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Pattern formation for adsorbate-induced phase transition model

By

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Abstract

Hildebrand [3] proposed an adsorbate-induced phase transition model. For this model, Takei et al. [6] showed several stationary patterns by the numerical simulations. We prove the existence of the corresponded patterns by the bifurcation theory from a constant solution. Moreover, the direction of the bifurcation branch near the bifurcation point is obtained. It is a pitchfork type for the stripe and square pattern, transcritical type for the hexagonal one.

§ 1. Introduction

Several people [5], [4], [3] [10] [11] proposed models which describe the process of pattern formation in the catalytic oxidation of CO molecules on a platinum surface. In this paper, we consider the model given in [3] as follows:

$$
\text{(P)} \quad \begin{cases} u_t = D\Delta u + \alpha \nabla\{u(1-u)\nabla\chi(\rho)\} - (ae^{\beta\chi(\rho)} + b)u + c & \text{in } \Omega \times (0,T), \\ \rho_t = \Delta\rho + df(u,\rho) - \varepsilon\left(\rho - \dfrac{1}{2}\right) & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial \rho}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\,\cdot\,,0) = u_0, \quad \rho(\,\cdot\,,0) = \rho_0 & \text{in } \Omega, \end{cases}
$$

where Ω is a bounded domain in $\boldsymbol{R}^2$ with the boundary $\partial\Omega$ and $a, b, c, d, D, \alpha, \beta, \varepsilon$ are positive constants. The unknown functions $u = u(x,t)$ and $\rho = \rho(x,t)$ denote the

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adsorbate coverage rate of the surface by CO molecules and the structural state of surface at a position $x \in \Omega$ and time $t \in [0, \infty)$, respectively. The functions $\chi(\rho)$ and $f(u, \rho)$ are defined by

$$\chi(\rho) = \rho^2(2\rho - 3), \quad f(u, \rho) = \rho(\rho + u - 1)(1 - \rho). \tag{1.1}$$

As shown in Tsujikawa and Yagi [9], Takei et al. [8] and [6], there exists a unique global solution of (P) and an exponential attractor of the corresponding dynamical system.

From the view point of the pattern formation, it is shown the existence of stationary spot solutions and its stability of (P) with $\varepsilon = 0$ in $\boldsymbol{R}$ and $\boldsymbol{R}^2$ by using the singular perturbation method [5], [4]. On the other hand, various types of stationary patterns by numerical computations in [11], [6] are obtained. They are stationary stripe, square and hexagonal patterns on the surface.

In this paper, we show the existence of these stationary solutions of (P) for $\varepsilon \geq 0$ by using the bifurcation from the constant solution. To do so, we consider the following nonlinear elliptic problem:

$$\text{(SP)} \begin{cases} D\Delta u + \alpha\nabla\{u(1-u)\nabla\chi(\rho)\} - (ae^{\beta\chi(\rho)} + b)u + c = 0 & \text{in } \Omega, \\ \Delta\rho + df(u, \rho) - \varepsilon\left(\rho - \dfrac{1}{2}\right) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial \rho}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

It is easy to show that a positive constant solution of (SP) satisfies

$$\begin{cases} u = \dfrac{c}{ae^{\beta\chi(\rho)} + b}, \\ df(u, \rho) = \varepsilon\left(\rho - \dfrac{1}{2}\right). \end{cases} \tag{1.2}$$

Lemma 1.1. *If positive constants a, b, c and β satisfy*

$$\frac{1}{2} < \frac{c}{a + b} \quad \textit{and} \quad \frac{c}{ae^{-\beta} + b} < 1, \tag{1.3}$$

there is only one positive constant solution (u^*, ρ^*) of (1.2) for any $\varepsilon > 0$, which satisfies

$$\frac{1}{2} < \rho^* < 1.$$

Proof. By (1.2), (1.3), we have $0 < u < 1$ and $0 < \rho \leq 1$. Since $-1 \leq \chi(\rho) < 0$ for $0 < \rho \leq 1$ by (1.1), then (1.2) implies

$$\frac{c}{a + b} < u = \frac{c}{ae^{\beta\chi(\rho)} + b} \leq \frac{c}{ae^{-\beta} + b} \quad \text{for } 0 < \rho \leq 1.$$

Hence (1.3) implies $0 < 1 - u < 1/2$. Since $f(u, 1-u) = f(u, 1) = 0$ and $f(u, \rho) > 0$ for $1 - u < \rho < 1$, then the second equation of (1.2) has only one solution $\rho = \rho^* \in (1/2, 1)$. For such a ρ^*, the first equation yields a solution $u = u^*$. $\square$

Here, we remark that there are positive constant solutions (u^*, v^*) which satisfy $0 < \rho^* \leq 1/2$ for suitable parameter ε and d [3].

In this paper, by the bifurcation theory of Crandall and Rabinowiz [1], we treat α as a bifurcation parameter and show the existence of stripe, square and hexagonal patterns bifurcating from the positive constant solution $(u^*,\ \rho^*)$ obtained in Lemma 1.1. Moreover, the direction of the bifurcation branch is obtained.

§ 2. Degenerate bifurcation at a simple eigenvalue

§ 2.1. Degenerate condition

To study several patterns including the hexagonal one, we set the following square domain Ω in $\boldsymbol{R}^2$:

$$\Omega = \left(0, \frac{\pi}{l}\right) \times \left(0, \frac{\pi}{\sqrt{3}l}\right), \tag{2.1}$$

where $l > 0$.

Let X and Y be Hilbert spaces defined by

$$X = H^2_\nu(\Omega) \times H^2_\nu(\Omega), \quad Y = L^2(\Omega) \times L^2(\Omega),$$

where $H^2_\nu(\Omega) = \left\{u \in H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\right\}$.

By defining the operator $F : X \times \boldsymbol{R} \to Y$ as

$$F(u, \rho, \alpha) = \begin{pmatrix} D\Delta u + \alpha \nabla\{u(1-u)\nabla\chi(\rho)\} - (ae^{\beta\chi(\rho)} + b)u + c \\ \Delta\rho + df(u, \rho) - \varepsilon\left(\rho - \dfrac{1}{2}\right) \end{pmatrix},$$

(SP) is rewritten as $F(u,\ \rho,\ \alpha) = \mathbf{0}$. Then, the linearized operator $F_{(u,\rho)}(u^*, \rho^*, \alpha)$ of $F(u,\ \rho,\ \alpha)$ at (u^*, ρ^*) with respect to $(u,\ \rho)$ is given by

$$F_{(u,\rho)}(u^*, \rho^*, \alpha)\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} D\Delta - B & -\alpha A\Delta + C \\ V & \Delta - W \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}, \tag{2.2}$$

where

$$\begin{aligned} &A = -u^*(1-u^*)\chi'(\rho^*), \quad B = ae^{\beta\chi(\rho^*)} + b, \quad C = -a\beta\chi'(\rho^*)e^{\beta\chi(\rho^*)}u^*, \\ &V = d\rho^*(1-\rho^*), \quad W = \varepsilon - df_\rho(u^*, \rho^*). \end{aligned} \tag{2.3}$$

Remark. *Since*

$$\chi'(\rho^*) < 0, \quad f_\rho(u^*, \rho^*) < 0 \quad \textit{for small } \varepsilon > 0,$$

A, B, C, V and W are positive constants.

In order to obtain the dimension and the base of Ker $F_{(u,\rho)}(u^*, \rho^*, \alpha)$, we consider the following problem:

$$\text{(L)} \quad \begin{cases} F_{(u,\rho)}(u^*, \rho^*, \alpha)\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in } \Omega \\ \dfrac{\partial h}{\partial \nu} = \dfrac{\partial k}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let

$$\phi_m(x) = \cos(lmx), \quad \psi_n(y) = \cos(\sqrt{3}lny). \tag{2.4}$$

Then, $\{\phi_m(x)\psi_n(y)\}_{m,n=0}^{\infty}$ is the completely orthonormal system of Hilbert space $H^2_\nu(\Omega)$. Therefore, a solution $(h(x,y), k(x,y))$ of (L) is represented by the following expansions:

$$h(x,y) = \sum_{m,n=0}^{\infty} h_{mn}\phi_m(x)\psi_n(y), \quad k(x,y) = \sum_{m,n=0}^{\infty} k_{mn}\phi_m(x)\psi_n(y). \tag{2.5}$$

Substituting these expansions into (L), we have

$$\begin{cases} \displaystyle\sum_{m,n=0}^{\infty} \left[\left\{Dl^2(m^2+3n^2)+B\right\}h_{mn} - \left\{\alpha Al^2(m^2+3n^2)+C\right\}k_{mn}\right]\phi_m(x)\psi_n(y) = 0, \\ \displaystyle\sum_{m,n=0}^{\infty} \left[Vh_{mn} - \left\{l^2(m^2+3n^2)+W\right\}k_{mn}\right]\phi_m(x)\psi_n(y) = 0. \end{cases}$$

Thus, if there is a nontrivial solution $(h_{mn},\ k_{mn})$ which satisfies

$$\begin{pmatrix} Dl^2(m^2+3n^2)+B & -\alpha Al^2(m^2+3n^2)-C \\ V & -l^2(m^2+3n^2)-W \end{pmatrix}\begin{pmatrix} h_{mn} \\ k_{mn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.6}$$

for some $(m,\ n,\ \alpha) \in \boldsymbol{N}^2 \times \boldsymbol{R}$, that is, $(m,\ n,\ \alpha)$ satisfies

$$\begin{vmatrix} Dl^2(m^2+3n^2)+B & -\alpha Al^2(m^2+3n^2)-C \\ V & -l^2(m^2+3n^2)-W \end{vmatrix} = 0,$$

then (L) has a nontrivial solution. Therefore, for any fixed $(m,\ n)$, there exists α such that dim Ker $F_{(u,\rho)}(u^*,\ \rho^*,\ \alpha) > 0$ if and only if

$$\alpha = \frac{\{Dl^2(m^2+3n^2)+B\}\{l^2(m^2+3n^2)+W\} - CV}{AVl^2(m^2+3n^2)} \ (=: \alpha(m,n)). \tag{2.7}$$

On the other hand, the map $(m,n) \mapsto \alpha(m,n)$ is not one to one because that $(m,n) \mapsto N(m,n) := m^2 + 3n^2$ is so. For instance, for any $(i,\ j) \in \boldsymbol{N}^2$, it holds that

$$N(i,i+2j) = N(2i+3j,j) = N(i+3j,i+j),$$

that is,

$$\alpha(i,i+2j) = \alpha(2i+3j,j) = \alpha(i+3j,i+j). \tag{2.8}$$

It follows from (2.8) that $\dim \mathrm{Ker}\, F_{(u,\rho)}(u^*,\ \rho^*,\ \alpha^*) = 2$ for $\alpha^* = \alpha(2,\ 0) = \alpha(1,\ 1)$ and $\dim \mathrm{Ker}\, F_{(u,\rho)}(u^*,\ \rho^*,\ \alpha^*) = 3$ for $\alpha^* = \alpha(1,\ 3) = \alpha(4,\ 2) = \alpha(5,\ 1)$.

Let $d(\alpha^*)$ be a number of elements in the set $\{(m,\ n) \in \boldsymbol{N}^2 |\ \alpha^* = \alpha(m,\ n)\}$. By using (2.8), we have

Lemma 2.1. *For any $\alpha = \alpha(m,\ n)$ given in (2.7),* $\dim$ Ker $F_{(u,\rho)}(u^*,\ \rho^*,\ \alpha(m,\ n))$ *is positive. Moreover, if $m+n$ is an even number and* Ker $F_{(u,\rho^*)}(u^*,\ \rho^*,\ \alpha^*) > 1$ *for any $\alpha^* = \alpha(m,\ n)$ give by (2.7), it holds that $d(\alpha^*) \neq 3\ell - 2$ for some integer $\ell \geq 2$. If $i = 0$ or $j = 0$ in (2.8), then $d(\alpha^*) = 3\ell - 1$ and, if $i \neq 0$ and $i \neq 0$, $d(\alpha^*) = 3\ell$ for some integer ℓ.*

Set

$$h(u) = u(1-u), \quad g(\rho) = ae^{\beta\chi(\rho)} + b, \quad z(\rho) = \rho - \frac{1}{2}.$$

Then, (SP) is rewritten as

$$\text{(SSP)} \begin{cases} D\Delta u + \alpha\nabla\{h(u)\nabla\chi(\rho)\} - g(\rho)u + c = 0 & \text{in } \Omega, \\ \Delta\rho + df(u,\rho) - \varepsilon z(\rho) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial\rho}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

§ 2.2. Local bifurcation of stripe and square types

If $\dim \mathrm{Ker}\, F_{(u,\rho)}(u^*,\rho^*,\alpha(m,n)) = 1$, the local bifurcation theorem by Crandall-Rabinowitz [1, Theorem 1.7] is applicable to our problem. In this case, we have

$$\mathrm{Ker}\, F_{(u,\rho)}(u^*,\rho^*,\alpha(m,n)) = \mathrm{span}\left\{\begin{pmatrix} \phi_m(x)\psi_n(y) \\ k_{mn}\phi_m(x)\psi_n(y) \end{pmatrix}\right\}, \tag{2.9}$$

where $\Phi(x,y) = \phi_m(x)\psi_n(y)$. Let h_{mn} be normalized as $h_{mn} = 1$. Then, k_{mn} is given by

$$k_{mn} = \frac{V}{l^2(m^2+3n^2)+W} \ (>0). \tag{2.10}$$

Since ϕ_0 and ψ_0 are constants, it follows from (2.9) that for $m = 0$ or $n = 0$, the bifurcating solution at $\alpha(m,n)$ corresponds to the stripe pattern. On the other hand, if $mn \neq 0$, the bifurcating solution corresponds to the square pattern [7].

Theorem 2.2. *If* $\dim \mathrm{Ker}\ F_{(u,\rho)}(u^*,\ \rho^*,\ \alpha(m,\ n)) = 1$*, there exists a positive constant* δ *such that for any* $s \in (-\delta, \delta)$*, there is a nonconstant solution* $(u(s), \rho(s), \alpha(s))$ *of* (SP) *in the neighborhood of* $(u^*, \rho^*, \alpha(m,n)) \in X \times \boldsymbol{R}$*, which satisfies*

$$\begin{pmatrix} u(s) \\ \rho(s) \\ \alpha(s) \end{pmatrix} = \begin{pmatrix} u^* \\ \rho^* \\ \alpha(m,n) \end{pmatrix} + s \begin{pmatrix} \Phi(x,y) \\ k_{mn}\Phi(x,y) \\ 0 \end{pmatrix} + s^2 \begin{pmatrix} \tilde{u}(s) \\ \tilde{\rho}(s) \\ \tilde{\alpha}(s) \end{pmatrix}, \tag{2.11}$$

where $(\tilde{u}(s), \tilde{\rho}(s), \tilde{\alpha}(s)) \in X \times \boldsymbol{R}$.

Proof. In order to verify all assumptions of the Crandall-Rabiniwitz Theorem [1, Theorem 1.7], in addition to (2.9), we have to show

$$F_{(u,\rho),\alpha}(u^*, \rho^*, \alpha(m,n)) \begin{pmatrix} \Phi(x,y) \\ k_{mn}\Phi(x,y) \end{pmatrix} \notin \mathrm{Range}\, F_{(u,\rho)}(u^*, \rho^*, \alpha(m,n)), \tag{2.12}$$

where $F_{(u,\rho),\alpha}(u^*,\ \rho^*,\ \alpha)$ is the Fréchet derivative of $F_{(u,\rho)}(u,\ \rho,\ \alpha)$ with respect to α at $(u^*,\ \rho^*)$. Since (2.2) implies

$$F_{(u,\rho),\alpha}(u^*, \rho^*, \alpha(m,n)) \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} -A\Delta k \\ 0 \end{pmatrix}, \tag{2.13}$$

it holds that

$$\begin{aligned} &F_{(u,\rho),\alpha}(u^*, \rho^*, \alpha(m,n)) \begin{pmatrix} \Phi(x,y) \\ k_{mn}\Phi(x,y) \end{pmatrix} \\ &= \begin{pmatrix} Ak_{mn}l^2(m^2+3n^2)\Phi(x,y) \\ 0 \end{pmatrix}. \end{aligned} \tag{2.14}$$

On the other hand, the adjoint operator L^*_{mn} of $F_{(u,\rho)}(u^*, \rho^*, \alpha(m,n))$ is given by

$$L^*_{mn} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} D\Delta - B & V \\ -\alpha(m,n)A\Delta + C\,\Delta - W \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix}.$$

As a similar way to Ker $F_{(u,\rho)}(u^*, \rho^*, \alpha)$, there is a positive constant k^*_{mn} such that

$$\mathrm{Ker}\, L^*_{mn} = \mathrm{span} \left\{ \begin{pmatrix} \Phi(x,y) \\ k^*_{mn}\Phi(x,y) \end{pmatrix} \right\}. \tag{2.15}$$

It follows from the Fredholm alternative (e.g., [2, Appendix D]) that

$$\mathrm{Range}\, F_{(u,\rho)}(u^*, \rho^*, \alpha(m,n)) = (\mathrm{Ker}\ L^*_{mn})^{\perp}. \tag{2.16}$$

By using (2.14) and (2.15), it is easy to show that

$$F_{(u,\rho),\alpha}(u^*, \rho^*, \alpha(m,n)) \begin{pmatrix} \Phi(x,y) \\ k_{mn}\Phi(x,y) \end{pmatrix} \notin (\mathrm{Ker}\ L^*_{mn})^{\perp}.$$

Then we get (2.12) by (2.16). Therefore, we can use the Crandall-Rabiniwitz Theorem to get a local branch $(u(s),\ \rho(s),\ \alpha(s))$ $(-\delta < s < \delta)$ of solutions of (SP), which is represented by

$$\begin{pmatrix} u(s) \\ \rho(s) \end{pmatrix} = \begin{pmatrix} u^* \\ \rho^* \end{pmatrix} + s \begin{pmatrix} \Phi(x,y) \\ k_{mn}\Phi(x,y) \end{pmatrix} + s^2 \begin{pmatrix} \tilde{u}(s) \\ \tilde{\rho}(s) \end{pmatrix} \tag{2.17}$$

and

$$\alpha(s) = \alpha(m,\ n) + s\hat{\alpha}(s). \tag{2.18}$$

Finally, we show $\hat{\alpha}(0) = 0$. Note that for $s \in (-\delta, \delta)$,

$$\begin{aligned} \langle \Phi, \tilde{u}(s) \rangle &:= \iint_{\Omega} \Phi(x,y)\, \tilde{u}(x,y,s)\, dxdy = 0, \\ \langle \Phi, \tilde{\rho}(s) \rangle &:= \iint_{\Omega} \Phi(x,y)\, \tilde{\rho}(x,y,s)\, dxdy = 0. \end{aligned} \tag{2.19}$$

For simplicity, we denote

$$k = k_{mn}, \quad \alpha^* = \alpha(m,n).$$

Substituting (2.17) into the first equation of (SSP), we have

$$D\Delta u(x) + \alpha(s)\nabla\{h(u(s))\nabla\chi(\rho(s))\} - g(\rho(s))u(s) + c = 0. \tag{2.20}$$

A second differentiation of (2.20) with respect to s gives

$$\begin{aligned} & D\Delta u''(s) + \alpha''(s)\nabla\{h(u(s))\nabla\chi(\rho(s))\} \\ & + 2\alpha'(s)\nabla\left[h'(u(s))u'(s)\nabla\chi(\rho(s)) + h(u(s))\nabla\{\chi'(\rho(s))\rho'(s)\}\right] \\ & + \alpha(s)\nabla[h''(u(s))u'(s)^2\chi'(\rho(s))\nabla\rho(s) + h'(u(s))u''(s)\chi'(\rho(s))\nabla\rho(s) \\ & \qquad + 2h'(u(s))u'(s)\{\chi''(\rho(s))\rho'(s)\nabla\rho(s) + \chi'(\rho(s))\nabla\rho'(s)\} \\ & \qquad + h(u(s))\{\chi'''(\rho(s))\rho'(s)^2\nabla\rho(s) + 2\chi''(\rho(s))\rho'(s)\nabla\rho'(s) \\ & \qquad + \chi''(\rho(s))\rho''(s)\nabla\rho(s) + \chi'(\rho(s))\nabla\rho''(s)\}] \\ & - g''(\rho(s))\rho'(s)^2u(s) - g'(\rho(s))\rho''(s)u(s) \\ & - 2g'(\rho(s))\rho'(s)u'(s) - g(\rho(s))u''(s) = 0. \end{aligned} \tag{2.21}$$

From (2.17) and (2.18), $(u(s),\ \rho(s),\ \alpha(s))$ satisfies

$$\begin{aligned} & (u(0), u'(0), u''(0)) = (u^*, \Phi, 2\tilde{u}(0)),\ (\rho(0), \rho'(0), \rho''(0)) = (\rho^*, k\Phi, 2\tilde{\rho}(0)), \\ & (\alpha(0),\ \alpha'(0)) = (\alpha^*,\ \hat{\alpha}(0)). \end{aligned} \tag{2.22}$$

Substituting $s=0$ into (2.21) and using (2.22) we get

$$
\begin{aligned}
&2D\Delta\tilde{u}(0)+2kh(u^*)\chi'(\rho^*)\hat{\alpha}(0)\Delta\Phi \\
&+2\alpha^*\{kh'(u^*)\chi'(\rho^*)+k^2h(u^*)\chi''(\rho^*)\}\nabla(\Phi\nabla\Phi)+2\alpha^*h(u^*)\chi'(\rho^*)\Delta\tilde{\rho}(0) \\
&-k^2u^*g''(\rho^*)\Phi^2-2u^*g'(\rho^*)\tilde{\rho}(0)-2kg'(\rho^*)\Phi^2-2g(\rho^*)\tilde{u}(0)=0.
\end{aligned}
\tag{2.23}
$$

By (2.19), the inner product of (2.23) and Φ implies

$$
\begin{aligned}
&2\lambda kh(u^*)\chi'(\rho^*)\|\Phi\|_2^2\hat{\alpha}(0) \\
=&2\alpha^*\{kh'(u^*)\chi'(\rho^*)+k^2h(u^*)\chi''(\rho^*)\}\{\langle\Phi,|\nabla\Phi|^2\rangle-\lambda\langle\Phi,\Phi^2\rangle\} \\
&-(k^2u^*g''(\rho^*)+2kg(\rho^*))\langle\Phi,\Phi^2\rangle,
\end{aligned}
\tag{2.24}
$$

where λ is a positive constant such that $-\Delta\Phi=\lambda\Phi$ in Ω.
Note that $\langle\Phi,\Phi^2\rangle=0$ and $\langle\Phi,|\nabla\Phi|^2\rangle=0$. Therefore, (2.24) implies $\hat{\alpha}(0)=0$, that is, there exists a function $\tilde{\alpha}(s)$ such that $\alpha(s)=\alpha(m,\ n)+s^2\tilde{\alpha}(s)$. Thus, the proof of this theorem is completed. □

§ 2.3. Local bifurcation of hexagonal type

Here, we only consider the case of $\dim \mathrm{Ker}\, F_{(u,\rho)}(u^*,\ \rho^*,\ \alpha(m,n))=2$. Especially, we treat $\alpha(2,0)=\alpha(1,1)$ from (2.8), which implies

$$
\begin{aligned}
&\mathrm{Ker}\, F_{(u,\rho)}(u^*,\rho^*,\alpha(2,0)) = \mathrm{Ker}\, F_{(u,\rho)}(u^*,\rho^*,\alpha(1,1)) \\
&=\mathrm{span}\left\{\begin{pmatrix}\phi_2(x)\,\psi_0(y)\\ k_{20}\,\phi_2(x)\,\psi_0(y)\end{pmatrix},\begin{pmatrix}\phi_1(x)\,\psi_1(y)\\ k_{11}\,\phi_1(x)\,\psi_1(y)\end{pmatrix}\right\}.
\end{aligned}
\tag{2.25}
$$

Since we can not directly apply the Crandall-Rabinowitz Theorem to this case because of

$$\dim \mathrm{Ker}\, F_{(u,\rho)}(u^*,\rho^*,\alpha(m,n))=2\neq 1,$$

we exchange the considered function space from $H^2_\nu(\Omega)$. Following the approach in Nishida et al. [7], we introduce the subspace $H^2_{\mathrm{hexa}}(\Omega)$ in $H^2(\Omega)$:

(2.26)

$$
\begin{aligned}
H^2_{\mathrm{hexa}}=\Bigg\{v(x,y)=&\sum_{m+n=\mathrm{even}}^{\infty}\beta_{mn}\Bigg(\phi_m(x)\psi_n(y)+\cos\frac{l(m-3n)x}{2}\cos\frac{\sqrt{3}l(m+n)y}{2} \\
&+\cos\frac{l(m+3n)x}{2}\cos\frac{\sqrt{3}l(m-n)y}{2}\Bigg):\sum_{m+n=\mathrm{even}}^{\infty}l^4(m^2+3n^2)^2\beta_{mn}^2<\infty\Bigg\}.
\end{aligned}
$$

In this case, a function belonging to $H^2_{\mathrm{hexa}}(\Omega)$ has $2\pi/3$-invariance around the center of the domain with respect to the rotation. In order to obtain the solution corresponding to the hexagonal pattern, it is enough to show the existence of the solution in $H^2_{\text{hexa}}(\Omega)$. It follows from (2.26) that

$$\begin{aligned}&\phi_2(x)\psi_0(y)+\cos(lx)\cos(\sqrt{3}ly)+\cos(lx)\cos(\sqrt{3}ly)\\ =&\,\phi_2(x)\psi_0(y)+2\phi_1(x)\psi_1(y)\end{aligned}$$

for $(m,n)=(2,0)$. Consequently, we take the coefficients $\{\beta_{mn}\,|\,m+n=\text{even}\}$ in (2.26) as

$$\beta_{mn}=\begin{cases}1 & \text{if}\quad (m,n)=(2,0),\\ 0 & \text{otherwise},\end{cases}$$

that is, $\phi_2(x)\psi_0(y)+2\phi_1(x)\psi_1(y)\in H^2_{\text{hexa}}$. Note that

$$\dim\operatorname{Ker}F_{(u,\rho)}(u^*,\rho^*,\alpha(1,1))=\dim\operatorname{Ker}F_{(u,\rho)}(u^*,\rho^*,\alpha(2,0))=2$$

in X from (2.25).

Let $\hat{\Phi}(x,y)=\phi_2(x)\psi_0(y)+2\phi_1(x)\psi_1(y)$. In order to apply the Crandall-Rabinowitz Theorem, we restrict the domain of F and rewrite it as $\tilde{F}$, that is,

$$\tilde{F}:H^2_{\text{hexa}}\times H^2_{\text{hexa}}\times\boldsymbol{R}\to Y,$$

$$\begin{aligned}\operatorname{Ker}\tilde{F}_{(u,\rho)}(u^*,\rho^*,\alpha(1,1))=&\;\operatorname{Ker}\tilde{F}_{(u,\rho)}(u^*,\rho^*,\alpha(2,0))\\ =&\;\operatorname{span}\left\{\begin{pmatrix}\hat{\Phi}(x,y)\\ k_{20}\hat{\Phi}(x,y)\end{pmatrix}\right\}\end{aligned}\tag{2.27}$$

and

$$\dim\operatorname{Ker}\tilde{F}_{(u,\rho)}(u^*,\rho^*,\alpha(1,1))=\dim\operatorname{Ker}\tilde{F}_{(u,\rho)}(u^*,\rho^*,\alpha(2,0))=1.$$

Theorem 2.3. *For $\alpha=\alpha(2,0)(=\alpha(1,1))$, there exists a positive constant δ such that for $s\in(-\delta,\delta)$ there is a nonconstant solution $(u(s),\ \rho(s),\ \alpha(s))$ of* (SP) *in the neighborhood of $(u^*,\rho^*,\alpha(2,0))\in H^2_{\text{hexa}}\times H^2_{\text{hexa}}\times\boldsymbol{R}$, which satisfies*

$$\begin{pmatrix}u(s)\\ \rho(s)\end{pmatrix}=\begin{pmatrix}u^*\\ \rho^*\end{pmatrix}+s\begin{pmatrix}\hat{\Phi}(x,y)\\ k_{20}\hat{\Phi}(x,y)\end{pmatrix}+s^2\begin{pmatrix}\tilde{u}(s)\\ \tilde{\rho}(s)\end{pmatrix},\tag{2.28}$$

where $(\tilde{u}(s),\tilde{\rho}(s))\in H^2_{\text{hexa}}\times H^2_{\text{hexa}}$.

Proof. Owing to (2.27), it suffices to show that

$$\tilde{F}_{(u,\rho),\alpha}(u^*,\rho^*,\alpha(2,0))\begin{pmatrix}\hat{\Phi}(x,y)\\ k_{20}\hat{\Phi}(x,y)\end{pmatrix}\notin\operatorname{Range}\tilde{F}_{(u,\rho)}(u^*,\rho^*,\alpha(2,0)).\tag{2.29}$$

From (2.13), we have

$$\tilde{F}_{(u,\rho),\alpha}(u^*,\rho^*,\alpha(2,0))\begin{pmatrix}\hat{\Phi}(x,y)\\ k_{20}\hat{\Phi}(x,y)\end{pmatrix}=\begin{pmatrix}4Ak_{20}\hat{\Phi}(x,y)\\ 0\end{pmatrix}. \tag{2.30}$$

On the other hand, the adjoint operator $\tilde{L}^*$ of $\tilde{F}_{(u,\rho)}(u^*,\rho^*,\alpha(2,0))$ is given by

$$\tilde{L}^*\begin{pmatrix}h\\ k\end{pmatrix}=\begin{pmatrix}D\Delta-B & V\\ -\alpha(2,0)A\Delta+C\,\Delta-W\end{pmatrix}\begin{pmatrix}h\\ k\end{pmatrix}.$$

As a similar way to $\tilde{L}^*_{mn}$ in Theorem 2.2, there is a positive constant $k^*>0$ such that

$$\text{Ker }\tilde{L}^*_{20}=\text{span}\left\{\begin{pmatrix}\hat{\Phi}(x,y)\\ k^*\hat{\Phi}(x,y)\end{pmatrix}\right\}. \tag{2.31}$$

By (2.30) and (2.31), it is easy to show that

$$\tilde{F}_{(u,\rho),\alpha}(u^*,\rho^*,\alpha(2,0))\begin{pmatrix}\hat{\Phi}(x,y)\\ k^*\hat{\Phi}(x,y)\end{pmatrix}\notin(\text{Ker }L^*_{20})^{\perp}. \tag{2.32}$$

Combining Fredholm alternative with (2.32) yields (2.29).

□

§ 3. Direction of the bifurcation

In order to show the type of the bifurcation, we show the expansion of $\alpha(s)$ with respect to s. Let $\alpha(s)=\alpha(m,n)+s\hat{\alpha}(s)$. If $\hat{\alpha}(0)\neq 0$, the bifurcation is **transcritical**. When $\hat{\alpha}(0)=0$, the bifurcation is **supercritical** if $\frac{d\hat{\alpha}}{ds}(0)>0$ and **subcritical** if $\frac{d\hat{\alpha}}{ds}(0)<0$.

§ 3.1. Transcritical bifurcation of hexagonal type

Theorem 3.1. *Let $\alpha(s)=\alpha(2,0)+s\hat{\alpha}(s)$ in Theorem 2.3. Then, $\hat{\alpha}(0)$ satisfies*

$$\begin{aligned}&8l^2kh(u^*)\chi'(\rho^*)\hat{\alpha}(0)\\ &=-4\alpha^*l^2\{kh(u^*)\chi(\rho^*)+k^2h(u^*)\chi''(\rho^*)\}-k^2u^*g''(\rho^*)-2kg(\rho^*).\end{aligned} \tag{3.1}$$

Proof. By the same process to obtain (2.24), we get

$$\begin{aligned}&2\lambda kh(u^*)\chi'(\rho^*)\|\hat{\Phi}\|_2^2\hat{\alpha}(0)\\ =&2\alpha^*\{kh'(u^*)\chi'(\rho^*)+k^2h(u^*)\chi''(\rho^*)\}\{\langle\hat{\Phi},|\nabla\hat{\Phi}|^2\rangle-\lambda\langle\hat{\Phi},\hat{\Phi}^2\rangle\}\\ &-(k^2u^*g''(\rho^*)+2kg(\rho^*))\langle\hat{\Phi},\hat{\Phi}^2\rangle.\end{aligned} \tag{3.2}$$

By (3.2) and

$$\langle \hat{\Phi}, \hat{\Phi}^2 \rangle = \frac{\sqrt{3}\pi^2}{2l^2}, \quad \langle \hat{\Phi}, |\nabla \hat{\Phi}|^2 \rangle, = \sqrt{3}\pi^2, \quad \|\hat{\Phi}\|_2^2 = \frac{\sqrt{3}\pi^2}{2l^2}, \quad \lambda = 4l^2, \tag{3.3}$$

(3.1) is obtained. □

Lemma 1.1 and Theorem 3.1 imply that

Remark. *Since* $h(u^*) > 0, \chi(\rho^*) < 0, \chi'(\rho^*) < 0, \chi''(\rho^*) > 0, g(\rho^*) > 0$ *and* $g''(\rho^*) > 0$, *the bifurcation of hexagonal type is generically transcritical.*

On the other hand, it follows from (2.8) that for $\alpha(1,\ 3) = \alpha(4,\ 2) = \alpha(5,\ 1)$, dim Ker $\tilde{F}_{(u,\rho)}(u^*,\ \rho^*,\ \alpha(1,\ 3)) = 1$ and

$$\text{Ker } \tilde{F}_{(u,\rho)}(u^*,\ \rho^*,\ \alpha(1,\ 3)) = \left\{ \begin{pmatrix} \Phi^*(x,\ y) \\ k_{13}\Phi^*(x,\ y) \end{pmatrix} \right\},$$

where $\Phi^*(x,\ y) = \phi_1(x)\psi_3(y) + \phi_4\psi_2(y) + \phi_5(x)\psi_1(y)$ and k_{13} given by (2.10).

Remark. *Theorems* 2.3 *and* 3.1 *hold for* $\alpha(1,\ 3) = \alpha(4,\ 2) = \alpha(5,\ 1)$.

§ 3.2. Pitchfork bifurcation of stripe and square types

In view of (2.11), we remember that the bifurcation of stripe or square pattern is pitchfork type. Then we need to investigate the sign of $\tilde{\alpha}(0)$ in (2.11) to know the direction of the pitchfork bifurcation. Here, $\alpha(m,n)$ and k_{mn} stand in (2.7) and (2.10). We obtain the exact expression of $\tilde{\alpha}(0)$ from the following Theorem 3.2 and Lemma 3.3.

Theorem 3.2. *Let $\tilde{\alpha}(s)$ be the function defined in (2.11). Then,*

$$\begin{aligned}
&6\lambda k h(u^*)\chi'(\rho^*)\|\Phi\|_2^2\,\tilde{\alpha}(0) \\
=& -6kg'(\rho^*)P - 6\alpha^* k h'(u^*)\chi'(\rho^*)Q \\
&- 6\{\lambda\alpha^* h'(u^*)\chi'(\rho^*) + \lambda\alpha^* k h(u^*)\chi''(\rho^*) + ku^* g''(\rho^*) + g'(\rho^*)\}R \\
&+ 6\alpha^* h'(u^*)\chi'(\rho^*)S \\
&- 3\alpha^* k\{h''(u^*)\chi'(\rho^*) - kh'(u^*)\chi''(\rho^*)\}\langle \Phi^2, |\nabla\Phi|^2\rangle \\
&- \{3\lambda\alpha^* k^2 h'(u^*)\chi''(\rho^*) + \lambda\alpha^* k^3 h(u^*)\chi'''(\rho^*) + k^3 u^* g'''(\rho^*) + 3k^2 g''(\rho^*)\}\|\Phi\|_4^4.
\end{aligned}$$

where

$$(P,Q,R,S) := (\langle \tilde{u}(0), \Phi^2\rangle, \langle \tilde{u}(0), |\nabla\Phi|^2\rangle, \langle \tilde{\rho}(0), \Phi^2\rangle, \langle \tilde{\rho}(0), |\nabla\Phi|^2\rangle). \tag{3.4}$$

Lemma 3.3. *The above (P, Q, R, S) satisfies*

$$\begin{pmatrix} L & 4D & M & N \\ 4\lambda^2 D & L & \lambda^2 N & M \\ 2df_u(u^*,\rho^*) & 0 & K & 4 \\ 0 & 2df_u(u^*,\rho^*) & 4\lambda^2 & K \end{pmatrix} \begin{pmatrix} P \\ Q \\ R \\ S \end{pmatrix} = \begin{pmatrix} H\langle \Phi^2, |\nabla\Phi|^2\rangle + J\|\Phi\|_4^4 \\ (\lambda H + J)\langle \Phi^2, |\nabla\Phi|^2\rangle \\ G\|\Phi\|_4^4 \\ G\langle \Phi^2, |\nabla\Phi|^2\rangle \end{pmatrix}, \tag{3.5}$$

where $L = -4\lambda D - 2g(\rho^*)$, $M = -4\lambda\alpha^* h(u^*)\chi'(\rho^*) - 2u^* g'(\rho^*)$, $N = 4\alpha^* h(u^*)\chi'(\rho^*)$, $K = 2\{df_\rho(u^*,\rho^*) - \varepsilon - 2\lambda\}$, $G = -d\{f_{uu}(u^*,\rho^*) + 2f_{u\rho}(u^*,\rho^*)k + f_{\rho\rho}(u^*,\rho^*)k^2\}$, $H = 4\alpha^*\{kh'(u^*)\chi'(\rho^*) + k^2 h(u^*)\chi''(\rho^*)\}$ and $J = k^2 u^* g''(\rho^*) + 2kg'(\rho^*)$.

Proof. It follows from (2.23) and $\hat{\alpha}(0) = 0$ that

$$\begin{aligned} &2D\Delta\tilde{u}(0) + 2\alpha^*\{kh'(u^*)\chi'(\rho^*) + k^2 h(u^*)\chi''(\rho^*)\}\nabla(\Phi\nabla\Phi) \\ &+ 2\alpha^* h(u^*)\chi'(\rho^*)\Delta\tilde{\rho}(0) \\ &- k^2 u^* g''(\rho^*)\Phi^2 - 2u^* g'(\rho^*)\tilde{\rho}(0) - 2kg'(\rho^*)\Phi^2 - 2g(\rho^*)\tilde{u}(0) = 0. \end{aligned} \tag{3.6}$$

The inner product of (3.6) and Φ^2 implies

$$\begin{aligned} &2D\langle\Delta\tilde{u}(0), \Phi^2\rangle + 2\alpha^*\{kh'(u^*)\chi'(\rho^*) + k^2 h(u^*)\chi''(\rho^*)\}\langle\nabla(\Phi\nabla\Phi), \Phi^2\rangle \\ &+ 2\alpha^* h(u^*)\chi'(\rho^*)\langle\Delta\tilde{\rho}(0), \Phi^2\rangle \\ &- k^2 u^* g''(\rho^*)\|\Phi\|_4^4 - 2u^* g'(\rho^*)\langle\tilde{\rho}(0), \Phi^2\rangle - 2kg'(\rho^*)\|\Phi\|_4^4 - 2g(\rho^*)\langle\tilde{u}(0), \Phi^2\rangle = 0. \end{aligned}$$

Since

$$\langle\Delta F, \Phi^2\rangle = 2\langle F, |\nabla\Phi|^2\rangle - 2\lambda\langle F, \Phi^2\rangle, \quad \langle\nabla(\Phi\nabla\Phi), \Phi^2\rangle = -2\langle\Phi^2, |\nabla\Phi|^2\rangle, \tag{3.7}$$

it holds that

$$\begin{aligned} &(-4\lambda D - 2g(\rho^*))P + 4DQ \\ &- (4\lambda\alpha^* h(u^*)\chi'(\rho^*) + 2u^* g'(\rho^*))R + 4\alpha^* h(u^*)\chi'(\rho^*)S \\ =&4\alpha^*\{kh'(u^*)\chi'(\rho^*) + k^2 h(u^*)\chi''(\rho^*)\}\langle\Phi^2, |\nabla\Phi|^2\rangle \\ &+ \{k^2 u^* g''(\rho^*) + 2kg'(\rho^*)\}\|\Phi\|_4^4. \end{aligned} \tag{3.8}$$

On the other hand, the inner product of (3.6) and $|\nabla\Phi|^2$ implies

$$\begin{aligned} &2D\langle\Delta\tilde{u}(0), |\nabla\Phi|^2\rangle + 2\alpha^*\{kh'(u^*)\chi'(\rho^*) + k^2 h(u^*)\chi''(\rho^*)\}\langle\nabla(\Phi\nabla\Phi), |\nabla\Phi|^2\rangle \\ &+ 2\alpha^* h(u^*)\chi'(\rho^*)\langle\Delta\tilde{\rho}(0), |\nabla\Phi|^2\rangle - k^2 u^* g''(\rho^*)\langle\Phi^2, |\nabla\Phi|^2\rangle \\ &- 2u^* g'(\rho^*)\langle\tilde{\rho}(0), |\nabla\Phi|^2\rangle - 2kg'(\rho^*)\langle\Phi^2, |\nabla\Phi|^2\rangle - 2g(\rho^*)\langle\tilde{u}(0), |\nabla\Phi|^2\rangle = 0. \end{aligned}$$

Since

$$\text{(3.9)}\quad \langle \Delta F, |\nabla \Phi|^2\rangle = 2\lambda^2\langle F, \Phi^2\rangle - 2\lambda\langle F, |\nabla\Phi|^2\rangle, \quad \langle \nabla(\Phi\nabla\Phi), |\nabla\Phi|^2\rangle = 2\lambda\langle \Phi^2, |\nabla\Phi|^2\rangle,$$

it holds that

$$\begin{aligned}
&4\lambda^2 DP - \{4\lambda D + 2g(\rho^*)\}Q \\
&\quad + 4\lambda^2\alpha^* h(u^*)\chi'(\rho^*)R - \{4\lambda\alpha^* h(u^*)\chi'(\rho^*) + 2u^* g'(\rho^*)\}S \\
=&4\lambda\alpha^*\{kh'(u^*)\chi'(\rho^*) + k^2 h(u^*)\chi''(\rho^*)\}\langle \Phi^2, |\nabla\Phi|^2\rangle \\
&\quad + \{k^2 u^* g''(\rho^*) + 2kg'(\rho^*)\}\langle \Phi^2, |\nabla\Phi|^2\rangle.
\end{aligned}\tag{3.10}$$

Next, substituting (2.11) into the second equation of (SSP), we have

$$\Delta\rho(s) + df(u(s), \rho(s)) - \varepsilon z(\rho(s)) = 0.$$

A second differentiation of this equation with respect to s gives

$$\begin{aligned}
&\Delta\rho''(s) + df_{uu}(u(s),\rho(s))u'(s)^2 \\
&+ 2df_{u\rho}(u(s),\rho(s))u'(s)\rho'(s) + df_{\rho\rho}(u(s),\rho(s))\rho'(s)^2 \\
&+ df_u(u(s),\rho(s))u''(s) + df_\rho(u(s),\rho(s))\rho''(s) - \varepsilon\rho''(s) = 0
\end{aligned}\tag{3.11}$$

by $z(\rho) = \rho - \frac{1}{2}$. Substituting $s = 0$ into (3.11) and (2.22), we get

$$\begin{aligned}
&2\Delta\tilde\rho(0) + df_{uu}(u^*,\rho^*)\Phi^2 + 2df_{u\rho}(u^*,\rho^*)k\Phi^2 + df_{\rho\rho}(u^*,\rho^*)k^2\Phi^2 \\
&+ 2df_u(u^*,\rho^*)\tilde u(0) + 2df_\rho(u^*,\rho^*)\tilde\rho(0) - 2\varepsilon\tilde\rho(0) = 0.
\end{aligned}\tag{3.12}$$

The inner product of (3.12) and Φ^2 implies

$$\begin{aligned}
&2df_u(u^*,\rho^*)P + 2\{df_\rho(u^*,\rho^*) - \varepsilon - 2\lambda\}R + 4S \\
=& - d\{f_{uu}(u^*,\rho^*) + 2f_{u\rho}(u^*,\rho^*)k + f_{\rho\rho}(u^*,\rho^*)k^2\}\|\Phi\|_4^4
\end{aligned}\tag{3.13}$$

by (3.7). Similarly, computing the inner product of (3.12) and $|\nabla\Phi|^2$ and using (3.9), we obtain

$$\begin{aligned}
&2df_u(u^*,\rho^*)Q + 4\lambda^2 R + 2\{df_\rho(u^*,\rho^*) - \varepsilon - 2\lambda\}R \\
=& - d\{f_{uu}(u^*,\rho^*) + 2f_{u\rho}(u^*,\rho^*)k + f_{\rho\rho}(u^*,\rho^*)k^2\}\langle \Phi^2, |\nabla\Phi|^2\rangle.
\end{aligned}\tag{3.14}$$

Therefore, (3.5) is shown by (3.8), (3.10), (3.13) and (3.14). □

Proof of theorem 3.1.

The third differentiation of (2.20) with respect to s implies

$$\begin{aligned}
& D\Delta u'''(s) + \alpha'''(s)\nabla\{h(u(s))\nabla\chi(\rho(s))\} \\
& + 3\alpha''(s)\nabla\left[h'(u(s))u'(s)\nabla\chi(\rho(s)) + h(u(s))\nabla\{\chi'(\rho(s))\rho'(s)\}\right] \\
& + 3\alpha'(s)\nabla[h''(u(s))u'(s)^2\nabla\chi(\rho(s)) + h(u(s))u''(s)\nabla\chi(\rho(s)) \\
& \qquad + 2h'(u(s))u'(s)\nabla\{\chi'(\rho(s))\rho'(s)\} \\
& \qquad + h(u(s))\nabla\{\chi''(\rho(s))\rho'(s)^2 + \chi'(\rho(s))\rho''(s)\}] \\
& + \alpha(s)\nabla[h'''(u(s))u'(s)^3\nabla\chi(\rho(s)) + 3h''(u(s))u'(s)u''(s)\nabla\chi(\rho(s)) \\
& \qquad + 3h''(u(s))u'(s)^2\nabla\{\chi'(\rho(s))\rho'(s)\} + h'(u(s))u'''(s)\nabla\chi(\rho(s)) \\
& \qquad + 3h'(u(s))u''(s)\nabla\{\chi'(\rho(s))\rho'(s)\} \\
& \qquad + 3h'(u(s))u'(s)\nabla\{\chi''(\rho(s))\rho'(s)^2 + \chi'(\rho(s))\rho''(s)\} \\
& \qquad + h(u(s))\nabla\{\chi'''(\rho(s))\rho'(s)^3 + 3\chi''(\rho(s))\rho'(s)\rho''(s) + \chi'(\rho(s))\rho'''(s)\}] \\
& - g'''(\rho(s))\rho'(s)^3u(s) - 3g''(\rho(s))\rho'(s)\rho''(s)u(s) - 3g''(\rho(s))\rho'(s)^2u'(s) \\
& - g'(\rho(s))\rho'''(s)u(s) - 3g'(\rho(s))\rho''(s)u'(s) - 3g'(\rho(s))\rho'(s)u''(s) \\
& - g(\rho(s))u'''(s) = 0.
\end{aligned} \tag{3.15}$$

It follows from (2.11) that

$$(u'''(0), \rho'''(0), \alpha'''(0)) = 6(\tilde{u}'(0), \tilde{\rho}'(0), \tilde{\alpha}'(0)).$$

Substituting $s = 0$ into (3.15), we have

$$\begin{aligned}
& 6\Delta\tilde{u}'(0) + 6kh(u^*)\chi'(\rho^*)\tilde{\alpha}(0)\Delta\Phi \\
& + 3\alpha^*kh''(u^*)\chi'(\rho^*)\nabla(\Phi^2\nabla\Phi) + 6\alpha^*kh'(u^*)\chi'(\rho^*)\nabla\{\tilde{u}(0)\nabla\Phi\} \\
& + 3\alpha^*k^2h'(u^*)\chi''(\rho^*)\nabla(\Phi\nabla\Phi^2) + 6\alpha^*h'(u^*)\chi'(\rho^*)\nabla\{\Phi\nabla\tilde{\rho}(0)\} \\
& + \alpha^*k^3h(u^*)\chi'''(\rho^*)\Delta\Phi^3 + 6\alpha^*kh(u^*)\chi''(\rho^*)\Delta\{\Phi\tilde{\rho}(0)\} \\
& + 6\alpha^*h(u^*)\chi'(\rho^*)\Delta\tilde{\rho}'(0) - k^3u^*g'''(\rho^*)\Phi^3 - 6ku^*g''(\rho^*)\tilde{\rho}(0)\Phi - 3k^2g''(\rho^*)\Phi^3 \\
& - 6u^*g'(\rho^*)\tilde{\rho}'(0) - 6g'(\rho^*)\tilde{\rho}(0)\Phi - 6kg'(\rho^*)\tilde{u}(0)\Phi - 6g(\rho^*)\tilde{u}'(0) = 0.
\end{aligned} \tag{3.16}$$

By the inner product of (3.16) and Φ and (2.19), it holds that

$$\begin{aligned}
& - 6kh(u^*)\chi'(\rho^*)\tilde{\alpha}(0)\langle\Delta\Phi, \Phi\rangle \\
={}& 3\alpha^*kh''(u^*)\chi'(\rho^*)\langle\nabla(\Phi^2\nabla\Phi), \Phi\rangle + 6\alpha^*kh'(u^*)\chi'(\rho^*)\langle\nabla\{\tilde{u}(0)\nabla\Phi\}, \Phi\rangle \\
& + 3\alpha^*k^2h'(u^*)\chi''(\rho^*)\langle\nabla(\Phi\nabla\Phi^2), \Phi\rangle + 6\alpha^*h'(u^*)\chi'(\rho^*)\langle\nabla\{\Phi\nabla\tilde{\rho}(0)\}, \Phi\rangle \\
& + \alpha^*k^3h(u^*)\chi'''(\rho^*)\langle\Delta\Phi^3, \Phi\rangle + 6\alpha^*kh(u^*)\chi''(\rho^*)\langle\Delta\{\Phi\tilde{\rho}(0)\}, \Phi\rangle \\
& - k^3u^*g'''(\rho^*)\|\Phi\|_4^4 - 6ku^*g''(\rho^*)\langle\tilde{\rho}(0), \Phi^2\rangle - 3k^2g''(\rho^*)\|\Phi\|_4^4 \\
& - 6g'(\rho^*)\langle\tilde{\rho}(0), \Phi^2\rangle - 6kg'(\rho^*)\langle\tilde{u}(0), \Phi^2\rangle.
\end{aligned} \tag{3.17}$$

Also, we note

$$\begin{aligned}
&\langle \nabla\{\tilde{u}(0)\nabla\varPhi\},\varPhi\rangle = -\langle \tilde{u}(0), |\nabla\varPhi|^2\rangle = -Q,\\
&\langle \nabla\{\varPhi\nabla\tilde{\rho}(0)\},\varPhi\rangle = \langle \tilde{\rho}(0), |\nabla\varPhi|^2\rangle - \lambda\langle\tilde{\rho}(0),\varPhi^2\rangle = S-\lambda R,\\
&\langle \Delta\{\varPhi\tilde{\rho}(0)\},\varPhi\rangle = -\lambda\langle\tilde{\rho}(0),\varPhi^2\rangle = -\lambda R.
\end{aligned}$$

Thus, (3.17) is rewritten as

$$\begin{aligned}
&6\lambda k h(u^*)\chi'(\rho^*)\|\varPhi\|_2^2\tilde{\alpha}(0)\\
=&-6kg'(\rho^*)P - 6\alpha^* k h'(u^*)\chi'(\rho^*)Q\\
&-6\{\lambda\alpha^* h'(u^*)\chi'(\rho^*) + \lambda\alpha^* k h(u^*)\chi''(\rho^*) + k u^* g''(\rho^*) + g'(\rho^*)\}R\\
&+6\alpha^* h'(u^*)\chi'(\rho^*)S\\
&+3\alpha^* k h''(u^*)\chi'(\rho^*)\langle\nabla(\varPhi^2\nabla\varPhi),\varPhi\rangle + 3\alpha^* k^2 h'(u^*)\chi''(\rho^*)\langle\nabla(\varPhi\nabla\varPhi^2),\varPhi\rangle\\
&+\alpha^* k^3 h(u^*)\chi'''(\rho^*)\langle\Delta\varPhi^3,\varPhi\rangle - k^3 u^* g'''(\rho^*)\|\varPhi\|_4^4 - 3k^2 g''(\rho^*)\|\varPhi\|_4^4.
\end{aligned} \tag{3.18}$$

By using

$$\langle\nabla(\varPhi^2\nabla\varPhi),\varPhi\rangle = -\langle\varPhi^2,|\nabla\varPhi|^2\rangle,\quad \langle\nabla(\varPhi\nabla\varPhi^2),\varPhi\rangle = -\langle\varPhi^2,|\nabla\varPhi|^2\rangle - \lambda\|\varPhi\|_4^4$$

and (3.18), the formulation of $\tilde{\alpha}(0)$ in this theorem is obtained.

□

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