RIMS Kôkyûroku Bessatsu **B3**
(2007), 131–137

Network entrainment: comparison of lattice and random networks

By

Hiroshi KORI*

Abstract

Entrainment of oscillator networks has been studied for the last few decades. In spatially distributed oscillators (including Belousov-Zhabotinsky reaction diffusion systems) the entrainment threshold for coupling intensity doesn't depend on the system length. In contrast, it has been recently found that in random networks the entrainment threshold grows exponentially with the system length [H. Kori and A.S. Mikhailov, Phys. Rev. Lett. **93**, 254101 (2004); Phys. Rev. E **74**, 066115 (2006)]. In this report, we briefly review the difference in the entrainment behavior between lattice and random networks.

§ 1. introduction

Pacemakers are wave sources in distributed oscillatory systems typically associated with a local group of elements having a higher oscillation frequency. Target patterns, generated by pacemakers, were the first complex wave patterns observed in the Belousov-Zhabotinsky system [1]. Pacemakers play an important role in functioning of the heart [2] and in the collective behavior of *Dictyostelium discoideum* [3]. They are also observed in large-scale ecosystems [4]. While the majority of related investigations have so far been performed for systems with local diffusive coupling between the elements, pacemakers can also operate in oscillator networks with complex connection topologies. One of the most intriguing examples is the circadian (i.e., approximately daily) clock in mammals (for details, see [5, 6]).

Is there any essential difference in the entrainment behavior from lattice and random oscillator networks? To answer this question, the entrainment behavior of random oscillator networks has been investigated [5, 6]. It was found there that the entrainment

2000 Mathematics Subject Classification(s): 2000 Mathematics Subject Classification(s):
Key Words: synchronization, complex networks
*Department of Mathematics, Hokkaido university, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan

threshold for coupling intensity between oscillators strongly depends on the *depth of a network*, defined as the mean forward distance from a pacemaker (i.e., source of external forcing) to the network nodes [5]. Interestingly, such a property is very different from that in spatially distributed oscillator systems, including the Belousov-Zhabotinsky system, where the entrainment threshold is independent of the system length. In this report, we summarize the entrainment behavior both in lattice and random oscillator networks to illustrate their essential differences in dynamical properties.

§ 2. model

We consider a system of N phase oscillators, one of them being a pacemaker. The basic model is given by a set of evolution equations for the pacemaker phase ϕ_1 and the oscillator phases ϕ_i $(2 \leq i \leq N)$,

$$\begin{aligned} \phi_1 &= (\omega + \Delta\omega)t, \\ \dot{\phi}_i &= \omega + \frac{\kappa}{z}\sum_{j=1}^{N} A_{ij}\Gamma(\phi_i - \phi_j). \end{aligned} \tag{2.1}$$

The topology of network connections is determined by the adjacency matrix $\mathbf{A}$ whose elements A_{ij} are either 1 or 0. The mean degree z is the average number of incoming connections per node, i.e. $z = \sum_{i,j} A_{ij}/N$. The element with $i = 1$ is special and represents a pacemaker. Its frequency is increased by $\Delta\omega$ with respect to the frequency ω of all other oscillators. The coupling between elements inside the network is characterized by the 2π-periodic function $\Gamma(x)$ and the (positive) coupling intensity coefficient κ. In absence of a pacemaker, such networks usually undergo autonomous perfect phase synchronization (i.e., $\phi_i = \phi_j$ for any i and j) if the coupling is attracting, i.e., if $(d/d\phi)\Gamma(\phi)|_{\phi=0} < 0$.

Without loss of generality, our model can be simplified. By going into a rotating frame, we have $\omega = 0$. Moreover, rescaled time $t' = t\ \Delta\omega$ and rescaled coupling strengths $\kappa' = \kappa/\Delta\omega, \mu' = \mu/\Delta\omega$ are introduced. After that, the model takes the form of Eq. (2.1) with $\Delta\omega = 1$ and $\omega = 0$ (below, we drop primes in the notations for the rescaled quantities).

The presence of a pacemaker imposes hierarchical organization in the network architecture, which plays a crucial role in determining the entrainment ability. For any node i, its distance l_i with respect to the pacemaker is defined by the length of the minimum forward path separating this node from the pacemaker. We define the element 1 have distances $l_1 = 1$. Among the rest elements, the elements receiving connections from this element 1 have distances $l_i = 2$, etc. Thus, the whole network is divided into a set of shells, each of which is composed of oscillators with distance h from the pacemaker. The shell population N_h is given by the number of the oscillators with distance

h. The depth L of a network is defined by the average distance from the pacemaker to the entire network, given as

$$L = \frac{1}{N}\sum_i l_i = \frac{1}{N}\sum_h hN_h. \tag{2.2}$$

Our focus is on the entrainment threshold $\kappa_{\rm cr}$, defined as the critical coupling intensity κ above which the whole network is entrained by the pacemaker (i.e., $\dot{\phi}_i = \Omega$ for all i). In particular, we are interested in the dependence of $\kappa_{\rm cr}$ on topological properties of the network structure.

§ 3. entrainment in random oscillator networks

We first summarize the results for standard random networks, also known as Erdös-Rényi (ER) networks [7, 8]. These networks are generated by independently assigning with probability p for any pair i and j of the network nodes a connection between the node i to the node j. Hence, elements $A_{ij} = A_{ji}$ of the adjacency matrix are chosen to be 1 with probability p and 0 otherwise, and the matrix $\mathbf{A}$ is symmetric. The mean degree z becomes approximately pN.

Such a oscillator network has been studied analytically for $1 \ll z \ll N$ [6]. It was found that for *any* coupling function with $\Gamma'(0) < 0$ (i.e., attracting coupling) the entrainment threshold has the following dependence

$$\kappa_{\rm cr} \sim z^{L-1}. \tag{3.1}$$

Thus, extremely strong coupling intensity is needed for the entrainment in a random network with a large depth. Moreover, Eq. (3.1) implies that the entrainment threshold increases with the system size. It is known that the typical depth of random networks is roughly $\ln(N/zN_1) + 1$ [9] and we may thus estimate

$$\kappa_{\rm cr} \sim N. \tag{3.2}$$

As explained in the next section, such a property is very different from that in lattice oscillator networks.

§ 4. entrainment in spatially distributed oscillators

In oscillator medium, such as BZ reaction diffusion systems, a pacemaker can entrain the whole system and this behavior is independent of its system size. As is known and explained in the following, certain nonlinearity is responsible for this type of the entrainment.

The entrainment takes place also in lattice oscillator networks, and its behavior is essentially the same as in the oscillator medium. The analytic treatment of lattice oscillator networks is more complicated than of continuous medium. Therefore, we do not try to make a rigorous theory for lattice networks. After a brief sketch of the entrainment behavior in lattice oscillator networks, we take a continuous limit of such a system, by which the entrainment behavior may be better understood. For both lattice oscillator networks and oscillator medium, it will be shown that the entrainment can take place regardless of the system size.

§ 4.1. 1D lattice oscillator network

We consider a 1D lattice network. For convenience, we replace the suffix i by x. The model (2.1) then is rewritten as

$$\begin{aligned} &\phi_1 = t \\ &\dot{\phi}_x = \kappa\{\Gamma(\phi_x - \phi_{x-1}) + \Gamma(\phi_x - \phi_{x+1})\}, \quad \text{for } x \geq 2. \end{aligned} \tag{4.1}$$

We adopt the Neumann boundary condition at $x = N$. We consider a sufficiently large system size and do not care dynamics near the boundary. Note that the network depth of this network is $L = N/2$. In our analysis, we employ the following particular coupling function:

$$\Gamma(\phi) = -\sin(\phi + \alpha) + \sin\alpha, \tag{4.2}$$

where α is a parameter. Note that $\Gamma(0) = 0$ for any α and $(d/d\phi)\Gamma(\phi)|_{\phi=0} < 0$ for $-\pi/2 < \alpha < \pi/2$.

We seek the entrainment solution that has a homogeneous phase difference between neighboring oscillators, i.e.,

$$\phi_x = t - dx, \tag{4.3}$$

where $d = \phi_i - \phi_{i+1}$. We call this solution the *phase wave* solution. Substituting this solution into Eq. (4.1), we obtain

$$2\kappa\sin\alpha(1 - \cos d) = 1, \tag{4.4}$$

from which d is found. Because $-1 \leq \cos d \leq 1$, the existence condition for this solution is $0 < \alpha \leq \pi$ and

$$\kappa \geq \kappa_{\text{cr}} \equiv 1/4\sin\alpha. \tag{4.5}$$

The stability analysis can be done as follows. We consider small perturbation from the solution,

$$\phi_x = t - dx + \epsilon\psi_x. \tag{4.6}$$

Substituting this into Eq. (4.1), linearizing it for small ϵ, we get

$$\dot{\psi}_x = -(\psi_x - \psi_{x-1})\cos(-d+\alpha) - (\psi_x - \psi_{x+1})\cos(d+\alpha). \tag{4.7}$$

We expand ψ_x as

$$\psi_x = \sum_p c_p e^{\lambda_p t + ipx}. \tag{4.8}$$

We then find

$$\begin{aligned}\lambda_p &= -(1-e^{-ip})\cos(-d+\alpha) - (1-e^{ip})\cos(d+\alpha)\\ &= -2\kappa\{\cos\alpha\cos d(1-\cos p) + i\sin\alpha\sin d\sin p\}.\end{aligned} \tag{4.9}$$

Thus, the phase wave solution is stable if $\cos\alpha > 0$. Together with the existence condition (4.5), it is found that the stable phase wave solution exists for $\cos\alpha > 0$ and $\kappa \geq \kappa_{\mathrm{cr}}$.

Importantly, the entrainment threshold κ_{cr} does *not* depend on the system size (or length) N in this lattice network [see Eq. (4.5)]. Direct numerical simulations of the model (4.1) support this observation. Such a property is actually shared also in continuum medium and may be better understood if we take the continuum limit of the model (4.1), as done in the next subsection.

§ 4.2. continuum medium

Here, we derive a continuum version of the model (4.1) and seek the phase wave solution of it. The space dimension is arbitrary. To begin with, we change the notation of our model:

$$\begin{aligned}\dot{\phi}_i &= \frac{\kappa}{l^2}\sum_{i'}\{\sin(\phi_{i'} - \phi_i - \alpha) + \sin\alpha\},\\ \phi_0 &= t,\end{aligned} \tag{4.10}$$

where $\sum_{i'}$ denotes the summation over the nearest neighbors of the oscillator i, and l is a lattice interval (which was unity in the lattice network). Now we take the continuous limit $l \to 0$ and $\phi_{i'} - \phi_i \to 0$ while keeping $(\phi_{i'} - \phi_i)/l$ finite. In the lowest order approximation, i.e. for small phase gradient $(\phi_{i'} - \phi_i)/l \ll 1$, the model (4.10) results in

$$\dot{\phi}(r,t) = \kappa\cos\alpha\nabla_r^2\phi + \kappa\sin\alpha(\nabla_r\phi)^2, \tag{4.11}$$

$$\phi(0,t) = t, \tag{4.12}$$

where $r \in \Re^D$ denotes the coordinate and a pacemaker is placed at $r = 0$.

We look for the phase wave solution, i.e., $\nabla_r \phi$ is constant, sourced from the pacemaker ($r = 0$). Substituting $\nabla_r \phi = d$ into Eq. (4.11) and putting $\dot{\phi} = 1$ (the entrainment condition), we obtain

$$d = \frac{1}{\kappa \sin \alpha}. \tag{4.13}$$

The stability of this solution is found straightforwardly: it is stable for $\cos \alpha > 0$. Thus, the stable phase wave solution exist for $\sin \alpha \neq 0$ and $\cos \alpha > 0$, and this condition is regardless of the system size N. Such a property is distinct from that in random networks.

Note that the nonlinear term $(\nabla_r \phi)^2$ effectively changes the system's base frequency when there is a constant phase gradient (i.e., a phase wave or a target pattern) sourced from the pacemaker [10]. This is the reason why non-zero $\sin \alpha$ is needed for the entrainment by the phase wave.

§ 5. discussion

As is briefly reviewed, the types of the entrainment behavior in random and lattice networks are very different. In lattice networks, the entrainment threshold does not depend on the network length if the coupling function admits the phase wave solution [e.g., Eq. (4.2) with $0 < \alpha < \pi/2$]. In contrast, in random networks, the entrainment threshold strongly depend on the network size and grows *exponentially* with the network depth.

What happens in networks whose property is between lattice and random networks? Is there well-defined transition somewhere in-between? Analysis of the entrainment behavior in small-world networks (e.g., Watts-Strogatz model [11]) would be of great interest. The study on this direction is now in progress by Naoki Masuda and H.K..

Financial support from the 21st Century COE program "Mathematics of Nonlinear Structures via Singularities" in Department of Mathematics, Hokkaido University, Japan is greatly acknowledged.

References

[1] A. N. Zaikin and A. M. Zhabotinsky, Nature **255**, 535 (1970).
[2] A. T. Winfree, *The Geometry of Biological Time* (Springer, New York, 1980).
[3] K. Lee, E. Cox, and R. Goldstein, Phys. Rev. Lett. **76**, 1174 (1996).
[4] B. Blasius, A. Huppert, and L. Stone, Nature **399**, 6734 (1999).
[5] H. Kori and A. S. Mikhailov, Phys. Rev. Lett. **93**, 254101 (2004).
[6] H. Kori and A. S. Mikhailov, Phys. Rev. E **74**, 066115 (2006).
[7] P. Erdös and A.Rényi, Pabl. Math. **6**, 290 (1959).

[8] B. Bollobás, *Random Graphs* (Academic Press, New York, 1985).
[9] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, Nucl. Phys. B **653**, 307 (2003).
[10] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, New York, 1984).
[11] D. J. Watts and S. H. Strogatz, Nature **393**, 440 (1998).