ASYMPTOTIC SOLUTIONS OF A DEGENERATE GARNIER SYSTEM OF THE FIRST PAINLEVE TYPE

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ASYMPTOTIC SOLUTIONS OF A DEGENERATE GARNIER SYSTEM OF THE FIRST PAINLEVÉ TYPE

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Abstract. We consider a degenerate Garnier system of the first Painlevé type on \( P^1(\mathbb{C}) \times P^1(\mathbb{C}) \). Around a singular locus of irregular type, we present a three-parameter family of solutions of it. Restriction of them to a certain hyperplane yields asymptotic solutions of the fourth order version of the first Painlevé equation.

1. Introduction

Suppose that the linear differential equation

\[
\frac{d^2 y}{dx^2} - \left( \sum_{k=1,2} \frac{1}{x - \lambda_k} \right) \frac{dy}{dx} - \left( 9x^5 + 9t_1x^3 + 3t_2x^2 + 3K_2x + 3K_1 - \sum_{k=1,2} \frac{\mu_k}{x - \lambda_k} \right) y = 0
\]

has non-logarithmic singular points at \( x = \lambda_1, \lambda_2 \). Then \( K_1 \) and \( K_2 \) are given by

\[
3K_1 = \sum_{k=1,2} \frac{\Pi_1(\lambda_k)}{\Pi_0(\lambda_k)} \left( \mu_k^2 - \frac{\mu_k}{\Pi_1(\lambda_k)} - 9\lambda_k^5 - 9t_1\lambda_k^3 - 3t_2\lambda_k^2 \right),
\]

\[
3K_2 = \sum_{k=1,2} \frac{1}{\Pi_0'(\lambda_k)} \left( \mu_k^2 - 9\lambda_k^5 - 9t_1\lambda_k^3 - 3t_2\lambda_k^2 \right)
\]

with

\[
\Pi_0(\xi) = (\xi - \lambda_1)(\xi - \lambda_2), \quad \Pi_1(\xi) = \xi - \lambda_1 - \lambda_2.
\]

The isomonodromic deformation with respect to the parameters \( t_1, t_2 \) yields the completely integrable Hamiltonian system

\[
\frac{\partial \lambda_j}{\partial t_k} = \frac{\partial K_k}{\partial \mu_j}, \quad \frac{\partial \mu_j}{\partial t_k} = -\frac{\partial K_k}{\partial \lambda_j} \quad (j, k = 1, 2);
\]
which is equivalent to

\[
\frac{\partial q_j}{\partial s} = \frac{\partial H_1}{\partial p_j}, \quad \frac{\partial p_j}{\partial s} = -\frac{\partial H_1}{\partial q_j}, \quad (j = 1, 2)
\]

(G)

\[
\frac{\partial q_j}{\partial t} = \frac{\partial H_2}{\partial p_j}, \quad \frac{\partial p_j}{\partial t} = -\frac{\partial H_2}{\partial q_j} \quad (j = 1, 2)
\]

with the Hamiltonians

\[
3H_1 := \left(q_2^2 - q_1 - \frac{s}{3}\right)p_1^2 + 2q_2p_1p_2 + p_2^2 + 9\left(q_1 + \frac{s}{3}\right)q_2 \left(q_2^2 - 2q_1 + \frac{s}{3}\right) - 3tq_1,
\]

\[
3H_2 := q_2p_1^2 + 2p_1p_2 + 9\left(q_2^4 - 3q_1q_2^2 + q_1^2 - \frac{s}{3}q_1 - \frac{t}{3}q_2\right).
\]

Here the new unknowns and variables are given by

\[
(q_1, q_2) = (\lambda_1\lambda_2 - t_1/3, \lambda_1 + \lambda_2),
\]

\[
(p_1, p_2) = \left(\frac{\mu_1 - \mu_2}{\lambda_2 - \lambda_1}, \frac{\lambda_1\mu_1 - \lambda_2\mu_2}{\lambda_1 - \lambda_2}\right),
\]

\[
(s, t) = (t_1, -t_2)
\]

(see [1]). This system may be regarded as a two-variable version of the first Painlevé equation PI. Let us consider (G) on \(P^1(\mathbb{C}) \times P^1(\mathbb{C}) \ (\ni (s, t))\). Then, (G) admits singular loci along \(s = \infty\) and \(t = \infty\). Restricting (G) to the complex line \(s = s_0 \ (\in \mathbb{C})\), we obtain the fourth order nonlinear differential equation

\[
q^{(4)} = 20qq'' + 10(q')^2 - 40q^3 - 8s_0q - \frac{8}{3}t \quad (\prime = d/dt, \ q := q_2),
\]

which belongs to the PI-hierarchy. For the PI-hierarchy written in the Hamiltonian form containing a large parameter, Y. Takei ([6]) constructed instanton-type formal solutions containing many free parameters, by reducing to the Birkhoff normal form. For Painlevé equations PI, ..., PV, two-parameter families of solutions were obtained near irregular singularities ([3], [4], [5], [7]). Furthermore, H. Kimura et al. ([2]) gave a reduction theorem for a class of Hamiltonian systems containing a Garnier system of PVI type around a regular singular locus.

In this paper, we give a three-parameter family of solutions of (G) near the singular locus \(t = \infty\), by constructing a canonical transformation which reduces (G) to

\[
\frac{\partial Q_j}{\partial s} = \frac{\partial L_1}{\partial P_j}, \quad \frac{\partial P_j}{\partial s} = -\frac{\partial L_1}{\partial Q_j},
\]

(G0)

\[
\frac{\partial Q_j}{\partial t} = \frac{\partial L_2}{\partial P_j}, \quad \frac{\partial P_j}{\partial t} = -\frac{\partial L_2}{\partial Q_j} \quad (j = 1, 2)
\]
with the Hamiltonians
\[
L_1 := \Lambda_1^{(1)}(t) Q_1 P_1 + \Lambda_2^{(1)}(t) Q_2 P_2,
\]
\[
L_2 := \Lambda_1^{(2)}(s, t) Q_1 P_1 + \Lambda_2^{(2)}(s, t) Q_2 P_2 + t^{-1} (\kappa_{20}(Q_1 P_1)^2 + \kappa_{11}Q_1 P_1 Q_2 P_2 + \kappa_{02}(Q_2 P_2)^2).
\]

Here
\[
\Lambda_1^{(1)}(t) := -(4\sqrt{5})^{-1} i \rho^3 t^{1/2}, \quad \Lambda_2^{(1)}(t) := (4\sqrt{5})^{-1} i \overline{\rho}^3 t^{1/2},
\]
\[
\Lambda_1^{(2)}(s, t) := -\rho t^{1/6} - (8\sqrt{5})^{-1} i \rho^3 st^{-1/2},
\]
\[
\Lambda_2^{(2)}(s, t) := -\overline{\rho} t^{1/6} + (8\sqrt{5})^{-1} i \overline{\rho}^3 st^{-1/2}
\]

with
\[
(1.2) \quad \rho := i r_0 e^{-i \omega}, \quad r_0 := 2^{3/4} 15^{1/12}, \quad \omega := \frac{1}{2} \tan^{-1}(1/\sqrt{5}) \in (0, \pi/4),
\]
\[
\kappa_{20} = (-7 + 2\sqrt{5} i)/24, \quad \kappa_{11} = 2\sqrt{30}/5, \quad \kappa_{02} = \overline{\kappa_{20}}.
\]

Furthermore, restricting solutions of \((G_0)\) to the hyperplane \(s = s_0\), we obtain asymptotic solutions of \((P\I)\) near \(t = \infty\). In the final section, we sketch the process of construction of the formal canonical transformation. We employ the standard method for obtaining such transformations ([2], [5], [6]).

2. Results

To state our results, we explain the following notation:

(a) Consider the matrix
\[
J_0 := \begin{pmatrix}
0 & 2\beta/3 & 0 & 2/3 \\
-6 & 0 & 18\beta & 0 \\
0 & 2/3 & 0 & 0 \\
18\beta & 0 & -9\beta^2 & 0
\end{pmatrix}
\]

with
\[
\beta := -15^{-1/3}.
\]

The eigenvalues of \(J_0\) are \(\pm \rho, \pm \overline{\rho}\), and we have
\[
T_0^{-1} J_0 T_0 = \text{diag}[-\rho, \rho, -\overline{\rho}, \overline{\rho}],
\]
where

\begin{align*}
T_0 &:= (2\sqrt{5})^{-1/2}D_0\Omega, \\
D_0 &:= \text{diag}\left[2^{-1/2}r_0^{1/2}\beta^{-1/2}, \frac{\sqrt{6}}{2}r_0^{1/2}\beta^{-1/2}, \frac{\sqrt{6}}{3}r_0^{1/2}\beta^{-1/2}, \frac{3}{2}r_0^{1/2}\beta^{1/2}\right], \\
\Omega &:= 
\begin{pmatrix}
\ne^{-3i\omega/2} & e^{-3i\omega/2} & e^{3i\omega/2} & e^{3i\omega/2} \\
-\ne^{-i\omega/2} & ie^{3i\omega/2} & ie^{-3i\omega/2} & -ie^{-3i\omega/2}
\end{pmatrix}.
\end{align*}

(b) We fix the arguments of the eigenvalues of $J_0$ in such a way that

\[-\pi < \arg(-\rho) < -\frac{\pi}{2} < \arg \bar{\rho} < 0 < \arg \rho < \frac{\pi}{2} < \arg(-\bar{\rho}) < \pi,\]

where $\arg \bar{\rho} = -\arg \rho$, $\arg(-\bar{\rho}) = \pi - \arg \rho$, $\arg(-\rho) = -\pi + \arg \rho$. Let $\Sigma_0$ be the sector in the $t$-plane defined by

\begin{equation}
\Sigma_0 : -\arg \rho < \frac{7}{6}\arg t < \pi - \arg \rho.
\end{equation}

(c) For an arbitrary sector $\Sigma$ in the $t$-plane, and for a function $f(s, t)$ holomorphic for $(s, t) \in \mathbb{C} \times \Sigma$, we write

\[f(s, t) \in \mathcal{A}(\Sigma),\]

if, for any positive number $R$, the function $f(s, t)$ admits the asymptotic representation

\[f(s, t) \sim \sum_{\nu \geq 0} f_\nu(s)t^{-\nu/6}\]

uniformly for $|s| < R$ as $t \to \infty$ through $\Sigma$, where $f_\nu(s)$ is an entire function of $s$.

(d) For a vector $\mathbf{v} = (v_1, ..., v_m)$, we denote by $^T\mathbf{v}$ the transpose of it, and for a multi-index $\mathbf{k} = (k_1, ..., k_m) \in (\mathbb{N} \cup \{0\})^m$, we write

\[|\mathbf{k}| := k_1 + \cdots + k_m, \quad \mathbf{v}^\mathbf{k} := v_1^{k_1} \cdots v_m^{k_m}.\]

The formal canonical transformation is given by the following:

**Theorem 2.1.** There exists a formal canonical transformation

\begin{equation}
T(q, p) = U(s, t, Q, P),
\end{equation}

\[q = (q_1, q_2), \quad p = (p_1, p_2), \quad Q = (Q_1, Q_2), \quad P = (P_1, P_2),\]

which reduces $(G)$ into $(G_0)$. The right-hand side of (2.5) is a formal series given by

\[U(s, t, Q, P) = T(u(s, t), v(s, t)) + t^{\Delta_0}T_0\left(\Gamma_0(s, t) + \sum_{|j|+|k| \geq 1} \Gamma_{jk}(s, t)Q^jP^k\right)^T(Q, P).\]
Here

(i) $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ satisfy

\[ t^{-2/3}u_1, t^{-1/3}u_2, t^{2/3}v_1, t^{1/3}v_2 \in A(\Sigma_0); \]

(ii) $\Gamma_0$ and $\Gamma_{jk}$ are 4 by 4 matrices such that each entry of

\[ \Gamma_0 = I(s) + O(t^{-1/6}), \quad t^{(7/12)(|j|+|k|-2)}\Gamma_{jk} \]

belongs to $A(\Sigma_0)$, where $I(s)$ is a diagonal matrix satisfying $\det I(s) \neq 0$ with entries entire in $s$;

(iii) $T_0$ is given by (2.3) and

\[ \Delta_0 := \text{diag}[1/12, -1/4, -1/12, 1/4]. \]

It is easy to see that system (G$_0$) admits a general solution:

\[ Q_1 = \Phi_1(C_1, C_2, s, t) := C_1t^{2\kappa_{20}C_1C_2 + \kappa_{11}C_3C_4} \exp \left( -\frac{6}{7} \rho t^{7/6} - \frac{i \rho^3 st^{1/2}}{4\sqrt{5}} \right), \]

\[ P_1 = \Psi_1(C_1, C_2, s, t) := C_2t^{-2\kappa_{20}C_1C_2 - \kappa_{11}C_3C_4} \exp \left( \frac{6}{7} \rho t^{7/6} + \frac{i \rho^3 st^{1/2}}{4\sqrt{5}} \right), \]

\[ Q_2 = \Phi_2(C_3, C_4, s, t) := C_3t^{2\kappa_{02}C_3C_4 + \kappa_{11}C_1C_2} \exp \left( -\frac{6}{7} \overline{\rho} t^{7/6} + \frac{i \overline{\rho}^3 st^{1/2}}{4\sqrt{5}} \right), \]

\[ P_2 = \Psi_2(C_3, C_4, s, t) := C_4t^{-2\kappa_{02}C_3C_4 - \kappa_{11}C_1C_2} \exp \left( \frac{6}{7} \overline{\rho} t^{7/6} - \frac{i \overline{\rho}^3 st^{1/2}}{4\sqrt{5}} \right), \]

where $C_j$ (1 ≤ $j$ ≤ 4) are integration constants. For generic values of $C_j$, there is no direction in $t$-plane along which $\Phi_1$, $\Psi_1$, $\Phi_2$, $\Psi_2$ are simultaneously bounded. In the sector $\Sigma_0$, however, there exists a curve tending to $\infty$ such that $\Phi_1(C_1, C_2, s, t)$, $\Psi_1(C_1, C_2, s, t)$, $\Phi_2(C_3, 0, s, t)$, and $\Psi_2(C_3, 0, s, t) \equiv 0$ are bounded. Indeed, along the ray $(7/6)\text{arg } t = \pi/2 - \text{arg } \rho$, we have $\text{arg}(\overline{\rho} t^{7/6}) = \pi/2 - 2 \text{arg } \rho$, implying that $\text{Re}( -\overline{\rho} t^{7/6}) < 0$. Substituting these into the formal transformation of Theorem 2.1, and rearranging the terms, we get a three-parameter family of solutions of (G$_0$). The asymptotic property is justified by using the method of successive approximation together with Borel-Ritt type reasoning (see [8]).

**Theorem 2.2.** Let $R_0$ be an arbitrary positive number, and let $\delta_0$ be an arbitrary small positive number. Then system (G) admits a family of solutions:

\[ T(q, p) = T(u(s, t), v(s, t)) \]

\[ + t^{\Delta_0} T_0 \Xi(s, t, t^{-1/12}\Phi_1(C_1, C_2, s, t), t^{-1/12}\Psi_1(C_1, C_2, s, t), t^{-1/12}\Phi_2(C_3, 0, s, t)), \]

\[ (C_1, C_2, C_3) \in \mathbb{C}^3, \quad |C_1C_2| < \delta_1 \]
for \((s,t)\) in the domain given by
\begin{align}
|(7/6)\arg t - (\pi/2 - \arg \rho)| < \delta_0, \\
|t^{-1/12}\Phi_1(C_1, C_2, s, t)| + |t^{-1/12}\Psi_1(C_1, C_2, s, t)| + |t^{-1/12}\Phi_2(C_3, 0, s, t)| < R_0, \\
|s| < R_0,
\end{align}
where \(\delta_1\) is a sufficiently small positive number. The vector function \(\Xi\) is written in the form
\[\Xi(s, t, X_1, Y_1, X_2) = t^{1/12} \left[ \Gamma_0(s, t) + F(s, t, X_1, Y_1, X_2) \right]^T(X_1, Y_1, X_2, 0),\]
where the vector function \(F\) admits the asymptotic representation
\[F(s, t, X_1, Y_1, X_2) \sim \sum_{\nu \geq 1} \Gamma_\nu(s, X_1, Y_1, X_2) t^{-\nu/6}\]
as \(t \to \infty\) through the sector (2.6) uniformly for \((s, X_1, Y_1, X_2)\) satisfying
\[|s| < R_0, \quad |t|^{1/6}|X_1Y_1| < \delta_1, \quad |X_1| + |Y_1| + |X_2| < R_0.\]
The coefficients \(\Gamma_\nu\) are vector functions whose entries are polynomials in \(X_1, Y_1, X_2\) with coefficients entire in \(s\).

Let us denote by \(q_2 = \chi(C_1, C_2, C_3, s_0, t)\) the second entry of each solution given above. For \(\text{(P1)}\) with \(s_0 \in \mathbb{C}\), we immediately obtain the following:

**Corollary 2.3.** Equation \(\text{(P1)}\) admits a family of solutions:
\[q = \chi(C_1, C_2, C_3, s_0, t), \quad (C_1, C_2, C_3) \in \mathbb{C}^3, \quad |C_1C_2| < \tilde{\delta}_1\]
for \(t\) in the domain given by (2.6) and (2.7) with \(s = s_0\), where \(\tilde{\delta}_1\) is a sufficiently small positive number depending on \(\delta_0\) and \(s_0\).

**Remark.** In the sectors
\[\Sigma_1 : -\pi - \arg \rho < \frac{7}{6} \arg t < -\arg \rho,\]
\[\Sigma_2 : -\pi + \arg \rho < \frac{7}{6} \arg t < \arg \rho,\]
\[\Sigma_3 : \arg \rho < \frac{7}{6} \arg t < \pi + \arg \rho,\]
we can construct analogous formal canonical transformations, which yield asymptotic solutions corresponding to the triples
\begin{align}
(i) & \quad \Phi_1(C_1, C_2, s, t), \quad \Psi_1(C_1, C_2, s, t), \quad \Psi_2(0, C_4, s, t), \\
(ii) & \quad \Phi_1(C_1, 0, s, t), \quad \Phi_2(C_3, C_4, s, t), \quad \Psi_2(C_3, C_4, s, t), \\
(iii) & \quad \Psi_1(0, C_2, s, t), \quad \Phi_2(C_3, C_4, s, t), \quad \Psi_2(C_3, C_4, s, t),
\end{align}
respectively.

3. Construction of the Formal Transformation

The construction of (2.5) is divided into several steps, and it is obtained by composing the transformations given in these steps. For the simplicity of description, in every step, we use the following common notation: we denote initial Hamiltonians and variables by $H_k$ and $q_j, p_j$ ($j, k = 1, 2$), namely an initial Hamiltonian system by

$$\frac{\partial q_j}{\partial s} = \frac{\partial H_1}{\partial p_j}, \quad \frac{\partial p_j}{\partial s} = -\frac{\partial H_1}{\partial q_j},$$
$$\frac{\partial q_j}{\partial t} = \frac{\partial H_2}{\partial p_j}, \quad \frac{\partial p_j}{\partial t} = -\frac{\partial H_2}{\partial q_j} \quad (j = 1, 2);$$

a canonical transformation by

(3.1) \quad (q_j, p_j) \mapsto (Q_j, P_j);

and the resultant Hamiltonians by $K_k$ ($k = 1, 2$) with variables $Q_j, P_j$. Note that transformation (3.1) is canonical, if

(3.2) \quad \sum_j dp_j \wedge dq_j = \sum_j dP_j \wedge dQ_j.

Then, $K_k$ are computed by using the identity

(3.3) \quad \sum_j dp_j \wedge dq_j - dH_1 \wedge ds - dH_2 \wedge dt
= \sum_j dP_j \wedge dQ_j - dK_1 \wedge ds - dK_2 \wedge dt.

In each step, our computation is concentrated on $H_2$. The corresponding expression of $H_1$ is derived by using the completely integrable condition.

3.1. Step 1. To eliminate the term $-3tq_2$ of $3H_2$, we put

\[ q_1 = Q_1 + \alpha t^{2/3}, \quad q_2 = Q_2 + \beta t^{1/3} \quad (\alpha, \beta \in \mathbb{C}). \]

It is easy to see that this is a canonical transformation. Substitution into $H_2$ for (G) yields

\[ 3\tilde{H}_2 := Q_2 P_1^2 + \beta t^{1/3} P_1^2 + 2 P_1 P_2 \]
\[ + 9 \left( Q_2^4 + 4\beta t^{1/3} Q_2^3 - 3Q_1 Q_2^2 + (6\beta^2 - 3\alpha) t^{2/3} Q_2^2 - 6\beta t^{1/3} Q_1 Q_2 + Q_1^2 \right. \]
\[ \left. + (-3\beta^2 + 2\alpha) t^{2/3} Q_1 - \frac{8}{3} Q_1 + (4\beta^3 - 6\alpha \beta - \frac{1}{3}) t Q_2 \right). \]
Choosing $\beta = -15^{-1/3}$, $\alpha = 3\beta^2/2$ (cf. (2.2)), and using (3.3), we have the Hamiltonian of the resultant system:

\begin{equation}
3K_2 = 3\tilde{H}_2 - 2\alpha t^{-1/3}P_1 - \beta t^{-2/3}P_2
= Q_2P_1^2 + \beta t^{1/3}P_1^2 + 2P_1P_2
+ 9\left(Q_2^4 + 4\beta t^{1/3}Q_2^3 - 3Q_1Q_2^2 + \alpha t^{2/3}Q_2^2 - 6\beta t^{1/3}Q_1Q_2 + Q_1^2\right)
- 3sQ_1 - 2\alpha t^{-1/3}P_1 - \beta t^{-2/3}P_2.
\end{equation}

3.2. **Step 2.** To make the quadratic part non-degenerate, we apply the sharing transformation

\[ q_1 = t^{1/12}Q_1, \quad p_1 = t^{-1/12}P_1, \quad q_2 = t^{-1/4}Q_2, \quad p_2 = t^{1/4}P_2, \]

which is canonical. Then by (3.4) and (3.3), we have

\begin{equation}
3K_2 = 9t^{-1}Q_1^4 + t^{-5/12}(Q_2P_1^2 + 36 \beta Q_2^3 - 27Q_1Q_2^2)
+ t^{1/6}(\beta P_1^2 + 2P_1P_2 + 9Q_1^2 - 54 \beta Q_1Q_2 + \frac{27}{2}\beta^2 Q_2^2)
+ t^{-1}\left(-\frac{1}{4}Q_1P_1 + \frac{3}{4}Q_2P_2\right) - 3st^{1/12}Q_1 - t^{-5/12}(3\beta^2 P_1 + \beta P_2).
\end{equation}

3.3. **Step 3.** Observe that the quadratic part of (3.5)

\[ t^{1/6}\left(\beta p_1^2 + 2p_1p_2 + 9q_1^2 - 54 \beta q_1q_2 + \frac{27}{2}\beta^2 q_2^2\right) \]

corresponds to the matrix $J_0$ (cf. (2.1)), which is a coefficient of the linear part of the Hamiltonian system for $H_2$. The matrix $T_0$ satisfying $T_0^{-1}J_0T_0 = \text{diag}[-\rho, \rho, -\bar{\rho}, \bar{\rho}]$ is chosen so that the transformation

\[ T(q_1,p_1,q_2,p_2) = T_0T(Q_1,P_1,Q_2,P_2) \]

is canonical, namely that it satisfies (3.2). Then we have

\[ 3K_2 = \frac{6}{5}r_0^{-2}\beta^{-2}t^{-1}K_2^{(4)} + \frac{\sqrt{6}r_0^{-3/2}\beta^{-1/2}}{(2\sqrt{5})^{1/2}}t^{-5/12}K_2^{(3)} + 3t^{1/6}K_2^{(2,1)} + \frac{\sqrt{3}}{8\sqrt{10}}t^{-1}K_2^{(2,2)} - \frac{3r_0^{-1/2}\beta^{1/2}}{(2\sqrt{5})^{1/2}}st^{1/12}K_2^{(1,1)} - r_0^{1/2}\beta^{3/2}t^{-5/12}K_2^{(1,2)}, \]
where $K_{2}^{(4)}$, ... are homogeneous polynomials in $Q_1$, $P_1$, $Q_2$, $P_2$ given by

\[
K_{2}^{(4)} := e^{2i \omega} Q_1^2 P_1^2 + e^{-2i \omega} Q_2^2 P_2^2 + 4 Q_1 Q_2 P_1 P_2 + \cdots,
\]

\[
K_{2}^{(3)} := (4 + \sqrt{5} i) e^{3i \omega/2} (Q_1 + P_1) Q_1 P_1 + (4 - \sqrt{5} i) e^{-3i \omega/2} (Q_2 + P_2) Q_2 P_2 +
2 \sqrt{6} i e^{3i \omega/2} (Q_1 + P_1) Q_2 P_2 - 2 \sqrt{6} i e^{-3i \omega/2} (Q_2 + P_2) Q_1 P_1 + \cdots,
\]

\[
K_{2}^{(2,1)} := - \rho Q_1 P_1 - \overline{\rho} Q_2 P_2,
\]

\[
K_{2}^{(2,2)} := - e^{-2i \omega} (Q_1^2 - P_1^2) - \cdots,
\]

\[
K_{2}^{(1,1)} := e^{-3i \omega/2} (Q_1 + P_1) + e^{3i \omega/2} (Q_2 + P_2),
\]

\[
K_{2}^{(1,2)} := a_1 (Q_1 - P_1) + a_2 (Q_2 - P_2) (a_1, a_2 \in \mathbb{C}).
\]

3.4. Step 4. We would like to eliminate the linear parts of the Hamiltonians $H_j$ ($j=1,2$). Using a classical result for nonlinear equations (see e.g. [8]), we can choose the canonical transformation

\[
q_1 = Q_1 + u_1, \quad q_2 = Q_2 + u_2, \quad p_1 = P_1 + v_1, \quad p_2 = P_2 + v_2
\]

(for $u_j, v_j$ cf. Theorem 2.1) such that $t^{1/12} u_j, t^{1/12} v_j \in A(\Sigma)$, where $\Sigma$ is some sector satisfying $\Sigma \supset \overline{\Sigma_0}$ and $|\Sigma| > \pi$ ($|\Sigma|$ denotes the opening of $\Sigma$). By this transformation, we have

\[
K_2 = \sum_{2 \leq |j| + |k| \leq 4} t^{1/6 - (7/12)(|j| + |k| - 2)} h_{jk}(s, t) Q^j P^k
\]

with $h_{jk}(s, t) \in A(\Sigma)$; in particular, for $|j| + |k| = 2$,

\[
t^{1/6} h_{(1,0,1,0)}(s, t) = \Lambda^{(2)}_1(s, t) + O(t^{-7/6}),
\]

\[
t^{1/6} h_{(0,1,0,1)}(s, t) = \Lambda^{(2)}_2(s, t) + O(t^{-7/6}),
\]

\[
t^{1/6} h_{jk}(s, t) = O(t^{-1/2}) \quad \text{(otherwise)}.
\]

By a further linear canonical transformation, we have

\[
h_{jk}(s, t) \equiv 0 \quad \text{for } |j| + |k| = 2, \quad (j, k) \neq (1, 0, 1, 0), (0, 1, 0, 1).
\]

Using the completely integrable condition and the fact $|\Sigma| > \pi$, we can check that the linear terms of $H_1$ are simultaneously eliminated by the transformation above. For example,

\[
K_1 = (\Lambda^{(1)}_1(t) + f_1(s) + O(t^{-1/6})) Q_1 P_1 + (\Lambda^{(1)}_2(t) + f_2(s) + O(t^{-1/6})) Q_2 P_2 + \cdots
\]

with some polynomials $f_j(s)$ ($j = 1, 2$).
3.5. Step 5. We eliminate higher order terms in $H_2$ for $j \neq k$. Suppose that $h_{jk}(s, t) \equiv 0$ for $j, k$ satisfying $j \neq k$ and $|j| + |k| \leq \iota_0 - 1$ ($\iota_0 \geq 3$). Put

$$W = Q_1 p_1 + Q_2 p_2 + \sum_{|j|+|k| = \iota_0, j \neq k} f_{jk}(s, t)Q^j p^k.$$  

Then

$$q_1 = W_{p_1}, \quad q_2 = W_{p_2}, \quad P_1 = W_{Q_1}, \quad P_2 = W_{Q_2}$$

is a canonical transformation, and

$$K_2 = H_2 - W_t = \cdots + \sum_{\iota_0} ((k_1 - j_1)\Lambda_1^{(2)} + (k_2 - j_2)\Lambda_2^{(2)}) f_{jk} Q^j p^k + \cdots$$

Choose $f_{jk}(s, t)$ so that

$$\partial f_{jk}/\partial t = h_{jk} + ((k_1 - j_1)\Lambda_1^{(2)} + (k_2 - j_2)\Lambda_2^{(2)}) f_{jk}.$$  

Since $j \neq k$, there exists $f_{jk}$ such that

$$t^{(7/12)(\iota_0 - 2)} f_{jk}, \quad t^{-1/6 + (7/12)(\iota_0 - 2)} (\partial f_{jk}/\partial t) \in \mathcal{A}(\Sigma_{jk}),$$

for some sector $\Sigma_{jk}$, $|\Sigma_{jk}| > \pi$. Thus we get the canonical transformation

$$T(q, p) = T(Q, P) + \sum_{|j| + |k| \geq \iota_0 - 1} \varphi_{jk}(s, t)Q^j P^k,$$

such that the coefficients of the terms in $K_2$ for $|j| + |k| = \iota_0$, $j \neq k$ vanish.

Applying the procedure above, we inductively obtain the required transformation.

3.6. Step 6. By a transformation of the form

$$T(q, p) = \begin{pmatrix} Q_1 \exp(S_x(Q_1 P_1, Q_2 P_2)) \\ Q_2 \exp(S_y(Q_1 P_1, Q_2 P_2)) \\ P_1 \exp(-S_x(Q_1 P_1, Q_2 P_2)) \\ P_2 \exp(-S_y(Q_1 P_1, Q_2 P_2)) \end{pmatrix},$$

$(S_x = \partial S/\partial x, S_y = \partial S/\partial y)$ with

$$S(x, y) = \sum_{|j| \geq 1} \psi_j(s, t)x^{j_1}y^{j_2}, \quad j = (j_1, j_2),$$

$$t^{d(j)} \psi_j \in \mathcal{A}(\Sigma_0),$$

$$d(j) := \begin{cases} 1/6 & |j| = 1, \\ 2/3 & |j| = 2, \\ (7/6)(|j|-2) & |j| \geq 3, \end{cases}$$
we get the reduced system \((G_0)\).

Composing the transformations given above, we obtain the formal power series in \(Q, P\) given in Theorem 2.1, whose coefficients are functions expressible by asymptotic series in \(t\) uniformly valid for \(s\).

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