectors in $\mathbb{R}^N$. Then, using the slip boundary condition (4.19), we can write

$$(\mathbf{S}\,\mathbf{v})\cdot\,\mathbf{n}=\mathbf{v}\cdot\,\mathbf{S}\,\cdot\,\mathbf{n}=\sum_{i=1}^{N-1}(\mathbf{v}\cdot\tau_{\mathbf{i}})\,\tau_{\mathbf{i}}\cdot\mathbf{S}\,\mathbf{n}=\beta\,\mathbf{u}\cdot\mathbf{v},$$

which is valid on the boundary $\partial\Omega\times(0,T)$. In consequence, using (2.5) and (2.7), we obtain for every $\mathbf{u}\in\mathcal{W}$ and $\mathbf{v}\in\mathbf{V}$

$$\int_\Omega \operatorname{div}\mathbf{S}\,\mathbf{v}\,d\mathbf{x}=-\nu\int_\Omega \mathbf{D}(\mathbf{u}):\mathbf{D}(\mathbf{v})\,d\mathbf{x}+\beta\int_{\partial\Omega}\mathbf{u}\cdot\mathbf{v}\,dS\,. \tag{4.20}$$

This motivates us for the following definition of weak solution for the problem (4.17)-(4.19).

Definition 4.1 *We say that* $\mathbf{u}$ *is a weak solution to the problem (4.17)-(4.19), if:*
1. $\mathbf{u}\in\mathbf{L}^2(0,T;\mathbf{V})\cap\mathbf{L}^\infty(0,T;\mathbf{H})$*;*
2. $\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0$*;*
3. For every $\mathbf{v}\in\mathbf{V}$

$$\frac{d}{dt}\int_\Omega \mathbf{u}(t)\cdot\mathbf{v}d\mathbf{x}+\nu\int_\Omega \mathbf{D}(\mathbf{u}(t)):\mathbf{D}(\mathbf{v})d\mathbf{x}+\int_\Omega(\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t)\cdot\mathbf{v}d\mathbf{x}+ \tag{4.21}$$

$$\alpha\int_\Omega |\mathbf{u}(t)|^{\sigma-2}\mathbf{u}(t)\cdot\mathbf{v}d\mathbf{x}=\int_\Omega \mathbf{g}(t)\cdot\mathbf{v}d\mathbf{x}+\beta\int_{\partial\Omega}\mathbf{u}(t)\mathbf{v}dS.$$

This definition is silent about the initial data $\mathbf{u}_0$ and the forces field $\mathbf{g}$. But, this will be clear when we bellow establish the existence result. The existence of such a weak solution is proved on the basis of the same results with prescribed forces field, *i.e.*, when one considers the problem (4.17)-(4.19) with $\alpha=0$. However, for this problem (with $\alpha=0$ in (4.17)), and, to the best of our knowledge, the global in time existence result is only proved [14, Theorem 2.3] for the 2-D case (see also Mucha and Sadowski [26, Theorem 2.1]). For the 3-D case, we only know existence results but locally in time (see Tani *et al.* [20, 31]). However, these results are inadequate for the localization in time effects we will establish in the next section. The main problem we face when we try to carry out the global in time existence result of the no-slip boundary conditions case to the slip conditions case, is because the space $\mathcal{V}=\{v\in C_0^\infty(\Omega):\operatorname{div}v=0 \ \text{ in } \ \Omega\}$ is not dense in V. This brings us problems when taking an orthonormal basis of $\mathbf{V}$ to form the approximate solutions in the Galerkin method. For $N=2$, the following auxiliary result replaces the density of $\mathcal{V}$ in the corresponding subspace of $\mathbf{H}_0^1(\Omega)$ from the no-slip boundary conditions case.

Lemma 4.1 *Assume $N=2$. There exists a basis $\{\mathbf{w}_1,\dots,\mathbf{w}_m,\dots\}\subset\mathbf{H}^3(\Omega)$ for $\mathbf{V}$, which satisfies*

$$\mathbf{w}_m\cdot\tau=\beta^{-1}\,(\mathbf{S}\,\mathbf{n})\cdot\tau\quad on\quad \partial\Omega\times(0,T)\,. \tag{4.22}$$

The basis $\{\mathbf{w}_1,\dots,\mathbf{w}_m,\dots\}$ is also an orthonormal basis for $\mathbf{H}$.

This result is proved in [14, Lemma 2.2] with $2(\mathbf{D}(\mathbf{w}_m)\mathbf{n})\cdot\tau+\alpha\,\mathbf{w}_m{\cdot}\tau=0$, and $\alpha>0$, instead of (4.22). But, mathematically the case $\alpha\leq 0$ does not offer any difficulty. If we had $\alpha=0$ (and $N=2$), then the problem could be overcame by choosing a basis for V such that their vectors have compact supports in Ω. That can be done by prescribing any function from $\mathbf{V}$ as $\nabla^{\perp}\phi=(-\frac{\partial\phi}{\partial x_2},\frac{\partial\phi}{\partial x_1})$ for a scalar function ϕ such that $\phi=0$ on $\partial\Omega$ (see [26, p. 1872]). However, for $N=3$, this is no longer possible and, to the best of our knowledge, an analogous result to Lemma 4.1 is yet not established. Therefore, at the moment, we are only able to establish the existence result for $N=2$. The proof is an adaptation of the corresponding proof for the no-slip boundary conditions case. For this, see Theorem 2.3 in Clopeau *et al.* [14], §6.3 in Ladyzhenskaya [23] and §III.3 in Temam [33].

Theorem 4.1 *Assume $N=2$ and let $\mathbf{u}_0\in\mathcal{W}$ and $\mathbf{g}\in\mathbf{L}^2(0,T;\mathbf{V}')$. Then, there exists, at least, a global in time weak solution to the problem (4.17)-(4.19) in the sense of Definition 4.1.*

PROOF. *1. Existence of approximate solutions.* We consider an $\mathbf{H}^2(\Omega)$ orthonormal basis $\{\mathbf{w}_1,\dots,\mathbf{w}_m,\dots\}$ for $\mathcal{W}$, which, from Lemma 4.1, is also a basis for $\mathbf{V}$. For each m, we search an approximate solution $\mathbf{w}_m$ of (4.21) in the form

$$\mathbf{w}_m=\sum_{i=1}^{m}c_{im}(t)\mathbf{w}_i\,, \tag{4.23}$$

where $c_{im}(t)$ are the functions we look for. These functions are founded by solving the following system of ordinary differential equations obtained from (4.21):

$$\frac{d}{dt}c_{jm}(t)+\nu\int_\Omega\mathbf{D}(\mathbf{u}_m(t)):\mathbf{D}(\mathbf{w}_j)d\mathbf{x}+\int_\Omega(\mathbf{u}_m(t){\cdot}\nabla)\mathbf{u}_m(t)\cdot\mathbf{w}_jd\mathbf{x}+ \tag{4.24}$$

$$\alpha\int_\Omega|\mathbf{u}_m(t)|^{\sigma-2}\mathbf{u}_m(t)\cdot\mathbf{w}_jd\mathbf{x}=\int_\Omega\mathbf{g}(t)\cdot\mathbf{w}_jd\mathbf{x}+\beta\int_{\partial\Omega}\mathbf{u}_m(t)\cdot\mathbf{w}_jdS\,;$$

$$c_{jm}(0)=\int_\Omega\mathbf{u}_0\cdot\mathbf{w}_jd\mathbf{x}\,; \tag{4.25}$$

for $j=1,\dots,m$. This problem has a unique solution $c_{jm}\in C^1([0,T_m))$, for some small interval of time $[0,T_m)\subset[0,T]$.
2. A priori estimates. After some calculations, we get the following inequality,

where we have used Cauchy's inequality with $\varepsilon = \nu/(2C_K)$, C_K is the Korn's inequality constant (3.15),

$$\|\mathbf{u}_m(t)\|^2_{\mathbf{L}^2(\Omega)} + 2\nu \int_0^t \|\mathbf{D}(\mathbf{u}_m(s))\|^2_{\mathbf{L}^2(\Omega)} ds + 2\alpha \int_0^t \|\mathbf{u}_m(s)\|^\sigma_{\mathbf{L}^\sigma(\Omega)} ds \leq \tag{4.26}$$

$$\|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \frac{\nu}{C_K} \int_0^t \|\mathbf{u}_m(s)\|^2_{\mathbf{H}^1(\Omega)} + \frac{C_K}{\nu} \int_0^t \|\mathbf{g}(s)\|^2_{V'} ds + 2\beta \int_0^t \|\mathbf{u}_m(s)\|^2_{\mathbf{L}^2(\partial\Omega)} ds$$

for $t < T_m$. If $\beta \leq 0$ and once that $\alpha \geq 0$, one can readily obtains, after using Korn's inequality (3.15),

$$\|\mathbf{u}_m(t)\|^2_{\mathbf{L}^2(\Omega)} + \nu \int_0^t \|\mathbf{D}(\mathbf{u}_m(s))\|^2_{\mathbf{L}^2(\Omega)} ds \leq \|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \frac{C_K}{\nu} \int_0^t \|\mathbf{g}(s)\|^2_{V'} ds \tag{4.27}$$

for $t < T_m$. If $\beta > 0$, then we use, before all, the trace inequality (3.13) with $q = 2$, in the last right-hand term of (4.26), as it is done in (5.45)-(5.46). Then, we use twice Cauchy's inequality: first as it was done in (4.26), but with $\varepsilon_1 > 0$; then with $\varepsilon_2 > 0$ in the term resulting from the application of trace inequality - both ε_1 and ε_2 are to be defined later on. After this, we use Korn's inequality (3.15) to obtain the following inequality

$$\|\mathbf{u}_m(t)\|^2_{\mathbf{L}^2(\Omega)} + 2\nu \int_0^t \|\mathbf{D}(\mathbf{u}_m(s))\|^2_{\mathbf{L}^2(\Omega)} ds \leq \tag{4.28}$$

$$\|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + C(\varepsilon_1, \varepsilon_2) \int_0^t \|\mathbf{D}(\mathbf{u}_m(s))\|^2_{\mathbf{L}^2(\Omega)} ds + C(\varepsilon_1) \int_0^t \|\mathbf{g}(s)\|^2_{V'} ds\,,$$

for $t < T_m$, and where $C(\varepsilon_1, \varepsilon_2) = 2\left[\varepsilon_1 + \varepsilon_2 \max(1, \varepsilon_2/4)\beta C_{tr}\right] C_K$, C_{tr} is the constant from trace inequality (3.13). Then, we choose ε_1 and ε_2: $\varepsilon_1 = \nu/(2C_K) - \varepsilon_2 \max(1, \varepsilon_2/4)\beta C_{tr}$; and $\varepsilon_2 > 0$ such that $\varepsilon_1 > 0$. Finally, we obtain (4.28) with different positive constants depending on ν, β, C_K and C_{tr}. Hence

$$\sup_{t\in[0,T]} \|\mathbf{u}_m(t)\|^2_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + C(\varepsilon_1)\|\mathbf{g}\|^2_{\mathbf{L}^2(0,T;\mathbf{V}')} \tag{4.29}$$

which implies that the element $\mathbf{u}_m \in \mathbf{L}^\infty(0, T; \mathbf{H})$ and the sequence $\mathbf{u}_m$ remains bounded in $\mathbf{L}^\infty(0, T; \mathbf{H})$.

On the other hand, if we replace t by T in (4.26), we obtain

$$\|\mathbf{u}_m(T)\|^2_{\mathbf{L}^2(\Omega)} + 2\nu \int_0^T \|\mathbf{D}(\mathbf{u}_m(s))\|^2_{\mathbf{L}^2(\Omega)} ds + 2\alpha \int_0^T \|\mathbf{u}_m(s)\|^\sigma_{\mathbf{L}^\sigma(\Omega)} ds \leq \tag{4.30}$$

$$\|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \frac{\nu}{C_K} \int_0^T \|\mathbf{u}_m(s)\|^2_{\mathbf{H}^1(\Omega)} + \frac{C_K}{\nu} \int_0^T \|\mathbf{g}(s)\|^2_{V'} ds + 2\beta \int_0^T \|\mathbf{u}_m(s)\|^2_{\mathbf{L}^2(\partial\Omega)} ds$$

Proceeding as before, we obtain (4.27) if $\beta \leq 0$, and (4.28) if $\beta > 0$, both with t replaced by T. Then, using here once more Korn's inequality (3.15), we get

$$\|\mathbf{u}_m(T)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\nu}{C_K} \int_0^T \|\mathbf{u}_m(s)\|^2_{\mathbf{H}^1(\Omega)} ds \leq \|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + C\|\mathbf{g}\|^2_{\mathbf{L}^2(0,T;\mathbf{V}')}\,, \tag{4.31}$$

where $C = C_K/\nu$ or $C = C(\varepsilon_1)$, when considering (4.27) or (4.28), respectively. This estimate enables us to say $\mathbf{u}_m \in \mathbf{L}^2(0,T;\mathbf{V})$ and the sequence $\mathbf{u}_m$ remains bounded in $\mathbf{L}^2(0,T;\mathbf{V})$.

3. Passing to the limit. In complete analogy with the no-slip case for the Navier-Stokes problem (with $\alpha = 0$ in (4.17)), we can pass to the limit in the equations satisfied by approximate solutions and the proof follows in a standard manner. Note that the limit solution $\mathbf{u}$ satisfies also to (4.31). □

Remark 4.1 *Justifying as in [33, p. 282], we can say that the weak solutions proved above are weakly continuous from* $[0,T]$ *onto* $\mathbf{H}$*. Moreover, if we assume more regularity on the data, we can obtain more regular solutions (*$N = 2$*): if* $\mathbf{f} \in \mathbf{H}^1(0,T;\mathbf{H})$ *and* $\mathbf{u}_0 \in \mathcal{W} \cap \mathbf{H}^2(\Omega)$*, then the solutions proved above satisfy* $\mathbf{u}' \in \mathbf{L}^2(0,T;\mathbf{V}) \cap \mathbf{L}^\infty(0,T;\mathbf{H})$ *(see [14, Theorem 2.3] and [33, Theorem 3.5]).*

In the next result, we establish the uniqueness of weak solutions for the problem (4.17)-(4.19) in the 2-D case.

Theorem 4.2 *Assume* $N = 2$ *and let* $\mathbf{u}_0 \in \mathcal{W}$ *and* $\mathbf{g} \in \mathbf{L}^2(0,T;\mathbf{V}')$*. Then, a weak solution of the problem (4.17)-(4.19) in the sense of Definition 4.1 is unique.*

PROOF. Let $\mathbf{v}$ and $\mathbf{w}$ be two weak solutions in the sense of Definition 4.1. Then, from (4.21), and arguing as in [33, Theorem 3.2], we get the following relation for $\mathbf{u} = \mathbf{v} - \mathbf{w}$:

$$\frac{d}{dt}\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + 2\nu\|\mathbf{D}(\mathbf{u}(t))\|^2_{\mathbf{L}^2(\Omega)} + I_1 = I_2 + I_3\,, \tag{4.32}$$

where

$$I_1 := 2\alpha\int_\Omega \left(|\mathbf{v}(t)|^{\sigma-2}\mathbf{v}(t) - |\mathbf{w}(t)|^{\sigma-2}\mathbf{w}(t)\right)\cdot\mathbf{u}(t)\,d\mathbf{x}\,.$$

$$I_2 := 2\int_\Omega [(\mathbf{w}(t)\cdot\nabla)\mathbf{w}(t) - (\mathbf{v}(t)\cdot\nabla)\mathbf{v}(t)]\cdot\mathbf{u}(t)\,d\mathbf{x} \quad \text{and} \quad I_3 := 2\beta\,\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\partial\Omega)}.$$

We use the following inequality [32] to prove that $I_1 \geq 0$: for every ξ, $\eta \in \mathbb{R}^N$, and $1 < \sigma < 2$

$$\left(|\xi|^{\sigma-2}\xi - |\eta|^{\sigma-2}\eta\right)\cdot(\xi-\eta) \geq (\sigma-1)|\xi-\eta|^2\left(|\xi|^\sigma + |\eta|^\sigma\right)^{\frac{\sigma-2}{\sigma}}\,.$$

On the other hand, it can be proved, in a standard manner, that

$$|I_2| = \left|2\int_\Omega(\mathbf{u}(t)\cdot\nabla)\mathbf{w}(t)\cdot\mathbf{u}(t)d\mathbf{x}\right| \leq \tag{4.33}$$

$$\frac{2\nu}{C_K}\|\mathbf{u}(t)\|^2_{\mathbf{H}^1(\Omega)} + \frac{C_K}{\nu}\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)}\|\mathbf{w}(t)\|^2_{\mathbf{H}^1(\Omega)}\,.$$

Then, using Korn's inequality (3.15), we obtain, from (4.32), the following relation if $\beta \leq 0$

$$\frac{d}{dt}\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} \leq \frac{C_K}{\nu}\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)}\|\mathbf{w}(t)\|^2_{\mathbf{H}^1(\Omega)}\,. \tag{4.34}$$

Integrating (4.34), using (4.31) for $\mathbf{w}$, and known that $\mathbf{u}(0) = \mathbf{0}$, we prove that $\mathbf{v} = \mathbf{w}$. If $\beta > 0$, we firstly apply the trace inequality (3.13) to I_3 and after Cauchy's inequality, to obtain

$$|I_3| \leq \frac{\nu}{C_K} \|\mathbf{u}(t)\|_{\mathbf{H}^1(\Omega)} + \frac{C_K C_{tr}\beta}{2\nu} \|\mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)} .$$

Then proceeding as above, replacing in (4.33) $2\nu/C_K$ by ν/C_K and C_K/ν by $C_K/2\nu$, we get

$$\frac{d}{dt}\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} \leq \frac{C_K}{2\nu} \left(\|\mathbf{w}(t)\|^2_{\mathbf{H}^1(\Omega)} + C_{tr}\beta\right) \|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)}$$

and the result follows in the same manner. □

Remark 4.2 *Arguing as in Ladyzhenskaya [23, §6.3] (see also Galdi [17]), we can prove that, for $N = 2$, the weak solution $\mathbf{u}$ to the problem (4.17)-(4.19) satisfies the following energy equality:*

$$\frac{d}{dt}\int_\Omega |\mathbf{u}(t)|^2 d\mathbf{x} + \nu \int_\Omega |\mathbf{D}(\mathbf{u}(t))|^2 d\mathbf{x} + \alpha \int_\Omega |\mathbf{u}(t)|^\sigma d\mathbf{x} \tag{4.35}$$

$$= \int_\Omega \mathbf{g}(t)\cdot\mathbf{u}(t) d\mathbf{x} + \beta \int_{\partial\Omega} |\mathbf{u}(t)|^2 dS.$$

In what concerns to the 3-D case, we conjecture that it is possible to prove the existence of, at least, a global in time weak solution in the sense of Definition 4.1. Indeed, in Busuioc and Ratiu [13, p. 1134] is used the Galerkin method with a special basis to prove the existence of a weak solution for a second grade fluid problem with slip boundary conditions. We think that this method can be applied, with some modifications, for the Navier-Stokes problem with slip boundary conditions (2.9). As for uniqueness, we know that, for the Navier-Stokes problem with no-slip boundary conditions and $N = 3$, is an open problem. Therefore, uniqueness of weak solutions for our problem (4.17)-(4.19) is also an open problem.

The results of this section could have been proved, at least for the 2-D case, if we have consider a forces field such that

$$\mathbf{f}(\mathbf{x}, t, \mathbf{u}) = -\alpha|\mathbf{u}|^{\sigma-2}\mathbf{u} + \mathbf{h}(\mathbf{x}, t, \mathbf{u})\,, \quad \mathbf{h}(\mathbf{x}, t, \mathbf{u})\cdot\mathbf{u} \leq 0\,,$$

and $\mathbf{h}$ a Carathéodory function. In this case, the proof would be carried out by using a truncation and approximation argument together with a fixed point theorem. To control the convergence of suitable approximations, we would have to add some extra assumptions on $\mathbf{h}(\mathbf{x}, t, \mathbf{u})$ for large values of $\mathbf{u}$. To prove the uniqueness, we would have to assume a non-increasing condition on $\mathbf{h}$. See the references [3, 4, 5] where this procedure was adopted for stationary problems with no-slip boundary conditions.

5 Extinction in time

The results proved in this section are valid for any dimension $N \geq 2$, though we are not able, at the moment, to prove the existence of, at least, a weak solution the problem (4.17)-(4.19) if $N \geq 3$. In such situations, we have conditional results, *i.e.*, if the weak solutions exist, then they will satisfy to the properties proved here. The notion of weak solution considered here is in the sense of Definition 4.1. For $N = 2$, the energy equality (4.35) holds and therefore the formalism of multiplying the momentum equation $(4.17)_2$ by a weak solution can be dropped. However, for $N = 3$, even for the no-slip boundary conditions case, (4.35) is no longer valid. Weak solutions, for the $3 - D$ problem satisfy to an energy inequality - the sign $=$ is replaced by $\leq$ (see Galdi [17]). Nevertheless, this does not change any of our conclusions. Therefore, we will adopt that formalism for any dimension $N \geq 2$.

Let us first note that replacing $\mathbf{v}$ by $\mathbf{u}$ in (4.20), we obtain

$$\int_\Omega \operatorname{div} \mathbf{S}\, \mathbf{u}\, d\mathbf{x} = -\nu \int_\Omega |\mathbf{D}(\mathbf{u})|^2\, d\mathbf{x} + \beta \int_{\partial\Omega} |\mathbf{u}|^2\, dS\,. \tag{5.36}$$

for every $\mathbf{u} \in \mathcal{W}$. Note that this formulae is independent of N.

Theorem 5.1 *Let $\mathbf{u}$ be a weak solution of problem (2.5)-(2.9) in the sense of Definition 4.1 for a general $N \geq 2$. Assume that the forces field $\mathbf{f}$ satisfies (2.10). Then, regardless the sign of β and what was the velocity at the initial instant of time, there exists a positive finite time t^* such that $\mathbf{u} = \mathbf{0}$ for almost all $t \geq t^*$.*

PROOF. We formally multiply (2.6) by $\mathbf{u}$, a weak solution to problem (2.5)-(2.9), and use the integration by parts formulae (5.36), to obtain the energy equality:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}|^2 d\mathbf{x} + \nu \int_\Omega |\mathbf{D}(\mathbf{u})|^2 d\mathbf{x} = \beta \int_{\partial\Omega} |\mathbf{u}|^2 dS + \int_\Omega \mathbf{f}\,.\mathbf{u} d\mathbf{x}\,. \tag{5.37}$$

Using (2.10) and Korn's inequality (3.15), we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}|^2 d\mathbf{x} + \frac{\nu}{C_K}\int_\Omega |\nabla\, \mathbf{u}|^2 d\mathbf{x} + C_{\mathbf{f}}\int_\Omega |\mathbf{u}|^\sigma d\mathbf{x} \leq \beta \int_{\partial\Omega} |\mathbf{u}|^2 dS\,. \tag{5.38}$$

<u>Let us first consider the case $\beta \leq 0$</u>. In this case, we obtain

$$\frac{d}{dt}\int_\Omega |\mathbf{u}|^2 d\mathbf{x} + C_2 \int_\Omega \left(|\nabla\, \mathbf{u}|^2 + |\mathbf{u}|^\sigma\right) d\mathbf{x} \leq 0\,, \quad C_2 = 2\min(\nu/C_K, C_{\mathbf{f}})\,. \tag{5.39}$$

Now, we recall that each component u_i, $i = 1, \ldots, N$, of $\mathbf{u}$ satisfies to (3.14). Thus, we can use inequality (3.12) with $N = 2$, $p = q = 2$ and $r = \sigma$, to obtain for each $i = 1, \ldots, N$

$$\|u_i\|^2_{L^2(\Omega)} \leq C\|\nabla u_i\|^{2\theta}_{L^2(\Omega)}\|u_i\|^{2(1-\theta)}_{L^\sigma(\Omega)}, \quad C = C(N, \sigma, \partial\Omega), \tag{5.40}$$

where

$$\theta = 1 - \frac{2\sigma}{\sigma(N-2) - 2N} \in (0,1) \Leftarrow N \geq 2, \quad 1 < \sigma < 2. \tag{5.41}$$

In particular, $1 < \sigma < 2$ implies

$$\theta = 1 - \frac{\sigma}{2} \in \left(0, \frac{1}{2}\right) \quad \text{for} \quad N = 2, \qquad \theta = 1 - \frac{2\sigma}{6-\sigma} \in \left(0, \frac{3}{5}\right) \quad \text{for} \quad N = 3.$$

We use the trivial inequalities $|u_i|^2 \leq |\mathbf{u}|^2$ and $|\nabla u_i|^2 \leq |\nabla \mathbf{u}|^2$, and sum up, between $i = 1$ and $i = N$, the resulting relation from (5.40). After that, we use the algebraic inequality (3.11) to obtain

$$\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \leq C \left(\|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}\|^\sigma_{\mathbf{L}^\sigma(\Omega)} \right)^\mu, \quad C = C(N, \sigma, \partial\Omega), \tag{5.42}$$

where, from (5.41),

$$\mu := \theta + \frac{2}{\sigma}(1-\theta) = 1 + \frac{2(2-\sigma)}{\sigma(N-2) - 2N} > 1 \Leftarrow N \geq 2, \quad 1 < \sigma < 2. \tag{5.43}$$

In particular,

$$\mu = 2 - \frac{\sigma}{2} \in \left(1, \frac{3}{2}\right) \quad \text{for} \quad N = 2, \qquad \mu = 1 + \frac{4-2\sigma}{6-\sigma} \in \left(1, \frac{7}{5}\right) \quad \text{for} \quad N = 3.$$

Then, conjugating (5.39) and (5.42), we obtain the homogeneous ordinary differential inequality

$$\frac{d}{dt} y(t) + C\,(y(t))^{\frac{1}{\mu}} \leq 0\,, \quad C = C(N, \sigma, \partial\Omega), \quad y(t) := \int_\Omega |\mathbf{u}|^2 d\mathbf{x}\,. \tag{5.44}$$

The usage of (5.43) and an explicit integration of (5.44) proves the existence of

$$t^* := \frac{\mu}{C(\mu - 1)} E(0)^{\frac{\mu-1}{\mu}} > 0, \quad C = C(N, \sigma, \partial\Omega)$$

and such that $\mathbf{u} = \mathbf{0}$ for all $t \geq t^*$.

Now we consider the case $\beta > 0$. Keeping in mind that each component u_i, $i = 1, \ldots, N$, of $\mathbf{u}$ satisfies to (3.14), we use here first the trace inequality (3.13), with $q = 2$, on the right-hand term of (5.38), to obtain

$$\|u_i\|^2_{L^2(\partial\Omega)} \leq C \|\nabla u_i\|^2_{L^2(\Omega)} \|u_i\|^2_{L^2(\Omega)}, \quad C = C(N, \partial\Omega). \tag{5.45}$$

Then we use, in the following order, Cauchy's inequality with a suitable ε, the inequalities $|u_i|^2 \leq |\mathbf{u}|^2$ and $|\nabla u_i|^2 \leq |\nabla \mathbf{u}|^2$, and sum up, between $i = 1$ and $i = N$, the resulting relation from (5.45). This leads us to the following inequality

$$\frac{d}{dt} \int_\Omega |\mathbf{u}|^2 d\mathbf{x} + C_1 \int_\Omega \left(|\nabla \mathbf{u}|^2 d\mathbf{x} + |\mathbf{u}|^\sigma \right) d\mathbf{x} \leq C_2 \int_\Omega |\mathbf{u}|^2 d\mathbf{x}\,, \tag{5.46}$$

where C_1 and C_2 are two different positive constants depending on N, ν, $\partial\Omega$ and β. Now, combining (5.46) and (5.42), we obtain the following nonhomogeneous ordinary differential inequality

$$y' + C_1 y^{\frac{1}{\mu}} \leq C_2 y\,, \qquad y(t) = \int_\Omega |\mathbf{u}|^2 d\mathbf{x}, \quad \mu > 1,$$

where now C_1 depends also on σ. Introducing successively the new variables

$$v = y e^{-C_2 t} \quad \text{and} \quad \iota = \frac{\mu}{C_2(\mu-1)}\left(1 - e^{-\frac{(\mu-1)C_2}{\mu}t}\right)$$

we came to the homogeneous ordinary differential inequality (5.44) for v, and the result follows as there.□

Remark 5.1 *This result can also be established for unbounded domains with compact boundaries as far the inequalities of Gagliardo-Nirenberg (3.12), traces (3.13) and Korn (3.15) hold. For instance, its validity extends to convex unbounded domains, but bounded, at least, in one direction.*

Now, we consider in the momentum equation a forces field which exhibits anisotropic feedback nonlinearities:

$$-\mathbf{f}(\mathbf{x},t,\mathbf{u})\cdot\mathbf{u} \geq \sum_{i=1}^{N} C_{\mathbf{f}}^i\,|u_i|^{\sigma_i} \quad \forall\, \mathbf{u}\in\mathbb{R}^N, \qquad \sigma_i \in (1,2), \tag{5.47}$$

for some non-negative constants $C_{\mathbf{f}}^i$, with $i = 1,\ldots,N$. Bellow we will prove that we can obtain the result of Theorem 5.1 if all the constants $C_{\mathbf{f}}^i$ are positive.

Theorem 5.2 *Let $\mathbf{u}$ be a weak solution of problem (2.5)-(2.9) in the sense of Definition 4.1 for a general $N \geq 2$. Assume that the forces field $\mathbf{f}$ satisfies (5.47) with $C_{\mathbf{f}}^i > 0$ for every $i = 1,\ldots,N$. Then, regardless the sign of β and what was the velocity at the initial instant of time, there exists a positive finite time t^* such that $\mathbf{u} = \mathbf{0}$ for almost all $t \geq t^*$.*

PROOF. Proceeding as in the proof of Theorem 5.1, we obtain the analogous of (5.38)

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}|^2 d\mathbf{x} + \frac{\nu}{C_K}\int_\Omega |\nabla\,\mathbf{u}|^2 d\mathbf{x} + \sum_{i=1}^{N} C_{\mathbf{f}}^i \int_\Omega |\mathbf{u}|^{\sigma_i} d\mathbf{x} \leq \beta\int_{\partial\Omega} |\mathbf{u}|^2 dS\,. \tag{5.48}$$

Moreover, (5.40) is valid here too and from this relation, using again (3.11), we can easily obtain, for each $i = 1,\ldots,N$,

$$\|u_i\|^2_{L^2(\Omega)} \leq C\left(\|\nabla\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \sum_{i=1}^{N}\|u_i\|^{\sigma_i}_{L^{\sigma_i}(\Omega)}\right)^{\mu_i}, \quad C = C(N,\sigma_i,\partial\Omega), \tag{5.49}$$

where μ_i is defined in (5.43) by replacing σ with σ_i. Now, we assume, with no loss of generality, that $\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} < 1$. Applying this assumption to (5.49) and adding up, between $i = 1$ and $i = N$, the resulting relations, we obtain

$$\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \leq C\left(\|\nabla\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \sum_{i=1}^{N} \|u_i\|^{\sigma_i}_{L^{\sigma_i}(\Omega)}\right)^{\mu}, \quad C = C(N, \sigma_1, \ldots, \sigma_N, \partial\Omega),$$

where $\mu = \min_{1\leq i\leq N} \mu_i$. The rest of the proof follows as in the proof of Theorem 5.1 either $\beta \leq 0$ or $\beta > 0$. $\square$

However, if in (5.47) we assume that, at least, one $C^i_{\mathbf{f}}$ is zero, we are not able to establish the same result, unless we improve the assumptions. For the sake of the exposition, let us assume that, additionally to (5.47), we have

$$C^N_{\mathbf{f}} = 0 \qquad \text{and} \quad C^i_{\mathbf{f}} > 0 \quad \text{for all} \quad i \neq N\,. \tag{5.50}$$

In this case, proceeding as in the proof of Theorem 5.2, we can get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} |\mathbf{u}|^2 d\mathbf{x} + \nu \int_{\Omega} |\nabla\,\mathbf{u}|^2 d\mathbf{x} + \sum_{i=1}^{N-1} C^i_{\mathbf{f}} \int_{\Omega} |\mathbf{u}|^{\sigma_i} d\mathbf{x} \leq \beta \int_{\partial\Omega} |\mathbf{u}|^2 dS\,. \tag{5.51}$$

And, for each $i = 1, \ldots, N-1$,

$$\|u_i\|^2_{L^2(\Omega)} \leq C\left(\|\nabla\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \sum_{i=1}^{N-1} \|u_i\|^{\sigma_i}_{L^{\sigma_i}(\Omega)}\right)^{\mu_*}, \tag{5.52}$$

where $C = C(N, \sigma_1, \ldots, \sigma_{N-1}, \partial\Omega)$, $\mu_* = \min_{1\leq\mu_i\leq N-1}$ and μ_i are defined also as in (5.43) replacing σ by σ_i. To estimate $\|u_N\|^2_{\mathbf{L}^2(\Omega)}$, we use the same arguments we have used in [2]. We introduce the hyperplane

$$\Omega(z) = \Omega \cap \{\mathbf{x} = (\mathbf{x}', x_N) \in \mathbb{R}^N : \mathbf{x}' = (x_1, \ldots, x_{N-1}) \quad \text{and} \quad x_N = z\} \subseteq \mathbb{R}^{N-1}$$

and we, additionally, assume that

$$\text{the domain } \Omega \text{ is convex, at least, in the } x_N \text{ direction.} \tag{5.53}$$

Then, we formally multiply (2.5) by a weakly free divergence vector $\mathbf{u}$ and integrate by parts over $\Omega(z)$, where we use (2.9), to obtain

$$-\frac{1}{2}\frac{\partial}{\partial z}\int_{\Omega(z)} u_N^2\, d\mathbf{x}' = \int_{\Omega(z)} \sum_{i=1}^{N-1} \frac{\partial u_N}{\partial x_i} u_i\, d\mathbf{x}'. \tag{5.54}$$

Next, the integration of (5.54) between x^0_N, chosen such that $(x_1, \ldots, x_{N-1}, x^0_N) \in \partial\Omega$, and $x_N \leq x^1_N$, with x^1_N also chosen such that $(x_1, \ldots, x_{N-1}, x^1_N) \in \partial\Omega$, lead us to

$$\int_{\Omega(z)} u_N^2\, d\mathbf{x}' - \int_{\Omega(x^0_N)} u_N^2\, d\mathbf{x}' = -2\int_{x^0_N}^{x_N} \int_{\Omega(z)} \sum_{i=1}^{N-1} \frac{\partial u_N}{\partial x_i} u_i\, d\mathbf{x}'\, dz. \tag{5.55}$$

To proceed with the same kind of arguments, we need that the second term of the left-hand member of (5.55) vanishes. This is equivalent to assume

$$\partial\Omega_N \text{ is orthogonal to the } x_N \text{ axis}, \tag{5.56}$$

where $\partial\Omega_N = \partial\Omega \cap \Omega(z)$. Note that $\mathbf{u}\cdot\mathbf{n} = 0$ and condition (5.56) imply that $u_N = 0$ on $\partial\Omega_N \times (0,T)$. In this case, we apply Hölder's inequality to the resulting equation of (5.55), to obtain

$$\|u_N\|^2_{2,\Omega(z)} \leq C\,\|\nabla u_N\|_{2,\Omega(z)} \sum_{i=1}^{N-1} \|u_i\|_{2,\Omega(z)}\,, \quad C = C(N).$$

Integrating the last inequality with respect to z and using, again, Hölder's inequality, we achieve to the estimate

$$\|u_N\|^2_{\mathbf{L}^2(\Omega)} \leq C\,\|\nabla u_N\|_{\mathbf{L}^2(\Omega)} \sum_{i=1}^{N-1} \|u_i\|_{\mathbf{L}^2(\Omega)}\,, \quad C = C(N).$$

Now, applying, for each $i = 1,\dots,N-1$, (5.52) and then (3.11), we came to the inequality

$$\|u_N\|^2_{L^2(\Omega)} \leq C\left(\|\nabla\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \sum_{i=1}^{N-1} \|u_i\|^{\sigma_i}_{L^{\sigma_i}(\Omega)}\right)^{\mu_N}, \tag{5.57}$$

where $C = C(N,\sigma_1,\dots,\sigma_{N-1},\partial\Omega)$ and

$$\mu_N = \frac{1}{2} + \frac{\mu_*}{2} > 1 \Leftarrow \mu_* = \min_{i=1,\dots,N-1} \mu_i > 1. \tag{5.58}$$

Again, assuming, with no loss of generality, that $\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} < 1$, we obtain, from (5.52) and (5.57),

$$\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \leq C\left(\|\nabla\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \sum_{i=1}^{N-1} \|u_i\|^{\sigma_i}_{L^{\sigma_i}(\Omega)}\right)^{\mu},$$

where $C = C(N,\sigma_1,\dots,\sigma_{N-1},\partial\Omega)$ and $\mu = \min(\mu_*,\mu_N) \equiv \min_{1\leq i\leq N}\mu_i > 1$. Therefore, proceeding as in the proof of Theorem 5.1, we are able to establish the following result.

Theorem 5.3 *Let* $\mathbf{u}$ *be a weak solution of problem (2.5)-(2.9) in the sense of Definition 4.1 for a general* $N \geq 2$. *Assume also that (5.53) and (5.56) are fulfilled. If the forces field* $\mathbf{f}$ *satisfies (5.47) and (5.50), then, regardless the sign of* β *and what was the velocity at the initial instant of time, there exists a positive finite time* t^* *such that* $\mathbf{u} = \mathbf{0}$ *for almost all* $t \geq t^*$.

Remark 5.2 *The extra conditions (5.53) and (5.56) restrict the validity of the aforementioned result to domains with flat boundaries* $\partial\Omega_N$ *perpendicular to the* x_N *axis, being* x_N *the direction where the absorption forces field is absent. Examples of such domains are parallelepipeds and cylinders - in the last case, is on the basis where condition (5.56) must be fulfilled.*

Remark 5.3 *We are not able to prove the same result if, in (5.47), more than one $C_{\mathbf{f}}^i$ is zero. One justification for that is because the problem is stated by only two equations: (2.5) and (2.6). The momentum equation (2.6) is used to establish, for each $i = 1, \ldots, N-1$, the estimates (5.52). And the continuity equation (2.5) is fundamental to establish the analogous estimate (5.57) for $i = N$.*

In the limit, if, in (5.47), all $C_{\mathbf{f}}^i$ are zero, we know that the best we can get is an exponential decay. See, *e.g.*, [22, Theorem 6.1], where this is proved for $N = 2$. Interesting is the fact that, in such situation, for certain positive values of β, we nor can even expect an exponential decay. In fact, from (5.49), with the assumption that all $C_{\mathbf{f}}^i$ in (5.47) are zero, we obtain

$$\frac{d}{dt}\int_{\Omega}|\mathbf{u}|^2 d\mathbf{x} + 2\nu\int_{\Omega}|\nabla\,\mathbf{u}|^2 d\mathbf{x} \leq 2\beta\int_{\partial\Omega}|\mathbf{u}|^2 dS\,. \tag{5.59}$$

We apply a vector version of (3.13), with $q = 2$ and $\alpha = 1/2$, to the right-hand term of (5.59) and then we apply Cauchy's inequality with $\varepsilon > 0$. Finally, we apply (3.12), with $p = q = 2$ and $\theta = 1$, to the resulting second left-hand term of (5.59), and we obtain

$$\frac{d}{dt}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + C_{GN}^{-2}(2\nu - 2C_{tr}\beta\,\varepsilon)\|\mathbf{u}|^2_{\mathbf{L}^2(\Omega)} \leq C_{tr}\frac{\beta}{2\varepsilon}\|\mathbf{u}|^2_{\mathbf{L}^2(\Omega)} \tag{5.60}$$

where C_{GN} and C_{tr} are the constants resulting from applying (3.12) and (3.13), respectively. Therefore, for positive values of β such that

$$\beta > \frac{2\nu C_{GN}^{-2}}{2C_{tr}C_{GN}^{-2}\varepsilon + C_{tr}/\varepsilon},$$

the solution of (5.60) does not have exponential decay. Note that this is independent of ε, which must be chosen such that $\nu - C_{tr}\beta\,\varepsilon > 0$.

We can gather the results of Theorems 5.1, 5.2 and 5.3 to give them a general presentation, but adding to the forces field a suitable field which vanishes in a short time. We consider a forces field which satisfies to

$$\mathbf{f}(\mathbf{x}, t, \mathbf{u}) = \mathbf{h}(\mathbf{x}, t, \mathbf{u}) + \mathbf{g}(\mathbf{x}, t), \tag{5.61}$$

where $\mathbf{h}$ stands for one of the following fields

$$\mathbf{h}(\mathbf{x}, t, \mathbf{u}) = -C_{\mathbf{f}}|\mathbf{u}|^{\sigma-2}\mathbf{u}, \quad \sigma \in (1, 2), \quad C_{\mathbf{f}} > 0; \tag{5.62}$$

$$\mathbf{h}(\mathbf{x}, t, \mathbf{u}) = -(C_{\mathbf{f}}^1|u_1|^{\sigma_1-2}u_1, \ldots, C_{\mathbf{f}}^1|u_N|^{\sigma_N-2}u_N),\ \sigma_i \in (1, 2),\ C_{\mathbf{f}}^i \geq 0, \tag{5.63}$$

and, at most, only one $C_{\mathbf{f}}^i$ is zero, $i = 1, \ldots, N$. The extra field $\mathbf{g}$ satisfies to

$$\|\mathbf{g}(\cdot, t)\|^2_{\mathbf{L}^2(\Omega)} \leq \epsilon\,(1 - t/t_{\mathbf{g}})_+^{\frac{1}{\mu-1}}\,, \quad \mu > 1, \tag{5.64}$$

for some positive constants ϵ and $t_{\mathbf{g}}$, and where $u_+ = \max(0, u)$.

Theorem 5.4 *Let* $\mathbf{u}$ *be a weak solution of problem (2.5)-(2.9) in the sense of Definition 4.1 for a general* $N \geq 2$. *Assume that the forces field* $\mathbf{f}$ *satisfies one of the following items:*
1. (5.61), (5.62) and (5.64);
2. (5.61), (5.63) with $C_{\mathbf{f}}^i > 0$ *for all* $i = 1, \ldots, N$, *and (5.64); or*
3. (5.61), (5.63) with (5.50), and (5.64). In this case assume moreover that (5.53) and (5.56) are fulfilled.
Then, regardless the sign of β *and what was the velocity at the initial instant of time, there exist constants* $\epsilon_0 > 0$ *and* $t^* \geq 0$ *such that* $\mathbf{u} = \mathbf{0}$ *for almost all* $t \geq t_{\mathbf{g}}$, *if* $\epsilon_0 \geq \epsilon > 0$ *and* $t_{\mathbf{g}} \geq t^*$.

PROOF. Proceeding, correspondingly for each item, as in the proofs of Theorems 5.1, 5.2 and 5.3, and using (5.64), we obtain the ordinary differential inequality if $\beta \leq 0$:

$$\frac{d}{dt}y(t) + C_1\,(y(t))^{\frac{1}{\mu}} \leq C_2\,(1 - t/t_{\mathbf{g}})_+^{\frac{1}{\mu-1}} \qquad \mu > 1, \quad y(t) := \int_\Omega |\mathbf{u}|^2 d\mathbf{x}\,, \quad (5.65)$$

where $C_1 = C_1(N, \sigma, \partial\Omega)$ and $C_2 = C_2(N, \sigma, \partial\Omega, \epsilon_0)$. The only difference is the estimate of the term involving $\mathbf{g}$. In that term, we use first Young's inequality with a suitable ε. And then, the resulting term $\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}$ is estimated in terms of $\|\nabla\,\mathbf{u}\|_{\mathbf{L}^2(\Omega)}$ by using Gagliardo-Nirenberg's inequality with $p = q = 2$ and $\theta = 1$. The analysis of (5.65) made in Lemma 2.3 and Remark 2.4 of [7] proves the theorem if $\beta \leq 0$. The choice of ϵ_0 and t^* can be done, *mutatis mutandis*, as it was in the proof of Theorem 7.1 in [7], p. 229. Proceeding analogously, we obtain for $\beta > 0$:

$$\frac{d}{dt}y(t) + C_1\,(y(t))^{\frac{1}{\mu}} \leq C_2 y(t) + C_3\,(1 - t/t_{\mathbf{g}})_+^{\frac{1}{\mu-1}}\;, \quad y(t) := \int_\Omega |\mathbf{u}|^2 d\mathbf{x}\,, \quad (5.66)$$

where $\mu > 1$, $C_1 = C_1(N, \sigma, \partial\Omega)$, $C_2 = C_2(N, \sigma, \partial\Omega)$ and $C_3 = C_3(N, \sigma, \partial\Omega, \epsilon_0)$. The same arguments we have used in the proof of Theorems 5.1 allow us to recover (5.65) and the result follows. □

Remark 5.4 *Note that* t^* *depends on* $y(0) = \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)}$ *and therefore we need the assumption that* $\|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)}$ *is finite.*

In what concerns to the analogous effects in space, *i.e.*, the existence of a subdomain $\Omega_0 \subset \Omega$ where $\mathbf{u} = \mathbf{0}$, is a delicate problem. So far we expect to publish elsewhere these results but for 2-D stationary problems.

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