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Bifurcation analysis to Swift-Hohenberg equation with perturbed boundary conditions

By

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Abstract

We consider the Swift-Hohenberg equation with perturbed boundary conditions. We don't a priori know the eigenfunctions for the linearized problem since the SO(2) symmetry of the problem is broken by perturbation. We show that how the neutral stability curves change and, as a result, how the bifurcation diagrams change by the perturbation of the boundary conditions.

§ 1. Introduction

When we study mathematical models which are derived from physical phenomena, we usually consider these models with natural boundary conditions, such as Neumann or periodic boundary conditions. If mathematical models are considered with periodic boundary conditions, then the solutions to these problem automatically have SO(2) symmetry, namely, the solutions are invariant under parallel displacement with respect to spacial variables. In addition, if these models are considered with Neumann boundary conditions, then the solutions to these problem can be extended to the solutions on the whole line which are periodic in space, and in this sense, the problem also has SO(2) symmetry. It is known that the symmetry of the solution restricts its bifurcation structure to certain types. In fact, non-uniform stationary solutions bifurcate from the uniform state as pitchfork generically in the case when they have SO(2) symmetry.

On the other hand, there are also important problems in which boundary conditions are not SO(2) symmetric, and we are interested in influence of the boundary conditions to the bifurcation structures. For instance, Dillon, Maini and Othmer [2]

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analyzed activator-inhibitor systems with boundary conditions which is not SO(2) symmetric in their study of biological pattern formation. Namely, they analyze the system of two component reaction-diffusion equations, where each equation satisfies different boundary conditions with each other. And they obtained the stationary solutions and bifurcation structures which are qualitatively different to those in SO(2) symmetric case by controlling the boundary conditions. In addition, we can also find the similar kind of studies in thermal convection problems. Mizushima and Nakamura [8] studied a convection problem with partially nonslip boundary conditions (which is also not SO(2) symmetric), and they found that the neutral stability curves are qualitatively different to those obtained in the case of SO(2) symmetry. They observed that the neutral stability curves avoid crossing at multiple critical points for different modes which have same parity. The same problem was studied further by Kato and Fujimura [6]. They also examined the avoided-crossing phenomena of the neutral stability curves, and moreover, they analyze the local bifurcation structures by deriving the amplitude equation. They observed that the local bifurcation structure is obtained as the imperfection of the pitchfork bifurcation from two pure mode solutions which have same parity.

In this paper, we consider the Swift-Hohenberg equation:

$$\frac{\partial w}{\partial t} = \left\{\nu - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2\right\} w - w^3,\ t > 0,\ x \in (0, L/2) \tag{1.1}$$

with the following boundary conditions.

$$\begin{aligned} &w(t,0) = w(t, L/2) = 0, \\ &\delta w_x(t,0) - w_{xx}(t,0) = 0, \\ &\delta w_x(t, L/2) + w_{xx}(t, L/2) = 0. \end{aligned} \tag{1.2}$$

Where $w = w(t,x)$ is real valued function, $\nu,\ L > 0$ and $\delta \geq 0$ are parameters. Swift-Hohenberg equation (1.1) is known to be a phenomenological model of the thermal convection problem. Moreover, the problem (1.1) with (1.2) is SO(2) symmetric when $\delta = 0$, however, it is not when $\delta > 0$.

Here, we show the global bifurcation structure of stationary solutions to (1.1) with (1.2) based on the numerical simulation by AUTO, the software package for continuation and bifurcation in finite dimensional ordinary differential equations in Fig.1. We can see that several solution branches are folded with loops when $\delta = 0.05$. When $\delta = 0$ the mixed mode branch bifurcates from pure mode branch as pitchfork bifurcation. On the other hand, when $\delta > 0$, some of the pitchfork bifurcations disappear since the problem losses SO(2) symmetry, and as a result, imperfections of the pitchfork bifurcation take place close to the intersection points between m-th and n-th branches. These imperfections are observed only when the sum $m + n$ is even.

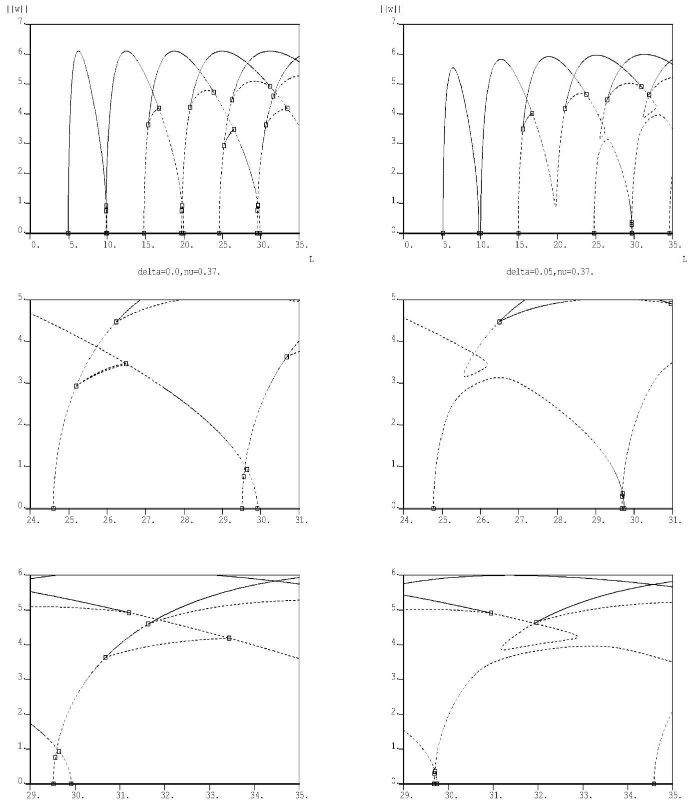

Figure 1. Bifurcation diagrams of (1.1) with (1.2) for $\nu = 0.37$. The horizontal and the vertical axis denote L and $||w||$, respectively. [Top left: $\delta = 0$], [Top right: $\delta = 0.05$], [Middle: Close up around the interaction point between third and fifth branches (Left: $\delta = 0$, Right: $\delta = 0.05$)], [Bottom: Close up around the interaction point between fourth and sixth branches (Left: $\delta = 0$, Right : $\delta = 0.05$)].

Motivated by these numerical results, we analyze the linearized eigenvalue problem, and moreover, we study the local bifurcation structures of stationary solutions to (1.1) with (1.2) by using the cubic normal forms which govern the dynamics of the critical modes.

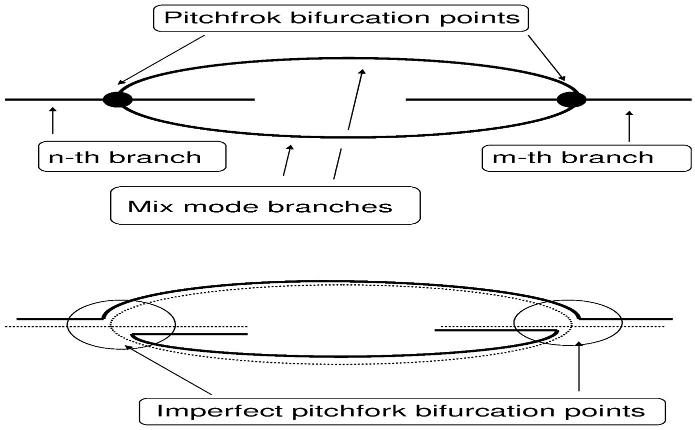

Figure 2. Schematic pictures of the bifurcation structures. [Above: The case $m+n$ is odd], [Below: The case $m+n$ is even].

§ 2. The symmetry and Linear stability analysis

In this section, we study the linear stability of the problem (1.1) with (1.2) around the trivial solution $w \equiv 0$. Namely, we examine how the neutral stability curves are modified by the perturbation of the boundary conditions with small parameter δ.

Let us first examine the case when $\delta = 0$, namely, we consider the linearized problem of (1.1) with the following boundary conditions:

$$w = w_{xx} = 0 \; at \; x = 0, L/2. \tag{2.1}$$

If w is a smooth solution of (1.1) with (2.1), then we can extend the solution in $(-L/2, L/2)$ by the following:

$$\hat{w}(t,x) := \begin{cases} w(t,x) & x \in (0, L/2), \\ -w(t,-x) & x \in (-L/2, 0). \end{cases}$$

Moreover, the solution can be extended as an L-periodic and smooth function. To be more precise, the solution w to (1.1) with (1.2) can be extended to the solution to (1.1) in $\mathbf{R}$ which satisfies:

$$w(t,x) = -w(t,-x), \quad w(t,x) = w(t, x+L).$$

This implies that linearized eigenfunctions and eigenvalues around the trivial solution

$(w \equiv 0)$ are given by

$$w_m := \sin\Big(\frac{2\pi}{L} mx\Big), \ \sigma_m := \nu - \Big(1 - \frac{4\pi^2}{L^2} m^2\Big)^2, \ m \in \mathbf{Z}. \tag{2.2}$$

Thus, we can conclude that neutral stability curves are given by the following:

$$C_m = \{ \ (L, \nu) \ ; \ \nu = \big(1 - \frac{4\pi^2}{L^2} m^2\big)^2 \}, \ m \in \mathbf{Z}.$$

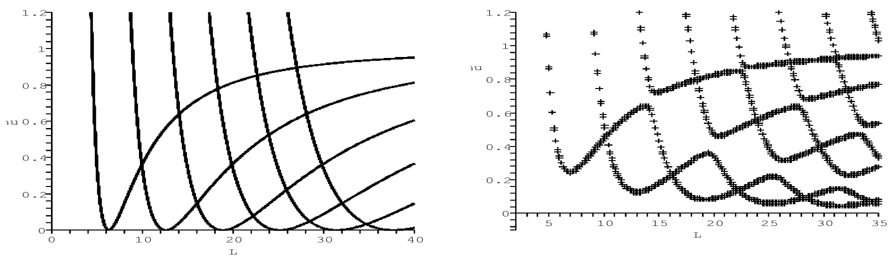

Figure 3. Neutral stability curves drawn in (L, ν)-plane. [Left: They correspond to the critical curves for $C_1, C_2, ..., C_6$ respectively from the left], [Right: The critical curves drown based on the numerical simulation when $\delta = 0.02$. The m-th and n-th curve avoid crossing when $m + n$ is even].

Let us consider that how the neutral stability curves are modified when $\delta > 0$. It should be noted that when $\delta > 0$, we can not extend the solutions as an L-periodic function. However, the problem is invariant under the mapping:

$$w(t, x) \to -w(t, x), \quad w(t, x) \to w(t, L/2 - x)$$

independent of δ. Let $(L^{m,n}, \nu^{m,n})$ be the intersection point of two neutral stability curves C_m and C_n,$(m \neq n)$, and let $B(r)$ be a ball in the parameter space $B(r) := \{(L, \nu); (\hat{L}^{m,n})^2 + (\hat{\nu}^{m,n})^2 < r^2\}$, where $\hat{L}^{m,n} := L - L^{m,n}$ and $\hat{\nu}^{m,n} := \nu - \nu^{m,n}$.

Theorem 2.1. *Let $m, n \in \mathbf{N}, (m \neq n)$. For sufficiently small $\delta > 0$, there exist a positive constant s such that the neutral stability curves $\mathfrak{N}^{m,n}$ in $B(s)$ are given as follows.*

$$\mathfrak{N}^{m,n} = \{(L, \nu) \in B(s); \varsigma_1 \xi_1^2 + \varsigma_2 \xi_2^2 + \varsigma_3 \xi_3^2 + O(s^3) = 0\}.$$

Here, $\xi_j = \vec{v}_j \cdot (\hat{\nu}^{m,n}, \hat{L}^{m,n}, \delta)$, for j=1,2,3, $\vec{v}_j$ and ς_j are eigenvectors and eigenvalues of the matrix $H_{m,n}$:

$$H_{m,n} = \begin{pmatrix} a_{m,n} & d_{m,n} & f_{m,n} \\ d_{m,n} & b_{m,n} & 0 \\ f_{m,n} & 0 & c_{m,n} \end{pmatrix} \tag{2.3}$$

where

$$a_{m,n} = \frac{-(-1)^{m+n}(n^2+m^2)^2\pi^2}{2mn}, \quad b_{m,n} = \frac{16(-1)^{m+n}mn(n^2-m^2)^2}{(n^2+m^2)^3},$$
$$c_{m,n} = \frac{32mn\{1-(-1)^{n+m}\}}{(m^2+n^2)}, \quad d_{m,n} = \frac{-(-1)^{m+n}\sqrt{2}(n^2-m^2)^2\pi}{mn\sqrt{n^2+m^2}},$$
$$f_{m,n} = \frac{2\sqrt{2}(-1)^{m+n}(n^2+m^2)^{\frac{3}{2}}\pi}{mn}.$$

Moreover, if $m+n$ is odd, it holds that $\varsigma_1\varsigma_2 < 0$ and $\varsigma_3 = 0$. And if $m+n$ is even, it holds that ς_1, $\varsigma_2 > 0$ and $\varsigma_3 < 0$.

Proof. The linearized eigenvalue problem of (1.1) with (1.2) is written as follows.

$$\begin{cases} \lambda w = \mathfrak{L}\, w, \\ w = \delta w_x \pm w_{xx} = 0 \ at\ x = 0, L/2. \end{cases}, \tag{2.4}$$

Here, $\mathfrak{L}$ denotes the linearized operator of the equation (1.1) around $w \equiv 0$, namely, it is defined as follows.

$$\mathfrak{L} := \Big\{\nu - \Big(1 + \frac{\partial^2}{\partial x^2}\Big)^2\Big\}.$$

We notice that problem (2.4) is self adjoint, that is, the following holds.

$$< \mathfrak{L}u, v >_{L^2} = < u, \mathfrak{L}v >_{L^2} .$$

Here, $< f, g >_{L^2}$ denotes the standard L^2 inner pruduct for real valued functions $f(x), g(x) \in L^2(0, L/2)$:

$$< f, g >_{L^2} := \int_0^{L/2} f(x)g(x)dx.$$

Therefore, all eigenvalues of the problem (2.4) are real. (The mathematical framework for a linear operator $\mathfrak{L}$ is given in Section3.)

We rewrite the linearized eigenvalue problem (2.4) as follows.

$$\frac{d}{dx}\, W = M(\lambda)\, W\ , \tag{2.5}$$

where,

$$W := \begin{pmatrix} w \\ w_1 \\ w_2 \\ w_3 \end{pmatrix}, \quad w_j = \frac{\partial^j w}{\partial x^j}, \ (j = 1, 2, 3) \tag{2.6}$$

and

$$M(\lambda) = \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ \nu - 1 & 0 & -2 & -\lambda \end{pmatrix} . \tag{2.7}$$

Since we are interested in the parameter region which gives 0-eigenvalue, we set $\lambda = 0$. The eigenvalues of $M(0)$ are given by

$$\Lambda_{\pm} := \sqrt{1 \pm \sqrt{\nu}\, i},\ (i = \sqrt{-1}).$$

Let $\vec{\zeta_j}$ $(j = 1, .., 4)$ be the eigenvectors of $M(0)$. Then, we obtain the general solution of (2.5) as follows.

$$W = c_1\vec{\zeta}_1 e^{\Lambda_+ x} + c_2\vec{\zeta}_2 e^{-\Lambda_+ x} + c_3\vec{\zeta}_3 e^{\Lambda_- x} + c_4\vec{\zeta}_4 e^{-\Lambda_- x}\ . \tag{2.8}$$

Here, c_j $(j = 1, 2, 3, 4)$ are arbitrary constants. We denote $C = (c_1, c_2, c_3, c_4)^t$. Then, from boundary conditions (1.2), we obtain the system of linear equations about C as follows.

$$P(L, \nu, \delta)\ C\ =\ 0\ , \tag{2.9}$$

where

$$P(L, \mu, \delta) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ e^{s_+} & e^{-s_+} & e^{s_-} & e^{-s_-} \\ \gamma_{-,+}\Lambda_+ & -\gamma_{+,+}\Lambda_- & \gamma_{-,-}\Lambda_- & -\gamma_{+,-}\Lambda_- \\ \gamma_{+,+}\Lambda_+ e^{s_+} & -\gamma_{-,+}\Lambda_+ e^{-s_+} & \gamma_{+,-}\Lambda_- e^{s_-} & -\gamma_{-,-}\Lambda_- e^{-s_-} \end{pmatrix}.$$

Here, $\gamma_{\pm,\pm} := \delta \pm \Lambda_{\pm}$ and $s_{\pm} := L\lambda_{\pm}/2$. Thus, the neutral stability curves are given as the set of parameters at which (2.9) has nontrivial solutions as follows.

$$\Sigma := \{(L, \nu, \delta) \in \mathbf{R}^3; g(L, \nu, \delta) = 0\} \tag{2.10}$$

Here,

$$g(L, \nu, \delta) := \det P(L, \nu, \delta)\ .$$

Let $1 \gg \delta > 0$ and $m, n \in \mathbf{Z}$. Without loss of generality, we assume $m > n$. Then, we obtain the Taylor expansion of $g(L, \nu, \delta)$ near (L, ν, δ)=$(L^{m,n}, \nu^{m,n}, 0)$ as follows.

$$\begin{aligned} g(L, \nu, \delta) = &(\hat{L}^{m,n}, \hat{\nu}^{m,n}, \delta) H_{m,n} (\hat{L}^{m,n}, \hat{\nu}^{m,n}, \delta)^t \\ &+ O(|(\hat{L}^{m,n}) + (\hat{\nu}^{m,n}) + \delta|^3). \end{aligned} \tag{2.11}$$

It should be noted that $H_{m,n}$ is Hesse matrix of $g(L, \nu, \delta)$ at $(L^{m,n}, \nu^{m,n}, 0)$. Let us examine the eigenvalues of $H_{m,n}$. For simplicity, we will omit the subscripts "$_{m,n}$" (For instance, $a = a_{m,n}$). If $m + n$ is even, we obtain the characteristic polynomial of $H_{m,n}$ as follows.

$$F(\varsigma) := -\varsigma^3 + (a + b)\varsigma^2 - (ab - f^2 - d^2)\varsigma - bf^2.$$

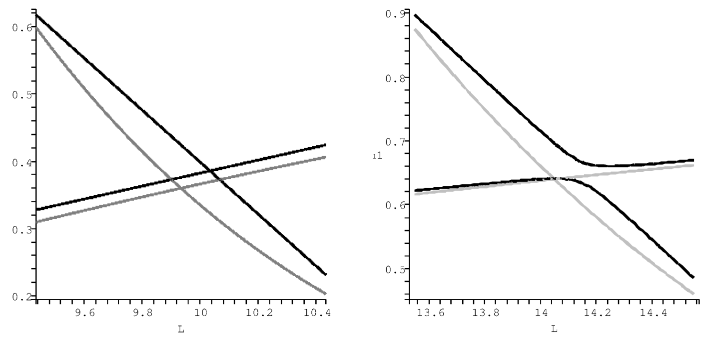

Figure 4. [Left : Neutral stability curves near $(L^{1,2}, \nu^{1,2})$], [Right : Neutral stability curves near $(L^{1,3}, \nu^{1,3})$]. The horizontal and vertical axis denote L and ν, respectively. Gray and black lines correspond to the case when $\delta = 0$ and $\delta = 0.05$, respectively.

We can see that $-bf^2 < 0$. And we have

$$\frac{dF}{d\varsigma} = -3\varsigma^2 + 2(a+b)\varsigma - (ab - f^2 - d^2) = 0,$$

$$\frac{df}{d\varsigma}(0) = -(ab - f^2 - d^2) > 0.$$

Thus, there exist a positive constant ς_* and a negative constant ς^* such that

$$\frac{dF}{d\varsigma}(\varsigma_*) = \frac{dF}{d\varsigma}(\varsigma^*) = 0.$$

Therefore, it follows that $H_{m,n}$ has two positive eigenvalues and third one is negative. On the other hand, if $m+n$ is odd, we have $\det H_{m,n} = 0$ and

$$F(\varsigma) = -\varsigma\{\varsigma^2 - (a+b+c)\varsigma + ab + bc + ca - f^2 - d^2\}$$

Here we notice that $a, c > 0$ and $b < 0$, therefore, it follows that $ca - f^2 < 0$. Thus, there exist a positive constant ς_{**} and a negative constant ς^{**} such that

$$\frac{dF}{d\varsigma}(\varsigma_{**}) = \frac{dF}{d\varsigma}(\varsigma^{**}) = 0.$$

Therefore, it follows that $H_{m,n}$ has a 0-eigenvalue and other two eigenvalues are opposite sign.

Let ς_j and $\vec{v}_j$, $j = 1, 2, 3.$ are eigenvalues and eigenvectors of $H_{m,n}$. Then, for small δ, there exist a positive constant s such that $g(L, \nu, \delta)$ is approximated as

$$g(L, \nu, \delta) = \varsigma_1\xi_1^2 + \varsigma_2\xi_2^2 + \varsigma_3\xi_3^2 + O((s+\delta)^3),$$

for $(L, \nu) \in B(s)$. Here, $\xi_j = \vec{v}_j \cdot (\hat{L}^{m,n}, \hat{\nu}^{m,n}, \delta)$. This completes the proof. □

Theorem2.1 tells us that the neutral stability curves are characterized around the multiple critical points $(L^{m,n}, \nu^{m,n})$ as follows. The neutral stability curves are homeomorphic to the set of two crossing lines in the case when the sum $m+n$ is odd. On the other hand, the neutral stability curves are homeomorphic to the set of two hyperbolae in the case when the sum $m+n$ is even(see Fig.4).

§ 3. Normal forms and Bifurcation Analysis

We shall study the local bifurcation structures around the degenerate critical points to understand the qualitative change of the bifurcation diagram which we saw in the section1.

As in Section2, $\mathfrak{L} : X \to L^2(0, L/2)$ denotes a linearized operator of the equation (1.1) with the boundary conditions (1.2), where the functional space X is defined as follows.

$$X := \{w \in H^4(0, L/2)\,;\ w(0) = w(L/2) = \delta w_x(0) - w_{xx}(0) = \delta w_x(L/2) + w_{xx}(L/2) = 0\}$$

It should be noted that $\mathfrak{L}$ is a sectorial operator, namely, $\mathfrak{L}$ generates an analytic semigroup. Moreover, for each $(L, \nu, \delta) \notin \Sigma$ (Σ is defined in (2.10)) , the operator $\mathfrak{L}^{-1} : L^2(0, L/2) \to L^2(0, L/2)$ is a bounded, linear, compact operator (see [4], [3] and [7] for the detail).

Let $\sigma_{l,\delta}$ and $\phi_{l,\delta}(x)$, $l \in \mathbf{N}$ be the linearized eigenvalues and eigenfuncions of (1.1) with (1.2). More precisely, $\phi = \phi_{l,\delta}(x)$ solves

$$\begin{aligned} \sigma\phi &= -\phi_{xxxx} - 2\phi_{xx} - (1-\nu)\phi\ , \\ \phi &= \delta\phi_x \pm \phi_{xx} = 0 \quad at \quad x = 0, L/2 \end{aligned} \tag{3.1}$$

with eigenvalues $\sigma = \sigma_{l,\delta}$ for $l \in \mathbf{N}$. We note that the eigenvalues $\sigma_{l,\delta}$ and eigenfunctions $\phi_{l,\delta}$ are numbered so that they coincide with those of their leading terms σ_l and w_l for $1 \gg \delta > 0$, $l \in \mathbf{N}$. Substituting the eigenfunction expansion:

$$w(t, x) = \sum_{l \in N} a_l(t)\phi_{l,\delta}(x)$$

into (1.1), we have

$$\dot{a}_j = \sigma_{j,\delta} a_j - \left\langle \Big(\sum_{l \in N} a_l \phi_{l,\delta} \Big)^3 \ ,\ \phi_{j,\delta} \right\rangle_{L^2} \Big/ ||\phi_{j,\delta}||^2_{L^2}\ ,\ j \in \mathbf{N}. \tag{3.2}$$

To study the local bifurcation structures around the degenerate critical point, we apply the center manifold theory.

Theorem 3.1. *Let $m, n \in \mathbf{N}$ $(n \neq m)$. There exist a positive constant ε such that for $\delta < O(\varepsilon^3)$, the cubic normal form of* (1.1) *with* (1.2) *on the center manifold are given as follows if* $(L, \nu) \in B(\varepsilon) \setminus B(\varepsilon^2)$.

$$(3.3)\qquad \begin{cases} \dot{a}_m = (\sigma_{m,\delta} + Aa_m^2 + Ba_n^2)a_m \\ \dot{a}_n = (\sigma_{n,\delta} + Ca_m^2 + Da_n^2)a_n \qquad \text{(if } m+n \text{ is odd).} \end{cases}$$

$$(3.4)\qquad \begin{cases} \dot{a}_m = (\sigma_{m,\delta} + Aa_m^2 + Ba_n^2)a_m + Ea_n^3 + Fa_n a_m^2 \\ \dot{a}_n = (\sigma_{n,\delta} + Ca_m^2 + Da_n^2)a_n + Ga_m^3 + Ha_n^2 a_m \qquad \text{(if } m+n \text{ is even).} \end{cases}$$

Moreover, the normal forms (3.3) and (3.4) are robust against perturbations in higher order terms.

Remark: We can not extend the result for a parameter region which is mush closer to the original degenerate critical point for m and n modes $(L^{m,n}, \nu^{m,n})$, since the eigenfunctions loose their regularity with respect to δ there. (See the proof for the detail.) Notice, however, that it is sufficient to understand the loop of the bifurcation diagram as we see below.

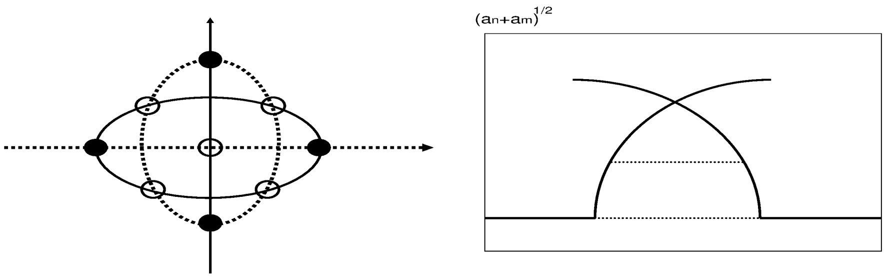

Figure 5. Left: Phase portraits of normal form (3.3). The horizontal and vertical axis denote a_m and a_n, respectively. Right: The bifurcation diagram of equilibriums of the normal form (3.3). The vertical axis denote $\sqrt{a_n^2 + a_m^2}$. Both figures correspond to the case when $m+n$ is odd.

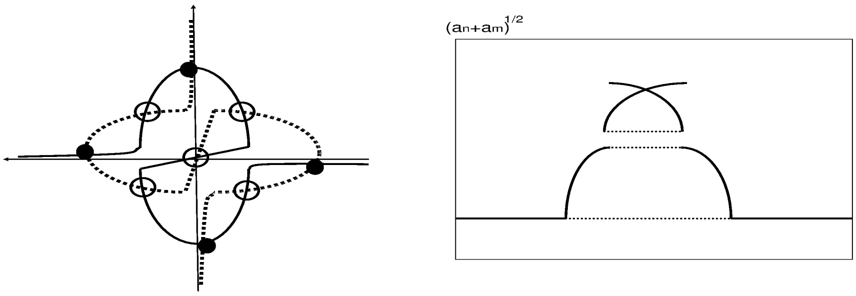

Figure 6. Left: Phase portraits of normal forms (3.4). The horizontal and vertical axis denote a_m and a_n, respectively. Right: The bifurcation diagram of equilibriums of the normal form (3.4). The vertical axis denote $\sqrt{a_n^2 + a_m^2}$. Both figures correspond to the case when $m + n$ is even.

By the normal form analysis, we can understand the local bifurcation structure of stationary solutions to (1.1) with (1.2). More precisely, when $m + n$ is odd, the bifurcation structure of the equilibrium to (3.3) is robust for small perturbation $\delta > 0$. On the other hand, if $m+n$ is even, the bifurcation structure of the equilibrium to (3.4) is modified under the perturbation with small $\delta > 0$. That is, when $m + n$ is even, we can see that the imperfection of the pitchfork bifurcation occur at secondary bifurcation points.

§ 4. Proof of Theorem3.1

We enumerate the lemmas to prove Theorem3.1.

Lemma 4.1. *Let $j, l \in \mathbf{N}$ and $j+l$ be even. Then, following equality holds.*

$$\sigma_{j,\delta} < \phi_{j,\delta}, \phi_{l,0} >_{L^2} = \sigma_{l,0} < \phi_{j,\delta}, \phi_{l,0} >_{L^2} + k_0 l \{(-1)^l \phi''_{j,\delta}(L/2) - \phi_{j,\delta}(0)''\}. \tag{4.1}$$

Here, $k_0 = 2\pi/L$.

Proof. It holds that

$$\sigma_{j,\delta} < \phi_{j,\delta}, \phi_{l,0} >_{L^2} = < \mathfrak{L}\phi_{j,\delta}, \phi_{l,0} >_{L^2}, \tag{4.2}$$

where $\mathfrak{L}$ is a linearized operator of (1.1). We have

$$< \mathfrak{L}\phi_{j,\delta}, \phi_{l,0} >_{L^2} = -\int_0^{L/2} \phi^{(4)}_{j,\delta} \sin(k_0 l x)\, dx - 2\int_0^{L/2} \phi''_{j,\delta} sin(k_0 l x)\, dx$$
$$-(1-\nu)\int_0^{L/2} \phi_{j,\delta}\phi_{l,0}\, dx.$$

$$\begin{aligned}
\int_0^{L/2} \phi^{(4)}_{j,\delta} \sin(k_0 l x)\, dx &= [\phi'''_{j,\delta} \sin(k_0 l x)]_0^{L/2} - k_0 l \int_0^{L/2} \phi'''_{j,\delta} \cos(k_0 l x)\, dx \\
&= -k_0 l [\phi''_{j,\delta} \cos(k_0 l x)]_0^{L/2} - k_0^2 l^2 \int_0^{L/2} \phi''_{j,\delta} \sin(k_0 l x)\, dx \\
&= -k_0 l \{(-1)^l \phi''_{j,\delta}(L/2) - \phi''_{j,\delta}(0)\} - k_0^2 l^2 [\phi'_{j,\delta} \sin(k_0 l x)]_0^{L/2} \\
&\quad + k_0^3 l^3 \int_0^{L/2} \phi'_{j,\delta} \cos(k_0 l x)\, dx \\
&= k_0^4 l^4 \int_0^{L/2} \phi_{j,\delta} \sin(k_0 l x)\, dx - k_0 l \{(-1)^l \phi''_{j,\delta}(L/2) - \phi''_{j,\delta}(0)\}
\end{aligned}$$

$$\begin{aligned}
\int_0^{L/2} \phi''_{j,\delta} \sin(k_0 l x)\, dx &= [\phi'_{j,\delta} \sin(k_0 l x)]_0^{L/2} - k_0 l \int_0^{L/2} \phi'_{j,\delta} \cos(k_0 l x)\, dx \\
&= -k_0 l [\phi_{j,\delta} \cos(k_0 l x)]_0^{L/2} - k_0^2 l^2 \int_0^{L/2} \phi_{j,\delta} \sin(k_0 l x)\, dx \\
&= -k_0^2 l^2 \int_0^{L/2} \phi_{j,\delta} \sin(k_0 l x)\, dx.
\end{aligned}$$

It is clear that (4.1) holds. □

Lemma 4.2. *For $l \in \mathbf{N}$, eigenfunctions $\phi_{j,\delta}$, $j = 1, 2, 3...$ satisfy the following properties.*

$$\phi_{2l-1,\delta}(x) = \phi_{2l-1,\delta}(L/2 - x),\ \phi_{2l,\delta}(x) = -\phi_{2l,\delta}(L/2 - x).$$

Proof. Let $j \in \mathbf{N}$ is even, then we have

$$\sigma_{j,\delta} < \phi_{j,\delta}, \phi_{j,0} >_{L^2} = \sigma_{j,0} < \phi_{j,\delta}, \phi_{j,0} >_{L^2} + k_0 j\{\phi''_{j,\delta}(L/2) - \phi''_{j,\delta}(0)\}. \tag{4.3}$$

And it is clear that $< \phi_{j,\delta}, \phi_{j,0} >_{L^2} \neq 0$. (3.1) is invariant under the mappings:

$$\phi \to -\phi \text{ and } \phi(x) \to \phi(L/2 - x).$$

Thus, for $j \in 2\mathbf{N}$, $\phi_{j,\delta}$ must satisfy

$$\begin{aligned} &(\mathrm{A}) : \phi_{j,\delta}(x) = \phi_{j,\delta}(L/2 - x), \\ &\quad \text{or} \\ &(\mathrm{B}) : \phi_{j,\delta}(x) = -\phi_{j,\delta}(L/2 - x), \end{aligned}$$

If $\phi_{j,\delta}$ satisfies (A), it follows that $\sigma_{j,\delta} = \sigma_{j,0}$. This contradics to Theorem2.1. Thus, $\phi_{j,\delta}$ must satisfy (B). Similarly, it holds that

$$\phi_{2l-1,\delta}(x) = \phi_{2l-1,\delta}(L/2 - x)$$

for $l \in \mathbf{N}$. □

Here, we give a proof of Theorem3.1.

Proof. First, we construct the center manifolds for (3.2) which are expressed as $a_l = h_l(a_m, a_n)$, $l \neq m, n$ for $|a_m|, |a_n| < O(\varepsilon)$, $|\sigma_{m,\delta}|, |\sigma_{n,\delta}| < O(\varepsilon^2)$, $\delta < O(\varepsilon^3)$. Let $l \in \mathbf{N}$, $l \neq m, n$. Then $h_l(a_n(t), a_m(t))$ solves the following equation.

$$\begin{aligned} \frac{\partial h_l}{\partial a_m}\dot{a}_m + \frac{\partial h_l}{\partial a_n}\dot{a}_n =& \sigma_{l,\delta} h_l(a_n, a_m) \\ & - \left\langle \Big(\sum_{j \in \mathbf{N}} a_j \phi_{j,\delta} \Big)^3, \phi_{l,\delta} \right\rangle_{L^2} / ||\phi_{l,\delta}||^2_{L^2}. \end{aligned}$$

And $h_l(a_n, a_m)$ satisfies

$$\frac{\partial h_l}{\partial a_m}(0,0) = \frac{\partial h_l}{\partial a_n}(0,0) = 0.$$

Thus, for $|a_m|, |a_n| < O(\varepsilon)$, $|\sigma_{m,\delta}|, |\sigma_{n,\delta}| < O(\varepsilon^2)$, $\delta < O(\varepsilon^3)$, we obtain

$$h_l(a_n, a_m) = \Big\langle \Big(\sum_{j\in\mathbf{N}} a_j\phi_{j,\delta} \Big)^3, \phi_{l,\delta} \Big\rangle_{L^2} \Big/ (\sigma_{l,\delta}||\phi_{l,\delta}||^2_{L^2}) + O(\varepsilon^4).$$

It follows that the equation (3.2) is reduced to

$$\begin{cases} \dot{a}_m = \sigma_{m,\delta}a_m - \big\langle (a_m\phi_{m,\delta} + a_n\phi_{n,\delta})^3, \phi_{m,\delta} \big\rangle_{L^2} / ||\phi_{m,\delta}||^2_{L^2} + O(\varepsilon^4) \\ \\ \dot{a}_n = \sigma_{n,\delta}a_n - \big\langle (a_m\phi_{m,\delta} + a_n\phi_{n,\delta})^3, \phi_{n,\delta} \big\rangle_{L^2} / ||\phi_{n,\delta}||^2_{L^2} + O(\varepsilon^4) \end{cases}$$

for $|a_m|, |a_n| < O(\varepsilon)$, $|\sigma_{m,\delta}|, |\sigma_{n,\delta}| < O(\varepsilon^2)$, $\delta < O(\varepsilon^3)$. Moreover, it holds that the normal form of (3.2) are invariant under the mapping $(a_m, a_n) \to (-a_m, -a_n)$ since (1.1) with (1.2) is invariant under the mapping $w(t,x) \to -w(t,x)$. More precisely, the nonlinear terms of the normal form are expressed as

$$\sum_{\substack{p,q,j\in\mathbf{N} \\ p+q=2j+1}} C^{(p,q)}\ a_n^p\ a_m^q\ .$$

Now we divide the proof into two parts.

Part1: We prove Theorem3.1 in the case when $m+n$ is odd. Without loss of generality, we assume that m is odd and n is even. We represent the eigenfunction expansion as follows:

$$\sum_{j\in\mathbf{N}} a_j\phi_{j,\delta} = \sum_{j_i\in\mathbf{N}} a_{2j_1-1}\phi_{2j_1-1,\delta} + \sum_{j_2\in\mathbf{N}} a_{2j_2}\phi_{2j_2,\delta}.$$

By the change of variable $x \to L/2 - x$, and using symmetry properties:

$$\phi_{2j,\delta}(x) = -\phi_{2j,\delta}(L/2 - x),\ \phi_{2l-1,\delta}(x) = \phi_{2l-1,\delta}(L/2 - x), \tag{4.4}$$

for $j, l \in \mathbf{N}$, we have

$$\begin{aligned} &\sum_{j_i\in\mathbf{N}} a_{2j_1-1}\phi_{2j_1-1,\delta}(L/2 - x) + \sum_{j_2\in\mathbf{N}} a_{2j_2}\phi_{2j_2,\delta}(L/2 - x) \\ &= \sum_{j_i\in\mathbf{N}} a_{2j_1-1}\phi_{2j_1-1,\delta}(x) + \sum_{j_2\in\mathbf{N}} (-a_{2j_2})\phi_{2j_2,\delta}(x). \end{aligned}$$

The equation (1.1) with (1.2) is invariant under the change of variable: $x \to L/2 - x$. It follows that the normal form is invariant under the mappings: $(a_m, a_n) \to (a_m, -a_n)$ and $(a_m, a_n) \to (-a_m, a_n)$. Thus, we obtain the normal form to (3.2) as follows.

$$\dot{a}_m = \sigma_{m,\delta}a_m + A\ a_m^3 + B\ a_n^2 a_m + \sum_{\substack{p_1,q_1,j_1\in\mathbf{N} \\ p_1+q_1=2j_1+1}} C_1^{(p_1,q_1)} a_n^{2p_1} a_m^{2q_1-1},$$

$$\dot{a}_n = \sigma_{n,\delta}a_n + C\ a_m^2 a_n + D\ a_n^3 + \sum_{\substack{p_2,q_2,j_2\in\mathbf{N} \\ p_2+q_2=2j_2+1}} C_2^{(p_2,q_2)} a_m^{2p_2} a_n^{2q_2-1}. \tag{4.5}$$

Thus, we obtain the leading terms of A, B, C and D as follows:

$$A = D = -\frac{3}{4} + O(\delta),$$
$$B = C = -\frac{3}{2} + O(\delta). \tag{4.6}$$

It is clear that A, B, C, $D < 0$ and $AD - BC < 0$ for $0 < \delta \ll 1$. Moreover, we can see that (3.3) is robust against the higher order terms since they are expressed as shown in (4.5).

Part2 In this part, we prove Theorem3.1 in the case when $m + n$ is even. Without loss of generality, we assume $m > n$. Then, the cubic normal form is given as follows:

$$\begin{cases} \dot{a}_m = \sigma_{m,\delta} a_m + A\ a_m^3 + B\ a_n^2 a_m + E\ a_n^3 + F\ a_m^2 a_n, \\ \dot{a}_n = \sigma_{n,\delta} a_n + C\ a_m^2 a_n + D\ a_n^3 + G\ a_m^3 + H\ a_n^2 a_m. \end{cases} \tag{4.7}$$

We notice that the coefficients A, B, C and D are also given as (4.6). In addition, E, F, G and H are given as follows:

$$E = - < \phi_{n,\delta}^3, \phi_{m,\delta} >_{L^2} / ||\phi_{m,\delta}||_{L^2}^2 \ , \tag{4.8}$$
$$F = -3 < \phi_{m,\delta}^2 \phi_{n,\delta}, \phi_{m,\delta} >_{L^2} / ||\phi_{m,\delta}||_{L^2}^2 \ , \tag{4.9}$$
$$G = - < \phi_{m,\delta}^3, \phi_{n,\delta} >_{L^2} / ||\phi_{n,\delta}||_{L^2}^2 \ , \tag{4.10}$$
$$H = -3 < \phi_{n,\delta}^2 \phi_{m,\delta}, \phi_{n,\delta} >_{L^2} / ||\phi_{n,\delta}||_{L^2}^2 \ . \tag{4.11}$$

We can find that $H = 3E$ and $F = 3G$. And it holds that $E = G = F = H = 0$ for $m \neq 3n$, $\delta = 0$ (if $m = 3n$, then $E = -1/4 + O(\delta)$). □

Finally, we derive the coefficients E, G formally. The problem (3.1) is equivalent to the following:

$$\sigma\phi = -k_0^4 \phi_{xxxx} - 2k_0^2 \phi_{xx} - (1 - \nu)\phi,$$
$$\phi = \delta\phi_x \pm k_0 \phi_{xx} = 0 \text{ at } x = 0, \pi.$$

For $1 \gg \delta > 0$, we set

$$\sigma = \sigma_0 + \delta\sigma_1, \quad \nu = \nu_0 + \delta\nu_1. \tag{4.12}$$

Here, σ_0, ν_0 satisfy following:

$$\sigma_0 = \nu_0 - (1 - m^2 k_0^2)^2.$$

We expand an unknown function ϕ as follows.

$$\phi = \phi_0 + \delta\phi_1 + \delta^2\phi_2 + \cdots\cdots. \tag{4.13}$$

Here, $\phi_0 = \sin(mx)$. By substituting (4.12) and (4.13) into (4.12), we obtain the eigenvalue problem in $O(\delta)$. And the solution to these problem is given as follows.

$$\begin{aligned}\phi_1 = &\frac{mk_0}{\pi(m^2k_0^2-1)}\left(x-\frac{\pi}{2}\right)\cos(mx) + \frac{-k_0(3m^2k_0^2-1)}{2\pi(m^2k_0^2-1)^2}\sin(mx)\\ &-\frac{mk_0\{(-1)^m+\cos(\lambda\pi)\}}{2(m^2k_0^2-1)\sin(\lambda\pi)}\sin(\lambda x) + \frac{mk_0}{2(m^2k_0^2-1)}\cos(\lambda x)\end{aligned}$$

where $\lambda = \sqrt{2-m^2k_0^2}/k_0$. As we remarked in Section3, ϕ_1 is undefined at the degenerate point: $k_0 = k_0^* := \sqrt{2/(m^2+n^2)}$ since $\sin(\lambda\pi)|_{k_0=k_0^*} = 0$. To avoid this difficulty, we introduce a small parameter ε as $\varepsilon^2 = |k_0 - 2\pi/L^{m,n}|$, and set $\delta = \varepsilon^3$, then we have

$$\phi = \sin(mx) + \varepsilon\frac{mn|m^2-n^2|\{(-1)^{m+n}+1\}}{\pi(m^2+n^2)^3}\sin(nx) + O(\varepsilon^2). \tag{4.14}$$

Using (4.14), we can find that $EG > 0$ for $m+n$ is even.

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