**On the Stokes geometry of the Noumi-Yamada system**

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# 1 Introduction

The Noumi-Yamada system $NY_m$ $(m = 2, 3, \dots)$ is a non-linear differential equation of $m+1$ unknown functions $u_0(t), \dots, u_m(t)$ of $t$ variable which has been found by Noumi-Yamada [NY]. To analyze it with WKB analysis, let us first recall the explicit form of the Noumi-Yamada system with a large parameter $\eta$. See Takei [T2] for the details. As the structure of $NY_m$ depends on the parity of $m$, we concentrate our attention, in most cases, to the case where $m$ is even.

Now, the system $NY_{2m}$ with a large parameter $\eta$ is of the following form:

$$(NY_{2m}) \qquad \eta^{-1}\frac{du_j}{dt} = u_j(u_{j+1} - u_{j+2} + \cdots - u_{j+2m}) + \alpha_j$$

$(j = 0, 1, 2, \dots, 2m)$, where $\alpha_j$ are formal power series of $\eta^{-1}$ with constant coefficients satisfying

$$\alpha_0 + \alpha_1 + \cdots + \alpha_{2m} = \eta^{-1}. \tag{1}$$

Hence $u_j$ may be assumed to satisfy the following normalization condition

$$u_0 + u_1 + \cdots + u_{2m} = t. \tag{2}$$

Here the indices $j$ of $u_j$ are considered to be elements of $\mathbb{Z}/(2m+1)\mathbb{Z}$, that is, $u_{j+2m+1} = u_j$.

The non-linear equation $NY_m$ describes the compatibility condition of a system of linear partial differential equations. The system is referred to as "a Lax pair" (as a kind of jargon). In our case it consists of a linear differential equation $NYL_m$ in $x$-variable that depends on a parameter $t$ (a deformation parameter) and another linear differential equation in $t$-variable that controls the isomonodromic deformation of $NYL_m$; the explicit form of $NYL_m$ is as follows.

$$(NYL_m) \qquad \frac{d\psi}{dx} = \eta A_t(x)\psi.$$

Here $\psi = {}^t(\psi_0(x), \dots, \psi_m(x))$, and $A_t(x)$ is a square matrix of the size $m+1$ with a parameter $t$ as follows:

$$A_t(x) = \frac{-1}{x}\begin{pmatrix} e_0 & u_1(t) & 1 & & & & \\ 0 & e_1 & u_2(t) & 1 & & & \\ & & \cdots & & & & \\ & & & \cdots & & & \\ & & & & e_{m-2} & u_{m-1}(t) & 1 \\ x & & & & & e_{m-1} & u_m(t) \\ xu_0(t) & x & & & & & e_m \end{pmatrix},$$

where $\{u_k(t)\}$ is a solution of $NY_m$ and $e_k$'s are some constants.

The principal aim of this article is to describe the Stokes geometry of $NY_m$ in terms of the Stokes geometry of the underlying linear equation $NYL_m$. The guiding principle on which we rely is as follows:

"**Stokes curves (in $t$-space) of a non-linear equation consist of points where the Stokes geometry (in $x$-space) of the underlying linear equation has some kinds of global topological changes**".

In order to extract concrete informations from this guiding principle, we first set a stage for the Stokes geometry of $NY_m$ by considering the linearization (the Fréchet derivative) of $NY_m$ at its 0-parameter solution (cf. [KKNT], [T2]). It is, however, disputable whether Fréchet derivative is an analytically meaningful object in this context, as we see below. Hence the terminology "formal Stokes geometry" is sometimes used to mean its ordinary Stokes geometry, namely the collection of ordinary Stokes curves and ordinary turning points of the Fréchet derivatives (cf. Section 8). Thus we use the Stokes geometry of $NYL_m$ as the most reliable object, and deduce informations on $NY_m$ from it using the above guiding principle. This approach was quite successful for a class of higher order Painlevé equations whose Lax pairs consist of second order equation ([KKNT]). But, $NYL_m$ is of higher order, and we encounter several troubles in putting the above idea into practice. Among other things, we have to take into account both ordinary turning points and virtual turning points, and hence both ordinary and new Stokes curves also, to find its complete Stokes geometry ([AKT]). This makes the Stokes geometry of $NYL_m$ quite complicated. Actually in his seminal papers [Sa1] and [Sa2], Sasaki noticed the following facts by the concrete computation for $m = 2$ and 4:

1. Although a point $t_0$ lies on a formal Stokes curve of $NY_m$, if $t_0$ is located far away from the turning point from which the Stokes curve emanates, no degeneration can be found in the Stokes geometry of $NYL_m$ , that is, no Stokes curves connect two ordinary turning points directly (note that, by a

result of [T2], if a point $t_0$ in a formal Stokes curve is near the turning point, we always observe some degeneration of the Stokes geometry of $NYL_m$).

2. Even if a point $t_0$ is situated apart from formal Stokes curves of $NY_m$, we sometimes observe some degeneration in the sense that that a Stokes curve of $NYL_m$ connects two virtual turning points directly. Moreover such virtual turning points of $NYL_m$ present some discontinuities called "napping" when the parameter $t$ moves near $t_0$; he related the napping phenomena of the virtual turning points with (the counter part of) the Nishikawa phenomena ([KKNT]) for the Noumi-Yamada system.

Furthermore in the first case, he found that a double turning point and a virtual turning point instead of an ordinary turning point are directly connected by a Stokes curve. Moreover he confirmed that Stokes curves of $NYL_m$ change discontinuously when $t$ crosses the Stokes curve.

Thus it seems possible that an appropriate interpretation of the guiding principle might be obtained if the situation where two turning points, either ordinary or virtual, are directly connected by a Stokes curve of $NYL_m$ is regarded as a degeneration. However, more detailed study of $NY_m$ have recently yielded several examples ([H1], pp. 116 - 124) where the straightforward interpretation of the above degeneration does not necessarily lead to the discontinuous behavior of Stokes curves of $NYL_m$ when the parameter moves around. Therefore we need some more precise language to define the degeneration of the Stokes geometry of $NYL_m$ properly.

Our detailed study of $NY_4$ ([H1], pp. 33 - 65) indicates that degeneration of the Stokes geometry in $NYL_m$ should be observed when two ordinary turning points are connected by a zigzag of "effective" Stokes segments, both ordinary and new; To facilitate detecting such a zigzag in a complicated Stokes geometry we introduce a graph theoretical notion "**an effective bidirectional binary tree**" which are determined by the configuration of turning points and Stokes curves, and investigate its basic properties. The above two phenomena found by Sasaki are interpreted as follows in terms of this language.

1. In all the examples we have so far checked, we can find an effective bidirectional binary tree in the Stokes geometry of $NYL_m$, if $t$ lies on a formal Stokes curve of $NY_m$. Moreover if $t$ moves apart from the Stokes curve of $NY_m$, the tree splits into some effective unidirectional binary trees, and some segments of these trees move discontinuously when $t$ moves across the Stokes curve ([H1, H3]).

2. The Nishikawa phenomenon is observed in the Stokes geometry of $NY_m$ when a new effective bidirectional binary tree is generated by a concatenation of

two other effective bidirectional binary trees which have the same leaf nodes ([H1, pp.53-78], see also [H3 and H4]).

The best way to understand the notion and properties of effective bidirectional binary trees is to become familiar with many concrete examples. In view of the limitation on the length of this article, we cannot include sufficiently many figures but we refer the reader to [H1]. We also omit the proofs of our claims in this paper and leave them to our forthcoming paper [H4].

In ending this introduction, we emphasize that most of the results in this paper are applicable to general higher order linear ordinary differential equations.

# 2 Preliminaries

First we recall briefly definitions of ordinary turning points and Stokes curves. Let $A(x)$ be an $m \times m$ matrix whose elements are rational functions of $x$, and we consider a linear differential system with a large parameter $\eta$ of unknown functions $v(x) = {}^t(v_1(x), \dots, v_m(x))$:

$$\eta^{-1}\frac{dv}{dx} = A(x)v. \tag{3}$$

Let $\Lambda(\lambda, x) = \det(\lambda I - A(x))$, and let $D(x)$ denotes the discriminant of $\Lambda(\lambda, x) = 0$. On a complex plane equipped with appropriate cut lines, let holomorphic functions $\lambda_1(x)$, $\lambda_2(x)$, $\dots$, $\lambda_n(x)$ of $x$ denote the roots of the algebraic equation $\Lambda(\lambda, x) = 0$ of $\lambda$. Hereafter we always assume the following conditions:

- (LA-1) $D(x)$ is not identically zero.
- (LA-2) For each $x$, the algebraic equation $\Lambda(\lambda, x) = 0$ has at most one double root, and the other roots are all simple.

**Ordinary turning points** are roots of the discriminant $D(x) = 0$. In particular, a simple (resp. double) root of $D(x) = 0$ is said to be a **simple (resp. double) turning point**. If two roots $\lambda_i$ and $\lambda_j$ of $\Lambda(\lambda, x) = 0$ merge at $x = x_0$, we say **the type of the turning point** $x_0$ is $(i, j)$. Remark that if $x_0$ is a simple turning point of the type $(i, j)$, $\lambda_i(x)$ and $\lambda_j(x)$ ramify at $x_0$ with the degree of ramification being 2. On the other hand, if $x_0$ is a double turning point, $\lambda_i(x)$ and $\lambda_j(x)$ are holomorphic functions near $x_0$.

Let $x_0$ be an ordinary turning point of the type $(i, j)$, and let $l : [0, 1) \to \mathbb{C}$ be a smooth curve in $\mathbb{C}$ with a starting point $x_0$. In what follows we assume the following condition (*):

- **Condition (*)** $\lambda_i(x)$ and $\lambda_j(x)$ can be analytically continued along $l$ from $x_0$, and their values never coincide on the curve $l$ except for the starting point $x_0$.

Owing to the condition (*), the following form $\omega$ defined near $x_0$

$$\omega = \mathrm{Im}\,(\lambda_i(x) - \lambda_j(x))dx \tag{4}$$

can be analytically continued along $l \setminus \{x_0\}$ as non-degenerate and smooth real analytic differential 1-form.

**Definition 2.1 (An integral curve)** A smooth curve $l$ with a starting point $x_0$ satisfying the condition $(*)$ and $l^*\omega = 0$ is called an integral curve emanating from an ordinary turning point $x_0$.

**Definition 2.2 (A Stokes curve)** A Stokes curve $l$ emanating from an ordinary turning point $x_0$ is maximal immersion $l$ from $[0, 1)$ or $[0, 1]$ to $\mathbb{C}$ whose restriction on $[0, 1)$ gives an integral curve emanating from $x_0$.

For any point $x$ in a Stokes curve $l$, if $l$ is an integral curve of a real differential 1-form (4) defined by roots $\lambda_{i'}$ and $\lambda_{j'}$ near $x$, we say **the type of the Stokes curve** $l$ at $x$ is $(i', j')$. Concerning the ending point of a Stokes curve emanating from an ordinary turning point $x_0$, we readily see that one of the following situations is observed:

1. No ending point. That is, the curve $l$ flows into a singular point of the equation or a point at infinity.

2. The ending point $x_1$ is a **simple** turning point $s$ and the type of $s$ and that of $l$ at $x_1$ have **one and only one common index**. In this case, we say that $l$ bifurcates at $x_1$. In fact, a semi-analytic set defined by $\mathrm{Im} \displaystyle\int_{x_0}^{x} \omega = 0$ bifurcates at $x_1$.

3. The ending point $x_1$ is an ordinary turning point $s$ and the type of $s$ is the same as that of $l$ at $x_1$. In this case, we often say that turning points $x_0$ and $x_1$ are connected by a Stokes curve $l$.

Remark that interior points of Stokes curves always form a smooth curve.

Next we will define virtual turning points. For the original definition of virtual turning points, we refer the reader to Aoki-Kawai-Takei [AKT]. Here we employ an alternative definition. Let $E$ be the set of singular points of the equation (the set $E = \{0\}$ in our case, i.e., for $NYL_m$).

**Definition 2.3 (Virtual turning points) ([T1])** A point $x_0$ is called a virtual turning point of the type $(i,\, j)$ $(i \neq j)$ if there exist a piecewise smooth closed path $C_{x_0}$ in $\mathbb{C} \setminus E$ with the starting and ending point $x_0$, and a continuous function $\mu(x)$ on $C_{x_0}$ that satisfy the following conditions.

1. For any $x \in C_{x_0}$, $\mu(x)$ is a root of the equation $\Lambda(\mu, x) = 0$. Moreover near the starting (resp. ending) point of $C_{x_0}$, $\mu(x) = \lambda_i(x)$ (resp. $\mu(x) = \lambda_j(x)$) holds.

2. The following equality holds:
$$\int_{C_{x_0}} \mu(x)dx = 0.$$

In general the above condition 1 implies $\lambda_j$ is an analytic continuation of $\lambda_i$ along $C_{x_0}$. However if the path $C_{x_0}$ passes an ordinary turning point $s$ of the type $(k, l)$ and $\mu(s) = \lambda_k(s) = \lambda_l(s)$ holds, then we exchange $\lambda_k$ and $\lambda_l$ at $s$.

Note that an ordinary turning point is, logically speaking, a virtual turning point in the above sense. But, for the sake of convenience we exclude ordinary turning points from the definition of virtual turning points. For any virtual turning point $v$, we can define integral curves and Stokes curves emanating from $v$ in the same way as in the case of ordinary turning points. A Stokes curve emanating from a virtual turning point is often called a **new Stokes curve**. From now on, by a turning point we mean either a virtual or an ordinary one.

Now let us consider the case where the equation contains a deformation parameter $t$, that is, the situation where the matrix $A(x;t)$ of (3) depends on a parameter $t$ holomorphically in a simply connected domain $V_t \subset \mathbb{C}_t$. We always assume that

- the conditions (LA-1) and (LA-2) hold for each fixed $t \in V_t$,
- the singular points of the equation do not move, and
- any ordinary turning point coincides with neither other turning points nor the singular points of the equation as $t$ moves in $V_t$. Furthermore the number of ordinary turning points remains constant.

The lemma below is fundamental in considering continuous deformations of the Stokes geometry.

**Lemma 2.4** On the above assumptions, we have:

1. Each ordinary turning point is a holomorphic function of $t$.

2. Any Stokes curve $l$ emanating from an ordinary turning point moves continuously when $t$ moves smoothly unless another ordinary turning points hit against the curve $l$.

For a virtual turning point $v$ at $t = t_0$, by a deformation of the closed path $C_v$ in Definition 2.3, we can find a virtual turning point $v(t)$ near $t = t_0$ satisfying that $v(t_0) = v$ and that $v(t)$ is holomorphic. The holomorphic function $v(t)$ is called **a germ of a virtual turning point** $v$ at $t = t_0$. Moreover $v(t)$ can be analytically continued to any points in $V_t$ as long as $v(t)$ never merge with ordinary turning points. The similar lemma holds for Stokes curves emanating from virtual turning points.

# 3 Constructible turning points

It is a hard task to locate virtual turning points using Definition 2.3 directly. However an excellent algorithm to find virtual turning points was presented in the paper [AKKSST]. It is as follows:

Let $x_0$ and $x_1$ be turning points, and $s_0$ (resp. $s_1$) a Stokes curve emanating from $x_0$ (resp. $x_1$). We assume $s_0$ and $s_1$ intersect at a point $x$ and the types of $s_0$ and $s_1$ at $x$ are $(i, j)$ and $(j, k)$ respectively. Note that the index $j$ is common in both types in this case. Let $l$ denote an integral curve of a real differential 1-form $\mathrm{Im}\,(\lambda_i - \lambda_k)dx$ passing through $x$.

**Theorem 3.1 [AKKSST] (Algorithm for locating VTP's)** If a point $v$ in the curve $l$ satisfies the following integral relation

$$\int_x^{x_0} \lambda_i - \lambda_j dx + \int_x^{x_1} \lambda_j - \lambda_k dx + \int_x^{v} \lambda_k - \lambda_i dx = 0,$$

then $v$ is a VTP, i.e., a virtual turning point. Here each integrations is performed along the integral curve designated above.

From now, when we say $x$ is an intersection point of two Stokes curves, we always assume the types of the Stokes curves at $x$ share one and only one common index. Remark that if the types are the same, both Stokes curves locally coincide near $x$. Incidentally we say that two objects in the Stokes geometry (i.e. turning points or Stokes curves) are **disjoint** if their types have no common indices.

**Definition 3.2** An intersection point $x$ of two Stokes curves is called regular if two curves intersect transversally at an interior point $x$ of the curves (when $x$ is a virtual turning point, it may be conventionally regarded as an interior point). Remark that the types of both curves at $x$ have one and only one common index by the definition.

Using the algorithm for locating VTP's, we can define the set of constructible turning points, which play an important role in exact WKB analysis. Let $U$ be a relatively compact connected open set with a smooth boundary in $\mathbb{C}$ which does not contain the singular points of the equation. Let $T(U)$ denote the set of all turning points in $U$. For any $v \in T(U)$, $s(v)$ denotes one of Stokes curves emanating from $v$.

**Definition 3.3** A subset $CT(U) \subset T(U)$ of **constructible turning points** is defined by the following conditions 1, 2 and 3:

1. $CT(U)$ contains all ordinary turning points in $U$.
2. For any Stokes curves $s(x_0)$ and $s(x_1)$ ($x_0$, $x_1 \in CT(U)$) and their regular intersection point $x$ in $U$, if a virtual turning point $v$ located by the above algorithm is in $U$, then $v$ belongs to $CT(U)$. Conventionally, we only consider the connected component containing $x$ in $U$ as the integral curve used in the algorithm.
3. $CT(U)$ is a minimal set satisfying conditions 1 and 2 above.

All virtual turning points $v \in CT(U)$ can be obtained by the repeated application of the algorithm starting from ordinary turning points, the minimal number of the applications of the algorithm to obtain $v$ is called **the level of a constructible turning point** $v$. In particular, the level of an ordinary turning point is 0. For each $k = 0, 1, 2 \ldots$, $CT_k(U)$ denote a subset of $CT(U)$ consisting of all constructible turning points whose levels are less than or equal to $k$.

Let $V$ be a subset of $T(U)$ and let $S(V, U)$ denote the set of all Stokes curves in $U$ emanating from $v \in V$. Here a Stokes curve in $U$ means a connected component containing $v$ in $U$. Then $G(V, U)$ designates the totality of the Stokes geometry in $U$ constructed by the data $V$ and $S(V, U)$; furthermore we say $G_k(U) = G(CT_k(U), U)$ is **the Stokes geometry of the level** $k$ **in** $U$.

From now on, by choosing an open set $U$ suitably we alway assume that the following finiteness condition is satisfied.

**Definition 3.4 (Finiteness condition)** In the Stokes geometry $G$, there are no cyclic Stokes curves (that is, a Stokes curve emanating from $v$ that returns to the same $v$). Moreover the number of Stokes curves in $G$ is finite and all Stokes curves in $G$ have finite length.

Let us consider the case where the equation has a deformation parameter $t$. For any $t \in V_t$, we can consider a set $CT_k(U; t)$ of all constructible turning points of the level $k$ in $U$ with the parameter $t$, and the Stokes geometry $G_k(U; t) = G(CT_k(U; t), U)$. For any $t_0 \in V_t$ and $v \in CT_k(U; t_0)$, **a germ** $v(t)$ **of**

**a constructible turning point** $v$ at $t_0$ can be defined in the same way as a germ of a virtual turning point was defined.

**Definition 3.5** For any $t_0 \in V_t$ and $v \in CT_k(U;t_0)$, the number of holomorphic functions $v(t)$ satisfying $v(t_0) = v$ and $v(t) \in CT_k(U;t)$ near $t = t_0$ is called a multiplicity of $v$ at $t_0$ in $CT_k(U;t)$.

We fix a smooth curve $\tau : [0,1] \to V_t$ in $V_t$. For any two points $t_1 = \tau(\theta_1)$, $t_2 = \tau(\theta_2)$ $(\theta_1 < \theta_2)$ in the curve $\tau$, $(t_1,\, t_2)$ (resp. $[t_1,\, t_2]$) denotes an open (resp. closed) segment $\{\tau(\theta);\, \theta_1 < \theta < \theta_2\}$ (resp. $\{\tau(\theta);\, \theta_1 \leq \theta \leq \theta_2\}$) of the curve $\tau$ respectively. Remark that the finiteness condition in this case should be the number and length of Stokes curves are uniformly bounded along the curve $\tau$. The following lemma implies that constructible turning points, generically speaking, have multiplicity 1 under the finiteness condition.

**Lemma 3.6** There exists a set $E = \{\tau(\theta_1), \ldots, \tau(\theta_r);\, \theta_1 < \cdots < \theta_r\}$ of finite points in the curve $\tau$ for which the following hold:

1. For any $t_0 \in \tau \setminus E$, all constructible turning points of $CT_k(U;t_0)$ have multiplicity 1 at $t_0$ in $CT_k(U;t)$, and
2. the number of $CT_k(U;t)$ is constant in each open segment $(\tau(\theta_i),\, \tau(\theta_{i+1}))$ $(0 \leq i \leq r)$ where we set $\theta_0 = 0$ and $\theta_{r+1} = 1$.

See [H4] for the proof.

**Remark 3.7** If a constructible turning point $v \in CT_k(U;t_0)$ is given, then $v$ defines a germ $v(t)$ at $t = t_0$. The germ $v(t)$ can be analytically continued to whole space $V_t$ as a virtual turning point (multi-valued); however, $v(t)$ is not necessarily a constructible virtual turning point when $t$ is far from $t_0$. Therefore the constructibility of virtual turning points is not continuous property.

# 4 Solid or dotted line condition

The following Fig. 1 is a numerically calculated example of the Stokes geometry (level 1) of $NYL_3$. In the figure, a fat dot denotes a turning point and each curve drawn by a thick line (resp. a thin line) designates an ordinary (resp. a new) Stokes curve. In particular, $s_1$ and $s_2$ are simple turning points, $d_1$ is a double turning point, and $v_1$ and $v_2$ are virtual turning points.

In the figure, the Stokes curves $s(v_1)$ emanating from the virtual turning point $v_1$ consists of solid and dotted line portions. In general, each Stokes curve consists of

- **solid line portions** where the curve is "**effective**", that is, where Stokes phenomena may be observed, and

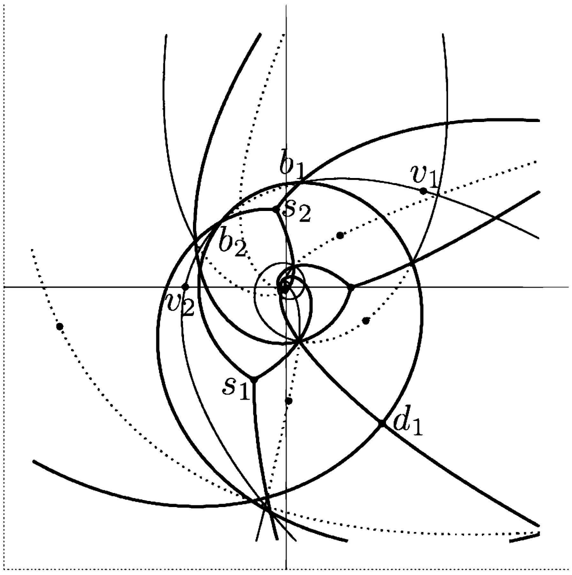


Figure 1: $NYL_3$ (Level 1).

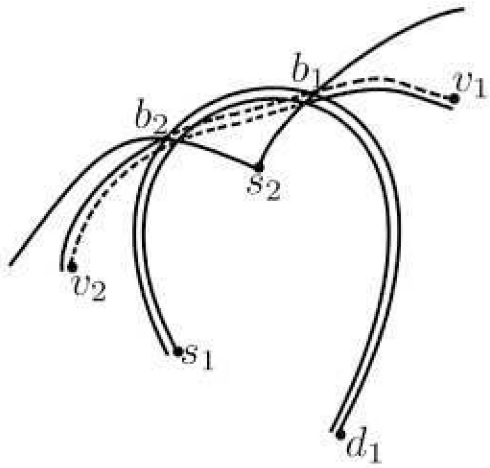


Figure 2: The overlap of curves.

- **dotted line portions** where Stokes phenomena never occur.

An algorithm to determine solid or dotted line portions of Stokes curves seems to be quite important. In this paper, we employ slightly modified version of the algorithm presented by Aoki-Kawai-Takei [AKT]. We will explain briefly the algorithm employed here.

We first remark that when the parameter $t$ lies on a Stokes curve of $NY_m$, we usually observe that two turning points are located in the same Stokes curve of $NYL_m$, that is, the Stokes curve emanating from one of the turning points geometrically overlaps with another Stokes curve. For example, the Stokes curve $s(d_1)$ emanating from the turning point $d_1$ and the curve $s(s_1)$ emanating from $s_1$ in Fig. 1 are geometrically the same. To determine the state (solid or dotted) of the overlapped Stokes curve correctly, it is necessary to apply the algorithm to each Stokes curve $s(d_1)$ and $s(s_1)$ separately. Therefore, in what follows, we consider **the Stokes curves $s(v_1)$ and $s(v_2)$ to be different if turning points $v_1$ and $v_2$ are different**, even if they geometrically coincide.
That means that a Stokes curve $s(v)$ is regarded as a pair $(v;\, l)$ of a turning point $v$ and an integral curve $l$ emanating from $v$. We denote by $[s(v)]$ the integral curve $l$ of a Stokes curve $s(v) = (v;\, l)$.

Let $l$ be a Stokes curve emanating from a turning point $x_1$, and $x$ a point in the curve $l$. Here a point of the Stokes curve $l$ means that of the underlying integral curve $[l]$ (note that $[l]$ itself is an immersion from $[0, 1)$ or $[0, 1]$ to $U$).

**Definition 4.1** Assume that the type of a Stokes curve $l$ at $x$ is $(i, j)$. When

$$\mathrm{Re} \int_{x_1}^{x} \lambda_i - \lambda_j dx < 0$$

holds, we say that the index $j$ dominates the index $i$ along the Stokes curve $l$, and the inequality of indices $i < j$ designates this situation.

The following ordered relation of two Stokes curves is one of the most important notions, and its importance was first noticed by [BNR]. Let $l_1$ and $l_2$ be Stokes curves, and $x$ an intersection point of the curves $[l_1]$ and $[l_2]$.

**Definition 4.2** Assume that the types of curves $l_1$ and $l_2$ at $x$ are $(i,\ j)$ and $(k,\ l)$ respectively. If the order relations of the following kind hold

$$i < j = k < l \quad \text{or} \quad j < i = k < l, \quad \ldots$$

(that is, one of the indices is less than the common index and the rest is greater than the common index), then we say that $l_1$ and $l_2$ form an **ordered crossing** at $x$.

When a Stokes curve intersects with an overlapping Stokes curve because of the overlap we must distinguish the curve which really intersects in the sense of the Stokes geometry. We will introduce a notion "combined" below. Let $v$, $v_1$, and $v_2$ be three turning points and $s(v)$, $s(v_1)$ and $s(v_2)$ their Stokes curves.

**Definition 4.3** We say that three curves $s(v)$, $s(v_1)$ and $s(v_2)$ are **combined** at a point $x$ if the curves $[s(v)]$, $[s(v_1)]$ and $[s(v_2)]$ intersect at $x$ and the types of the curves $s(v_1)$, $s(v_2)$ and $s(v)$ at $x$ are $(i, j)$, $(j, k)$ and $(i, k)$ respectively for mutually different indices $i$, $j$ and $k$, and if the following integral relation

$$\int_{x}^{v_1} \lambda_i - \lambda_j dx + \int_{x}^{v_2} \lambda_j - \lambda_k dx + \int_{x}^{v} \lambda_k - \lambda_i dx = 0$$

holds.

In Fig. 1, the Stokes curves $s(d_1)$, $s(s_2)$ and $s(v_1)$ are combined at the intersection point $b_1$ and $b_2$ respectively. On the other hand, the Stokes curves $s(d_1)$, $s(s_2)$ and $s(v_2)$ are not combined either at $b_1$ or at $b_2$. In the same way, the Stokes curves $s(s_1)$, $s(s_2)$ and $s(v_2)$ are combined at each intersection point, but, the Stokes curves $s(s_1)$, $s(s_2)$ and $s(v_1)$ are not combined.

**Remark 4.4** If there exist turning points with higher multiplicity, the definition of "combined" should be modified. However all turning points of $CT_k(U; t)$ have simple multiplicity at generic $t$ by the lemma in the previous section. Therefore the above definition is sufficient for our purpose. See Honda [H3] for the case with higher multiplicity.

In the same situation as above, we introduce the following.

**Definition 4.5 (Coherent)** We say that $s(v)$ is **coherent** at $x$ with respect to $s(v_1)$ and $s(v_2)$ if the following conditions are fulfilled:

1. $s(v)$, $s(v_1)$ and $s(v_2)$ are combined at $x$.
2. $s(v_1)$ and $s(v_2)$ form an ordered crossing at $x$.

If at a point $x$ of a Stokes curve $s(v)$, $s(v)$ becomes coherent with respect to two other Stokes curves, then we say that $x$ is a **coherent point** of $s(v)$.

For example, in Fig. 1, the Stokes curve $s(v_1)$ is coherent at $b_2$ with respect to $s(d_1)$ and $s(s_2)$, but $s(v_1)$ is not coherent at $b_1$ with respect to the same curves. The Stokes curves $s(v_2)$ is coherent at $b_1$ with respect to $s(s_1)$ and $s(s_2)$, but $b_2$ is not a coherent point of $s(v_2)$.

Let $G(V, U)$ be a Stokes geometry satisfying the finiteness condition (here $V$ is a subset of turning points $T(U)$ in $U$).

**Condition 4.6 (Solid or dotted line condition)** Solid or dotted line portions of Stokes curves of $G(V, U)$ should be determined so that the following two conditions are satisfied. For each Stokes curve $s(v)$ in $G(V, U)$ $(v \in V)$,

1. the state of the curve $s(v)$ in a neighborhood of $v$ is
   (a) solid if $v$ is an ordinary turning point.
   (b) dotted if $v$ is a virtual turning point.
2. the state of $s(v)$ should be converted at a point $x$ of the curve $s(v)$ if and only if there are turning points $v_1$ and $v_2 \in V$ satisfying
   (a) $s(v)$ is coherent at $x$ with respect to $s(v_1)$ and $s(v_2)$, and
   (b) $s(v_1)$ and $s(v_2)$ are solid lines near $x$.

A point $x$ of a Stokes curve $s(v)$ which satisfies the above condition 2 (that is, a point where the state of $s(v)$ should be converted) is called an **effective coherent point** of $s(v)$. For the properties and details of the above "solid or dotted line condition", in particular the uniqueness of a solution satisfying the condition, see Honda ([H2, Section 6], [H3]). We say a turning point $v$ is **not effective** if all Stokes curves emanating from $v$ consists of dotted line portions only. In the examples of the Stokes geometry given in [H1] and this article, all non effective turning points and their Stokes curves are not printed.

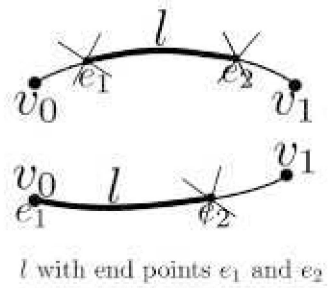


Figure 3: Bidirectional segments.

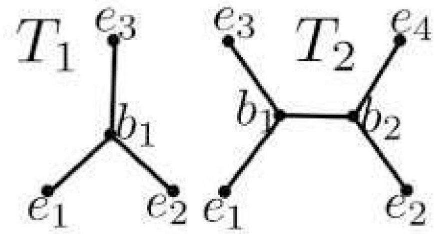


Figure 4: Binary trees $T_1$ and $T_2$.

# 5 A bidirectional binary tree

We are ready to explain the notion of (effective) bidirectional binary trees.

Let $v_0$ and $v_1$ be turning points in $U$, and assume that $v_0 \in [s(v_1)]$, $v_1 \in [s(v_0)]$ and the type of $s(v_0)$ and that of $s(v_1)$ are the same.

**Definition 5.1 (A bidirectional segment)** A segment $l = ([l];\, v_0,\, v_1)$ is called a **bidirectional segment** between $v_0$ and $v_1$ if the following two conditions are satisfied:

1. $[l]$ is a connected closed subset of $[v_0, v_1]$, and it is not a point.

2. The end points of $[l]$ are $v_0$, $v_1$ or intersection points of the Stokes curves with other Stokes curves.

Moreover we say that a bidirectional segment $l = ([l];\, v_0,\, v_1)$ is **effective** when $[l]$ is contained in the closure of solid line portions of $s(v_i)$ for each $i = 0, 1$.

Now let us recall the definition of a binary tree in the graph theory.

**Definition 5.2 (A binary tree)** A binary tree $T = (B,\, E,\, L)$ consists of $E$: a set of leaf nodes, $B$: a set of branching nodes, and $L$: a set of edges whose end points are $B \cup E$. These data should satisfy the following:

1. The degree of each leaf node is one (the degree of a node $p$ is the number of edges with end point $p$).

2. The degree of each branching node is three.

3. For any two nodes in $B \cup E$, they are connected by a path and such a path is unique. Here a path is by definition a set of edges in which each edge never appear twice.

An edge is often said to be a segment in our context.

The **degree** of a binary tree $T$ is by definition the number of leaf nodes. We also define the **depth** of a binary tree $T$ to be the number of edges of a maximal path in the tree $T$. For example, the degree of the binary trees $T_1$ (resp. $T_2$) in Fig. 4 is 3 (resp. 4), and the depth of $T_1$ (resp. $T_2$) is 2 (resp. 3). The tree $T_2$ consists of 4 leaf nodes ($e_1$, $e_2$, $e_3$ and $e_4$), 2 branching nodes ($b_1$, $b_2$) and 5 edges ($e_1b_1$, $e_2b_1$, $e_3b_2$, $e_4b_2$ and $b_1b_2$).

Let $l = ([l];\, v_1,\, v_2)$ be a bidirectional segment and $b$ an end point of $[l]$. The set $[v_1,\, v_2] \setminus \{$the interior of $[l]\}$ consists of two connected components. We define $\tau(l,\, b)$ and $\tau^a(l,\, b)$ as:

- $\tau(l,\, b)$ denotes one of the turning point $v_1$ or $v_2$ which belongs to the connected component containing $b$.
- $\tau^a(l,\, b)$ denotes the other turning point.

Intuitively $\tau(l,\, b)$ is a turning point $v_1$ or $v_2$ which is located in the same side as $b$ with respect to $l$.

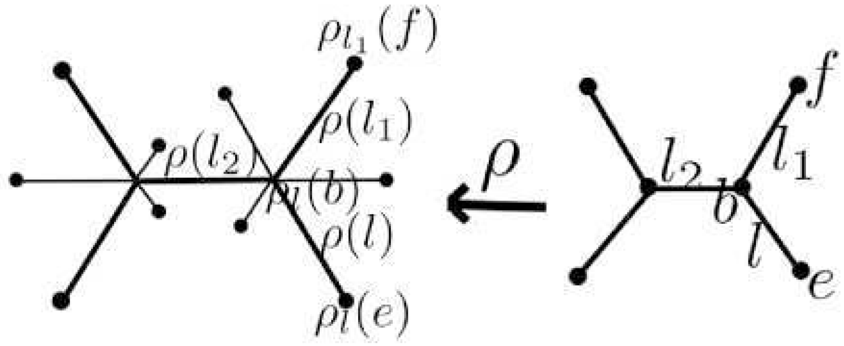


Figure 5: A bidirectional binary tree.

**Definition 5.3 (A bidirectional binary tree)** A tree $T = (B,\, E,\, L,\, \rho)$ is called a **bidirectional binary tree** in the Stokes geometry $G(V, U)$ if the following conditions are satisfied:

1. $(B,\, E,\, L)$ is a binary tree.
2. There exist a set $\hat{L}$ of bidirectional segments in $G(V, U)$, a bijective map $\rho : L \to \hat{L}$ and a family of topological immersions $\{\rho_l\}_{l \in L}$ which satisfy the following:

(a) $[\rho(l)]$ is the image of a topological immersion $\rho_l : l \to [\rho(l)]$ for each $l \in L$, and

(b) $\{\rho_l\}_{l \in L}$ preserves connectedness of the graph, that is, if $l_1, l_2 \in L$ share a branching node $b \in B$, then $\rho_{l_1}(b) = \rho_{l_2}(b)$ holds.

3. For each edge $l \in L$, its end point $p \in B \cup E$ and the corresponding bidirectional segment $\rho(l) = ([\rho(l)];\, v_0,\, v_1) \in \hat{L}$ , if the end point $\rho_l(p)$ of the segment $\rho(l)$ coincides with a turning point $v_0$ or $v_1$ (see the lower example in Fig. 3), then $p$ is a leaf node and $\rho_l(p)$ should be an ordinary turning point. If $\rho_l(p)$ is different from either $v_0$ nor $v_1$, $p$ is a branching node.

4. Let $b$ be a branching node and let $l$, $l_1$ and $l_2$ be the edges with end point $b$ in common. Let $v_l$, $v_{l_1}$ and $v_{l_2}$ respectively denote the turning point $\tau(\rho(l),\, \rho_l(b))$, $\tau^a(\rho(l_1),\, \rho_l(b))$ and $\tau^a(\rho(l_2),\, \rho_l(b))$. The Stokes curve $s(v_l)$ is coherent at $\rho_l(b)$ with respect to Stokes curves $s(v_{l_1})$ and $s(v_{l_2})$ (cf. Fig. 6).

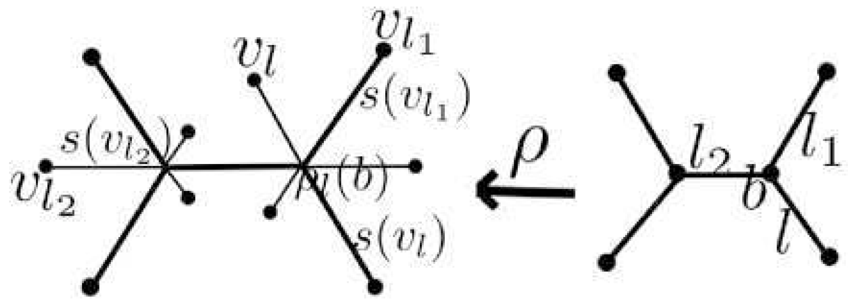


Figure 6: The condition 4.

When all bidirectional segments of $\hat{L}$ are effective, we say $T$ is **effective**.

In what follows, we identify the bidirectional segment $\rho(l) \in \hat{L}$ in the Stokes geometry $G$ with the edge $l \in L$ of the graph $(B,\, E,\, L)$ by the maps $\rho$ and $\rho_l$.

We will give a few examples of bidirectional binary trees. For details, see Honda [H1] pp. 33-52 (Fig. III-1-1 $\sim$ Fig. III-1-20) where the reader can observe an effective bidirectional binary tree "grows" so that its degree increases from 2 to 5; for example the simple turning point $a$ in Fig. III-1-3 hits the Stokes curve connecting $b_1$ and $b_2$ in Fig. III-1-2 so that the corresponding tree "grows" from the tree consisting of 2 leaf nodes ($b_1$ and $b_2$) to that consisting of 3 leaf nodes ($b_1$,

$b_2$ and $b_3$) in Fig. III-1-4, and so on. We show below the "growing" of the tree from degree 3 to degree 4; the tree $T_3$ clearly visualizes the importance of virtual turning points and new Stokes curves as we discuss below. The reader readily finds how complicated the actual world is; without introducing the graph-theoretic notions, it should be formidable to trace the change of the configurations.

The following Fig. 7 to Fig. 12 describe the Stokes geometry of $NYL_4$ where the deformation parameter $t$ is one of 3 points $t_1$, $t_2$ and $t_3$ lying on a formal Stokes curve of $NY_4$ (the point $t_1$ is the nearest to the turning point and $t_3$ is the farthest). In all figures below, the turning points $s_1$, $s_2$, $s_3$ and $s_4$ (resp. $v_1$, $v_2$, $\dots$, $v_6$) are ordinary (resp. virtual).

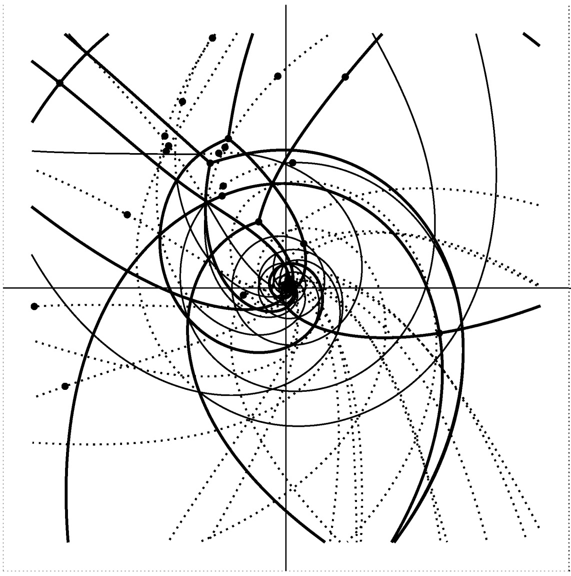

Figure 7: The Stokes geometry of $NYL_4$ at $t = t_1$ ([H1] Fig. III-1-6).

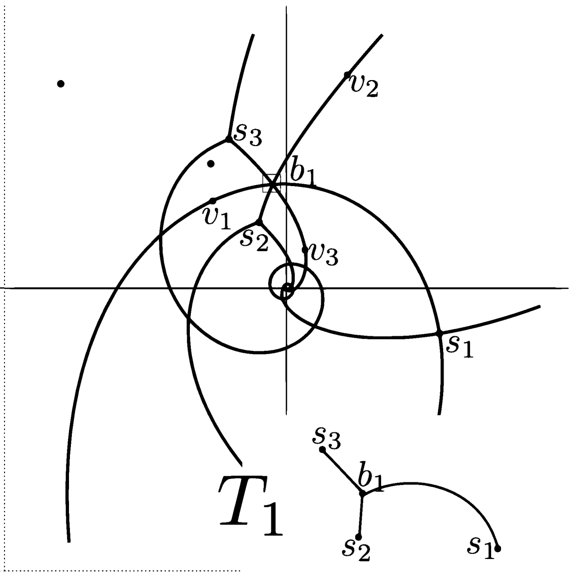


Figure 8: Only Stokes curves related to the tree $T_1$ have been drawn ($t = t_1$).

The tree $T_1$ (resp. $T_2$) of Fig. 8 (resp. Fig. 10) is an effective bidirectional binary tree with degree 3. The tree $T_1$ consists of 3 leaf nodes ($s_1$, $s_2$ and $s_3$), 1 branching node ($b_1$) and 3 bidirectional segments ($s_1b_1$, $s_2b_1$ and $s_3b_1$). For example, the bidirectional segment $s_1b_1$ lies in a common portion of two Stokes curves $s(s_1)$ emanating from $s_1$ and $s(v_1)$ emanating from $v_1$. In this way, each bidirectional segment of $T_1$ lies in a common portion of two Stokes curves. An important feature of $T_1$ is, for example, that the Stokes curve $s(v_1)$ is coherent at the branching node $b_1$ with respect to the two Stokes curves $s(s_2)$ and $s(s_3)$. In the same way, $s(v_2)$ (resp. $s(v_3)$) is also coherent at the branching node $b_1$ with respect to two Stokes curves $s(s_1)$ and $s(s_3)$ (resp. $s(s_1)$ and $s(s_2)$).

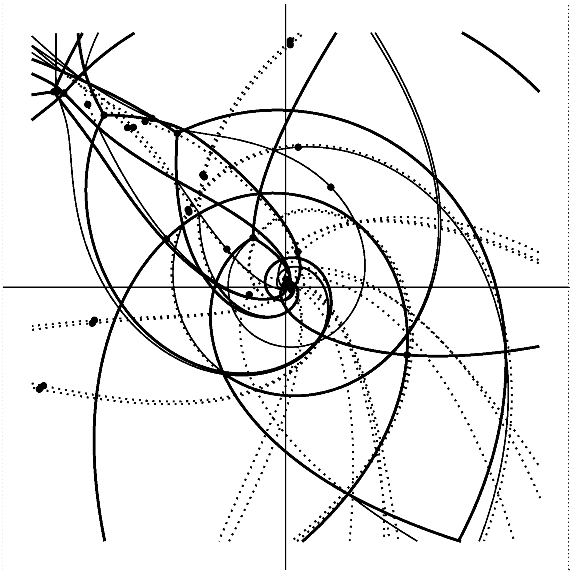

Figure 9: The Stokes geometry of $NYL_4$ at $t = t_2$ ([H1] Fig. III-1-7).

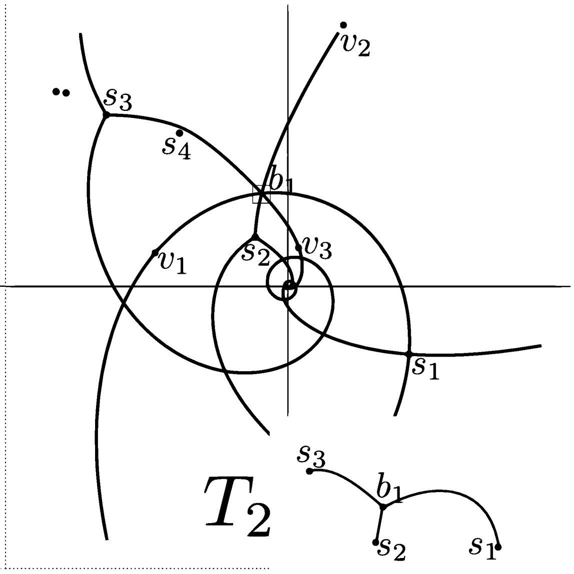


Figure 10: Only Stokes curves related to the tree $T_2$ have been drawn ($t = t_2$).

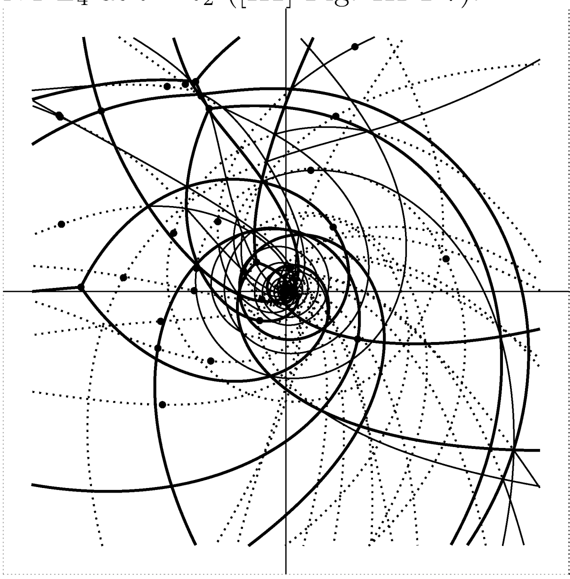

Figure 11: The Stokes geometry of $NYL_4$ at $t = t_3$ ([H1] Fig. III-1-9).

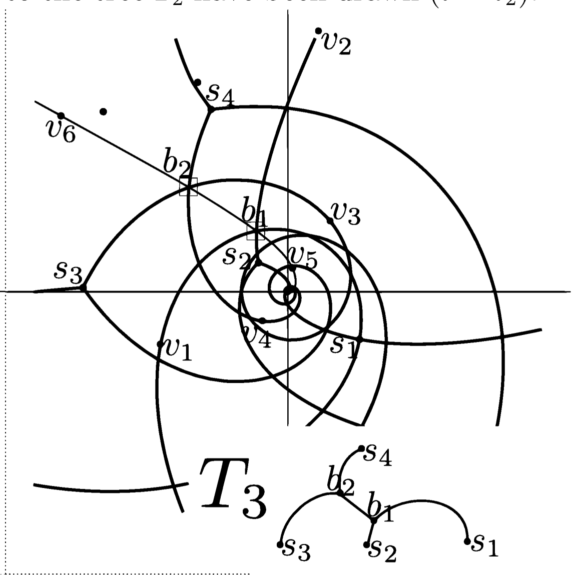


Figure 12: Only Stokes curves related to the tree $T_3$ have been drawn ($t = t_3$).

Remark that in Fig. 10, the simple turning point $s_4$ is located quite close to the bidirectional segment $s_3b_1$ of the tree $T_2$. In fact, when the parameter $t$ moves from $t_2$ to $t_3$ along the formal Stokes curve of $NY_4$, the turning point $s_4$ crosses the segment $s_3b_1$ and the tree $T_2$ grows.

The degree of the tree $T_3$ of Fig. 12 becomes 4 because the turning point $s_4$ joins in the tree as a new leaf node after $s_4$ hits against the segment of the tree. The tree $T_3$ consists of 4 leaf nodes ($s_1$, $s_2$, $s_3$ and $s_4$), 2 branching nodes ($b_1$ and $b_2$), and 5 bidirectional segments ($s_1b_1$, $s_2b_1$, $s_3b_2$, $s_4b_2$ and $b_1b_2$). In particular, the segment $b_1b_2$ is in a common portion of two new Stokes curves $s(v_5)$ and $s(v_6)$. Therefore new Stokes curves and virtual turning points play an important role in the tree $T_3$. Branching nodes $b_1$ and $b_2$ enjoy the same special order relations as the branching node $b_1$ of the tree $T_1$.

# 6 Several properties of trees

Let $T = (B, E, L, \rho)$ be a bidirectional binary tree. To define integration on a tree, we make some preparations. We first equip each segment of $T$ with an arbitrary orientation. We introduce some conventions related to the orientation of $T$.

1. For any bidirectional segment $l = ([l]; v_{l,s}, v_{l,e})$, we always choose turning points $v_{l,s}$ and $v_{l,e}$ so that the direction $v_{l,s} \to v_{l,e}$ along the curve is coincident with the orientation given in the tree $T$.

2. If a segment $[l]$ is an integral curve of a real differential 1-form $\omega = \mathrm{Im}(\lambda_{i_s(l)} - \lambda_{i_e(l)})dx$, we always choose two indices $i_s(l)$ and $i_e(l)$ satisfying

$$\mathrm{Re} \int_l (\lambda_{i_s(l)} - \lambda_{i_e(l)})dx > 0.$$

We define the **total integral value** $\Phi(T)$ of a bidirectional binary tree $T$ as follows:

$$\Phi(T) = \sum_{l \in L} \int_l (\lambda_{i_s(l)} - \lambda_{i_e(l)})dx.$$

The following lemma gives us the most basic property of a bidirectional binary tree.

**Lemma 6.1 (cf. [Sa2, p.74 (Lemma 1)], [AKSST, Appendix B])** For any bidirectional segment $l$ of $T$, we have

$$\Phi(T) = \int_{v_{l,s}}^{v_{l,e}} (\lambda_{i_s(l)} - \lambda_{i_e(l)})dx.$$

In particular、the integral value of the right hand side of the above equality does not depend on a choice of segments $l$ of the tree $T$.

Now let us consider deformations of bidirectional binary trees. The differential system under consideration has a deformation parameter $t$ in a simply connected domain $V_t$ which satisfies the same condition as that given in Section 2. Moreover we assume that the Stokes geometry $G_k(U;t)$ of the level $k$ satisfies the finiteness condition.

We fix a point $t_0 \in V_t$, and let $T = (B, E, L, \rho)$ be a bidirectional binary tree in $G_k(U;t_0)$. By the definition of a bidirectional binary tree, we can change the path of the total integral value of $T$ to a closed path which passes all leaf nodes of $T$. Therefore the total integral value $\Phi(T)$ of the tree $T$ defines a germ of holomorphic function $\phi_T(t)$ at $t = t_0$ with $\phi_T(t_0) = \Phi(T)$. Moreover $\phi_T(t)$ can be analytically continued to any point in $V_t$.

We introduce the following condition:

**Condition (#)**: For any branching node $b$ of $T$, all segments of $T$ sharing the end point $b$ mutually intersect transversally at $b$.

Let $H \subset V_t$ denote an analytic set near $t_0$ defined by

$$H = \{t \in V_t;\, \mathrm{Im}\,\phi_T(t) = 0\}.$$

**Proposition 6.2** Assume that $T$ satisfies the condition (#) and the degree of $T$ is less than or equal to $k+2$. Then there exist a neighborhood $W \subset V_t$ of $t_0$ and a bidirectional binary tree $T(t) = (B(t), E(t), L(t), \rho(t))$ in $G_k(U;t)$ for any $t \in H \cap W$ which satisfy:

1. $T(t_0) = T$, and when $t$ moves continuously along $H \cap W$, the tree $T(t)$ is also deformed continuously, that is, the following hold:
   (a) The binary tree $(B(t), E(t), L(t))$ dose not depend on $t$
   (i.e. $(B(t), E(t), L(t)) = (B, E, L)$) and,
   (b) for each edge $l \in L$, the topological immersion $\rho(t)_l : l \to [(\rho(t))(l)]$ is a continuous function of $t$.
2. For any $t \in H \cap W$, we have $\phi_T(t) = \Phi(T(t))$ where $\Phi(T(t))$ is the total integral value of the tree $T(t)$.

Next we investigate the effectiveness of deformed trees. Since the effectiveness of a tree depends on global informations of the Stokes geometry, we first introduce a notion of stability of the Stokes geometry. Let $\tau : [0,1] \to V_t$ be a smooth curve, and $t_1 = \tau(\theta_1)$ and $t_2 = \tau(\theta_2)$ two points in the curve $\tau$. Let $I = (t_1, t_2)$ denote an open segment $\{\tau(\theta);\, \theta_1 < \theta < \theta_2\}$ of the curve $\tau$. In the following two definitions, the types of both objects in question are assumed to be not disjoint.

**Definition 6.3** The Stokes geometry $G_k(U;t)$ is said to be **geometrically stable** in an open segment $I$ if the following conditions are satisfied when $t$ moves in the open segment $I$.

1. (Stability of turning points) No turning point coincides with an ordinary turning point nor a boundary point of $U$.

2. (Stability of Stokes curves) Ordinary turning points never hit against any Stokes curves. A Stokes curve which terminates at an ordinary turning point continues to terminate at the same turning point.

3. (Stability of intersection points) Any two Stoke curves are neither tangent nor intersecting at a point of the boundary of $U$. Furthermore each Stokes curve is not tangent with the boundary of $U$.

Remark that under the geometrical stability assumption, all intersection points of Stokes curves continuously move.

**Definition 6.4** The Stokes geometry $G_k(U;t)$ is said to be **stable** in an open segment $I$ if the following conditions hold:

1. $G_k(U;t)$ is geometrically stable in $I$, and

2. for any Stokes curve $l$ in $G_k(U;t)$, no coherent point of the curve $l$ collides with any other intersection points of the curve $l$ with other Stokes curves when $t$ moves in the segment $I$.

Let $v$ be a turning point in $CT_k(U;t_0)$ for some $t_0 \in I$. We also say that $v$ is **stable as a constructible turning point** of the level $k$ in $I$ if the germ $v(t)$ of $v$ at $t_0$ belongs to $CT_k(U;t)$ for any $t \in I$ (that is, no "napping" occurs when t moves in $I$).

From our experience ([H1]) the stability of the Stokes geometry is violated when an ordinary turning point crosses a Stokes curve (cf. Case A in Section 7). Although we encounter somewhat more delicate situations (cf. Case B and Case C in Section 7), we can show the following:

**Theorem 6.5** Assume that each Stokes curve does not continue to be tangent with the other Stokes curves nor the boundary of $U$ for all $t \in \tau$. There exists a set $E = \{\tau(\theta_1), \ldots, \tau(\theta_r);\, \theta_1 < \cdots < \theta_r\}$ of finite exceptional points in the curve $\tau$ for which the following hold:

1. The Stokes geometry $G_k(U;t)$ is stable in each open segment $(\tau(\theta_i),\ \tau(\theta_{i+1}))$ $(0 \le i \le r)$ where we set $\theta_0 = 0$ and $\theta_{r+1} = 1$, and
2. any turning point of the Stokes geometry $G_k(U;t)$ is stable as a constructible turning point of the level $k$ in each open segment $(\tau(\theta_i),\ \tau(\theta_{i+1}))$.

Let $\phi(t)$ be a holomorphic function in $V_t$ and $\tau : [0,1] \to V_t$ a smooth curve. We assume $\tau \subset \{t \in V_t;\ \mathrm{Im}\,\phi(t) = 0\}$. The following theorem is fundamental for deformations of effective bidirectional binary trees.

**Theorem 6.6** Let us assume that the Stokes geometry $G_k(U;t)$ is stable in $I = (\tau(0),\ \tau(1))$. If there is an **effective** bidirectional binary tree $\hat{T}$ in $G_k(U;t_1)$ for some $t_1 \in I$ satisfying the following:

1. The degree of $\hat{T}$ is less than or equal to $k+2$, and
2. the germ $\phi_{\hat{T}}(t)$ of the total integral value of the tree $\hat{T}$ coincides with $\phi(t)$ near $t = t_1$.

Then there exists an **effective** bidirectional binary tree $\hat{T}(t)$ in $G_k(U;t)$ for any $t \in I$ which satisfies the following conditions:

1. $\hat{T}(t_1) = \hat{T}$, and $\hat{T}(t)$ is deformed continuously when $t$ moves in $I$.
2. We have $\Phi(\hat{T}(t)) = \phi(t)$ for any $t \in I$.

See [H4] for the proof.

**Remark 6.7** We can weaken the stability condition of the theorem so that it suffices to consider only objects of $G_k(U;t)$ connected with the effective bidirectional binary tree by a "Stokes walk path" (for the Stokes walk path, see Honda [H3]).

The condition (†) below helps simplify complicated discussions about the Stokes geometry. We will use the condition in the next section. Let $T$ be a bidirectional binary tree (not necessarily effective).

**Condition (†)**: For any bidirectional segments $l = ([l]; v_{l,s}, v_{l,e})$ of a bidirectional binary tree $T$ with end points $p_s$ and $p_e$ where the direction $p_s \to p_e$ coincides with the orientation given in the tree $T$, the following two conditions hold.

1. In an open segment $(p_s,\ p_e)$, there are no effective coherent points of the underlying Stokes curves $s(v_{l,s})$ and $s(v_{l,e})$.
2. In an open segment $(v_{l,s},\ p_s)$ (resp. $(p_e,\ v_{l,e})$), the state of the underlying Stokes curve $s(v_{l,s})$ (resp. $s(v_{l,e})$) near $p_s$ (resp. $p_e$) is dotted.

For a bidirectional binary tree $T$ satisfying the condition (†), it is easy to prove that the tree $T$ becomes effective by the definition of trees. The assumption seems to be too restrictive. In the examples of the Noumi-Yamada system given in [H1], however, almost all bidirectional binary trees have no effective coherent points in the open segments $(v_{l,s},\, p_s)$ and $(p_e,\, v_{l,e})$, and hence the condition 2 is satisfied.

# 7 Deformations near exceptional points

Let $T$ be an effective bidirectional binary tree in $G_k(U; t_0)$ for some $t_0 \in V_t$. Set $\phi(t) = \phi_T(t)$ (remark that $\phi(t)$ is holomorphic in $V_t$) and let $\tau : [0, 1] \to V_t$ be a smooth curve contained in $\{t \in V_t;\, \mathrm{Im}\, \phi(t) = 0\}$ with $\tau(0) = t_0$. Let a point $t_1 = \tau(\theta_1)$ $(\theta_1 > 0)$ be the first exceptional point given by Theorem 6.5. Then effective bidirectional binary trees $T(t)$ in $G_k(U; t)$ for $t \in [t_0, t_1)$ are deformations of the tree $T$. We assume that the level $k$ is sufficiently large.

In this section, we will study deformations $T(t)$ of the tree $T$ around the exceptional point $t_1$. Since the problem has global nature, it seems to be difficult to investigate the problem without any assumptions. As the first step of our study, we assume the following:

**Assumption 7.1** Bidirectional binary trees in question satisfy the condition (†) near the exceptional points.

Here "trees in question" mean bidirectional binary trees whose germs of total integral values are equal to $\phi(t)$ if exist. Under the assumption, the problem is reduced to the existence of a bidirectional binary tree itself. When $\theta$ tends to $\theta_1$ $(\theta < \theta_1)$, for the tree $T(\tau(\theta))$ in $G_k(U; \tau(\theta))$ we find one of the following four cases at $\theta = \theta_1$.

- **(Case A)** An ordinary turning point hits against a segment of the tree.
- **(Case B)** The length of a segment of the tree becomes zero.
- **(Case C)** At a branching node $b$ of the tree, some edges of the tree with an end point $b$ become tangent each other.
- **(Case D)** For a segment $l = ([l]; v_s, v_e)$ of the tree, the portion $[v_s, v_e]$ of the Stokes curve touches the boundary of $U$.

To avoid difficulties related to the case D above, we also assume the following condition:

**Assumption 7.2** All turning points and portions of Stokes curves that are necessary to construct bidirectional binary trees are always contained in $U$.

In what follows, we will investigate the problem for Case A and Case B. By the following Lemma 7.3 below, it is enough to consider local deformations of the tree near the segment in question.

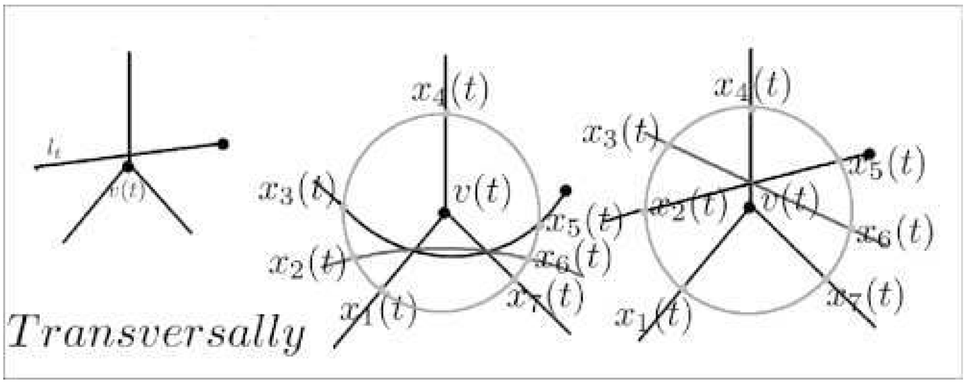


Figure 13: A turning point hits against a Stokes curve.

Let $l(t)$ be a Stokes curve and $v(t)$ a simple turning point. We assume the type of $l(t)$ and that of $v(t)$ have one and only one common index, and when $t$ tends to $t_1$, the turning point $v(t)$ hits against the curve $l(t)$. We draw a sufficiently small circle with the center at $v(t)$. The circle intersects with Stokes curves at 7 points $x_1(t), \ldots, x_7(t)$ (see Fig. 13). See [AKSST, Appendix A] for its proof.

**Lemma 7.3** Assume the Stokes curve $l(t)$ and $v(t)$ intersect transversally. Then all intersection points $x_1(t), \ldots, x_7(t)$ are real analytic functions near $t = t_1$.

The above lemma implies that although the direction of the Stokes curve $l(t)$ abruptly changes in Fig. 13 (we here assume that the turning point from which $l(t)$ emanates is in the right hand side of Fig. 13) when $t$ moves near $t_1$, if we take new Stokes curves into accounts, all curves of Fig. 13 move continuously outside the circle (see [AKSST, Section 1]). Moreover the constructibility of virtual turning points is preserved. Therefore the problem can be reduced to the inside of the circle.

In what follows, we investigate cases A and B. We assume that the several cases are not observed at the same time.

## 7.1 Case A

Let us consider a situation where an ordinary turning point $v$ hits against a bidirectional segment $l = ([l]; v_0, v_1)$ of the tree at $t = t_1$. We assume that $v$ is neither

$v_0$ nor $v_1$, and that the bidirectional segment $l$ and $v$ intersect transversally. The case A is classified into the following 4 sub-cases, A-1, A-2, A-3 and A-4.

- **(Case A-1)** The type of $v$ and that of $l$ are disjoint.

  Disjoint situations do not cause any problem: the effective bidirectional binary tree $T(t)$ remains intact near $t = t_1$.

- **(Case A-2)** The type of $l$ and that of $v$ are the same.

  The bidirectional binary tree $T(t)$ vanishes at $t = t_1$. The reverse order of examples in [H1] pp. 53-65 (Fig. III-2-12 $\sim$ Fig. III-2-1) is considered to give a corresponding example. This situation is related to a counterpart of the Nishikawa phenomenon ([KKNT]) in the Noumi-Yamada system.

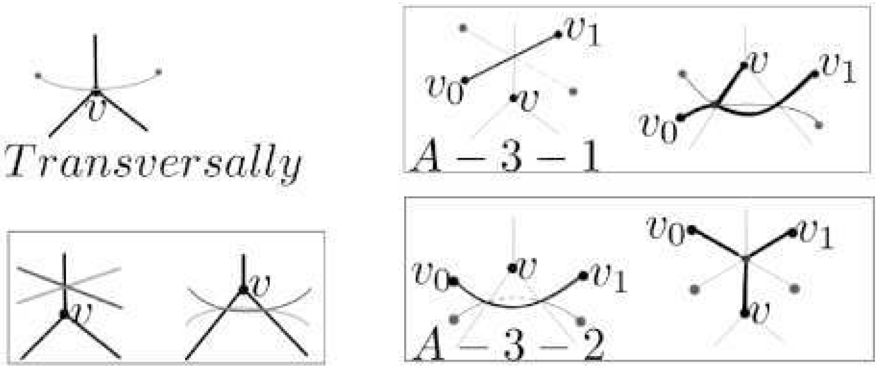


Figure 14: Case A-3.

- **(Case A-3)** $v$ is a simple turning point, and the type of $v$ and that of $l$ have one and only one common index.

  Let us consider the case A-3 in detail. An effective bidirectional binary tree $T(t)$ only exists for $t = \tau(\theta)$ $(\theta > \theta_1)$ when the geometry near $v$ is graphically equivalent to **A-3-1** or **A-3-2** given in the Fig. 14. After $v$ hits against $l$, $v$ becomes a new leaf node of $T(t)$, and the degree of $T(t)$ increases. The equality $\phi(t) = \Phi(T(t))$ still holds near $t = t_1$ where $\Phi(T(t))$ is the total integral value of the tree $T(t)$. Concrete examples of A-3-1 and A-3-2 are given in [H1] pp. 68-79 (Fig. IV-1-1 $\sim$ Fig. IV-3-4).

- **(Case A-4)** $v$ is a double turning point, and the type of $v$ and that of $l$ have one and only one common index.

  The effective bidirectional binary tree $T(t)$ continues to exist near $t = t_1$. Examples of this case are given in [H1] pp. 93-102 (Fig. V-2-1 $\sim$ Fig. V-2-10).

## 7.2 Case B

Let us consider a situation where the length of a bidirectional segment $l$ of the tree becomes zero at $t = t_1$ . The case B is classified into the two sub-cases B-1 and B-2.

- **(Case B-1)** One of the end points of $l$ is a leaf node.

  Remark that the leaf node $v$ of $l$ is an ordinary turning point.

  - **(Case B-1-1)** The leaf node $v$ is a double turning point.
    An effective bidirectional binary tree $T(t)$ exists for $t = \tau(\theta)$ $(\theta > \theta_1)$. The degree of $T(t)$ remains constant, and $\phi(t) = \Phi(T(t))$ still holds. Examples of (B-1-1) are given in [H1] pp. 93-102 (Fig. V-2-1 $\sim$ Fig. V-2-10).
  - **(Case B-1-2)** The leaf node $v$ is a simple turning point.
    An effective bidirectional binary tree $T(t)$ exists for $t = \tau(\theta)$ $(\theta > \theta_1)$. Since the case B-1-2 is a reverse case of A-3, the leaf node $v$ will be removed from the bidirectional binary tree $T(t)$, and the degree of $T(t)$ decreases. However $\phi(t) = \Phi(T(t))$ still holds. Examples of B-1-2 are given in [H1] pp. 83-92 (Fig. V-1-1 $\sim$ Fig. V-1-11).

- **(Case B-2)** Both end points of $l$ are branching nodes.

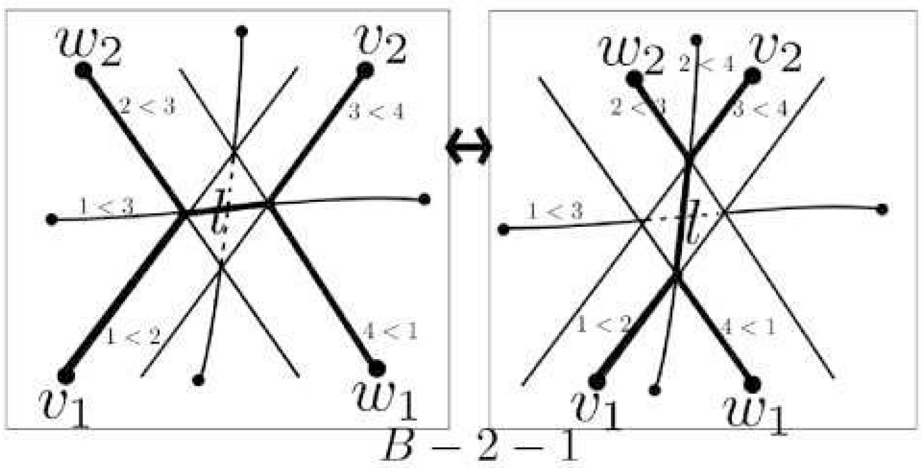


Figure 15: Case B-2-1.

As we have not yet studied all the cases exhaustively, we content ourselves with the discussion about some basic case B-2-1 below.

– **(Case B-2-1)** Assume that the geometry near the segment $l$ is graphically equivalent to Fig. 15, where the type of $v_1$ and that of $v_2$ (resp. $w_1$ and $w_2$) are disjoint. If they are not disjoint, the case A also occurs at the same time. Remark that order relations of curves are uniquely determined in this case.

  Although the segment $l$ will be replaced by another segment, an effective bidirectional binary tree $T(t)$ exists for $t = \tau(\theta)$ $(\theta > \theta_1)$. The degree of $T(t)$ remains constant, and $\phi(t) = \Phi(T(t))$ still holds. Examples of B-2-1 are given in [H1] pp. 103-110 (Fig. V-3-1 $\sim$ Fig. V-4-4).

Summing up, for all sub-cases of the case B investigated here an effective bidirectional binary tree $T(t)$ exists for $t = \tau(\theta)$ $(\theta > \theta_1)$. The author conjectures that this fact always holds for the case B.

# 8 Formal Stokes curves of $NY_m$

We will apply the results obtained so far to the Noumi-Yamada system $NY_m$. Throughout this section, we always assume Assumptions 7.1 and 7.2.

Let $\tau : [0, 1] \to \mathbb{C}_t$ be a portion of formal Stokes curve of $NY_m$ emanating from a turning point $s = \tau(0)$. The formal Stokes curve $\tau$ is, by definition, the curve defined by

$$\mathrm{Im} \int_s^t (\mu_+ - \mu_-)dt = 0,$$

where $\mu_\pm$ are two roots of the characteristic equation of the linearized system of $NY_m$ which merge at the turning point $s$, and we label $\mu_+$ and $\mu_-$ so that

$$\mathrm{Re} \int_s^t (\mu_+ - \mu_-)dt > 0$$

holds for $t = \tau(\theta)$ $(\theta > 0)$.

By a result of Takei [T2], for sufficiently small $\theta > 0$ there exists an effective bidirectional binary tree $T(\tau(\theta))$ of the degree two in the Stokes geometry $G_k(U; \tau(\theta))$ of $NYL_m$ which satisfies an integral relation

$$\Phi(T(\tau(\theta))) = \frac{1}{2} \int_s^{\tau(\theta)} (\mu_+ - \mu_-)dt.$$

There are finitely many exceptional points $E = \{\tau(\theta_1), \dots, \tau(\theta_r); \theta_1 < \cdots < \theta_r\}$ in the Stokes curve $\tau$, and we obtain the following :

**Theorem 8.1** Assume that at each exceptional point in $E$, only one of the cases A-1, A-3-1, A-3-2, A-4, B-1 or B-2-1 occurs, that is, we assume several cases do not occur at the same time. Then there is an effective bidirectional binary tree $T(\tau(\theta))$ in $G_k(U; \tau(\theta))$ for any $\theta > 0$, and we have

$$\Phi(T(\tau(\theta))) = \frac{1}{2}\int_s^{\tau(\theta)} (\mu_+ - \mu_-)dt.$$

**Remark 8.2** In general, the tree $T(t)$ grows or shrinks at an exceptional point in $E$. Furthermore if the case A-2 occurs at an exceptional point $\tau(\theta_j) \in E$ for some $j$, the tree vanishes at that point and the portion $\{\tau(\theta);\, \theta > \theta_j\}$ of the formal Stokes curve $\tau$ is considered to become a dotted line. In fact, we can find the similar situations for a Stokes curve generated by Nishikawa phenomena. The author does not know such a thing really happens for a Stokes curve emanating from a turning point of the first or the second kind in the sense of [KKNT].

# 9 Some comments

The study of degeneration of the Stokes geometry in view of effective bidirectional binary trees has just begun. Although the title of this paper is "On the Stokes geometry of the Noumi-Yamada system", almost all facts are established in a general framework. The reason why the author has chosen the title is that the initial purposes of effective bidirectional binary trees were to understand very complicated Stokes geometry of the Noumi-Yamada system well and clearly explain the mechanisms of several strange phenomena of the system. For the Noumi-Yamada system, the initial purposes seem to have been achieved in a rather satisfactory manner. However there still remain many problems to be attacked. In particular, to handle the major behavior of effective bidirectional binary trees near exceptional points is an important and further problem.

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