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The geometric Iwasawa conjecture from a viewpoint of the arithmetic topology

By

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Abstract

For a local system on a complete hyperbolic threefold which is compact or of finite volume, the twisted Alexander polynomial and the Ruelle-Selberg L-functions are defined. Under a cohomological assumption, we have shown that their order and leading constants of the Taylor expansion at the origin are almost identical. These results may be considered as a solution of a geometric analogue of the Iwasawa conjecture in the algebraic number theory. We will interpret these results from a viewpoint of the arithmetic topology.

§1. Introduction

In the conference we have talked a geometric analog of the Iwasawa conjecture. Let $X$ be a finite CW-complex of dimension three with a fixed base point $x_0$ such that there is a surjective homomorphism

$$
\pi_1(X, x_0) \rightarrow \mathbb{Z}
$$

and $\rho$ a unitary representation of the fundamental group. The kernel of $\epsilon$ determines an infinite cyclic covering $X_\infty$ of $X$ and $H.(X_\infty, \mathbb{C}), H.(X_\infty, \rho), H'(X_\infty, \mathbb{C})$ and $H'(X_\infty, \rho)$ become $\Lambda$-modules. Here $\Lambda$ is the group ring $\mathbb{C}[\mathbb{Z}]$. Note that it is isomorphic to the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$ in a non-canonical way.

Suppose that all the dimensions of $H.(X_\infty, \mathbb{C})$ and $H.(X_\infty, \rho)$ are finite. Then due to Milnor [4] it is known that $H^1(X_\infty, \rho)$ becomes a finite dimensional complex vector space. The twisted Alexander polynomial $A^1_\rho(t)$ is defined to be the characteristic polynomial $\det[t-\tau^*]$ of the action of a generator $\tau \in \mathbb{Z}$ on $H^1(X_\infty, \rho)$, which generates

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the characteristic ideal $\mathrm{Char}_\Lambda(H^1(X_{\infty}, \rho))$. It will play the same role as the Iwasawa polynomial in the original Iwasawa theory.

In order to introduce a counterpart of the $p$-adic zeta function, we assume that $X$ admits a hyperbolic structure of a finite volume. Then the Ruelle $L$-function $R_\rho(s)$ is defined to be

$$R_\rho(s) = \prod_{\gamma} P_\gamma(s)^{-1}, \quad P_\gamma(s) = \det[1 - \rho(\gamma)e^{-sl(\gamma)}],$$

where $\gamma$ runs through the set of prime closed geodesics of $X$ and $l(\gamma)$ denotes its length. It absolutely convergents if $\Re s$ is sufficiently large.

When $X$ is compact, Fried has shown it is meromorphically continued on the whole plane. Moreover if $H^0(X, \rho)$ vanishes, he has also shown the order of $R_\rho(s)$ at the origin is $2 \dim H^1(X, \rho)$ ([1]).

Suppose $H^0(X_{\infty}, \rho)$ vanishes. In [6], using Fried’s result we have shown that if the action of $\tau^*$ on $H^1(X_{\infty}, \rho)$ is semisimple the identity

$$(1) \quad 2\text{ord}_{t=1} A^1_\rho(t) = \text{ord}_{s=0} R_\rho(s),$$

holds, which may be considered as a geometric analog of the Iwasawa main conjecture. But since the compact case corresponds to a Galois representation which is unramified everywhere, it is desirable to generalize the result to a non-compact case. In fact when $X$ has only one cusp and when $\rho$ is a unitary character, we have shown the same identity holds if a certain condition of $\rho$ at the cusp is satisfied [7].

In this report we will interpret (1) from a viewpoint of the arithmetic topology. Let $\gamma$ be a prime closed geodesic of $X$ such that $\epsilon(\gamma) \neq 0$. (Such a geodesic will be mentioned as $\epsilon$-inert.) Then it is easy to to see that its inverse image in $X_{\infty}$ becomes the infinite cyclic covering of $\gamma$. Let $X_{\infty}(\gamma)$ be its complement. Then we will show an identity of fractional ideals of $\Lambda$:

$$\mathrm{Char}_\Lambda(H^1(X_{\infty}(\gamma), \rho)) \cdot (\det[1 - \rho(\gamma)t^{-|\epsilon(\gamma)|}])^{-1} = \mathrm{Char}_\Lambda(H^1(X_{\infty}, \rho)).$$

This is a partial solution of the geometric Iwasawa conjecture. In fact substituting $t = \exp\left(\frac{l(\gamma)}{|\epsilon(\gamma)|} s\right)$ it shows us that $P_\gamma(s)^{-1}$ divides $\mathrm{Char}_\Lambda(H^1(X_{\infty}, \rho))$ in $\mathbb{C}[s]$, which also follows from (1). Moreover a topological Euler system which enjoys the same properties as the original one([5]) will be constructed and we will show a formal result which should be compared with [5]Thorem 2.3.3.
§ 2. A review of our results

Let $X$ be a connected finite CW-complex with a fixed base point $x_0$ and $\Gamma$ its fundamental group. Let $\rho$ be its unitary representation of finite dimension and $V_\rho$ the representation space. Suppose that there is a surjective homomorphism

\[(2) \quad \Gamma \to Z.\]

By the Galois theory $\ker \epsilon$ determines the infinite cyclic covering $X_\infty \to X$. In the following we will identify the group ring $\mathbb{C}[\mathbb{Z}]$ with the ring $\Lambda = \mathbb{C}[t, t^{-1}]$ of Laurent polynomials of complex coefficients. (Note that such an isomorphism is not canonical.) Thus (2) induces a ring homomorphism

$$\mathbb{C}[\Gamma] \to \Lambda.$$  

Let $\tilde{X}$ be the universal covering of $X$. Then the chain complex $(C.(\tilde{X}), \partial)$ of complex coefficients is a complex of free $\mathbb{C}[\Gamma]$-module of finite rank and so is the cochain complex $(C^\cdot(\tilde{X}), d)$.

Following [2] let us consider a complex of finite dimensional vector spaces over $\mathbb{C}$:

$$C.(X, \rho) = C.(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} V_\rho,$$

and a complex of free $\Lambda$-modules of finite rank:

$$C.(X_\infty, \rho) = C.(\tilde{X}) \otimes_{\mathbb{C}[\ker \epsilon]} V_\rho \simeq C.(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} (V_\rho \otimes_{\mathbb{C}} \Lambda).$$

Similarly we set

$$C^\cdot(X, \rho) = C^\cdot(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} V_\rho,$$

and

$$C^\cdot(X_\infty, \rho) = C^\cdot(\tilde{X}) \otimes_{\mathbb{C}[\ker \epsilon]} V_\rho \simeq C^\cdot(\tilde{X}) \otimes_{\mathbb{C}[\Gamma]} (V_\rho \otimes_{\mathbb{C}} \Lambda),$$

which are the dual complex of $C.(X, \rho)$ over $\mathbb{C}$ and of $C.(X_\infty, \rho)$ over $\Lambda$, respectively. The homology or cohomology group of each complex will be denoted by

$$H.(X, \rho), H.(X_\infty, \rho),$$

and

$$H^\cdot(X, \rho), H^\cdot(X_\infty, \rho).$$
Note that both of $H.(X_{\infty}, \rho)$ and $H^\prime(X_{\infty}, \rho)$ are finitely generated $\Lambda$-modules.

Let $Y$ be a connected subcomplex of $X$. Suppose that there is a connected subcomplex $Y_{\infty}$ of $X_{\infty}$ which is an infinite cyclic covering of $Y$ and that the diagram:

$$
Y_{\infty} \to X_{\infty} \\
\downarrow \quad \downarrow \\
Y \to X
$$

induces an isomorphism

$$\text{Gal}(Y_{\infty}/Y) \simeq \text{Gal}(X_{\infty}/X) \simeq \mathbb{Z}.$$

Then we have a complex of free $\Lambda$-modules of finite rank:

$$C.(Y_{\infty}, \rho) = C.\left(\widehat{Y}\right) \otimes_{C[\pi_1(Y)]} (V_\rho \otimes_{\mathbb{C}} \Lambda).$$

Note that this is a subcomplex of $C.(X_{\infty}, \rho)$ whose quotient $C.(X_{\infty}, Y_{\infty}, \rho)$ is also a complex of free $\Lambda$-modules of finite rank. Taking the dual over $\Lambda$ we have an exact sequence of bounded complexes of free $\Lambda$-modules:

$$0 \to C^\prime(X_{\infty}, Y_{\infty}, \rho) \to C^\prime(X_{\infty}, \rho) \to C^\prime(Y_{\infty}, \rho) \to 0.$$

Thus we have an exact sequence of finitely generated $\Lambda$-modules:

$$(3) \quad H^q(X_{\infty}, Y_{\infty}, \rho) \to H^q(X_{\infty}, \rho) \to H^q(Y_{\infty}, \rho) \to H^{q+1}(X_{\infty}, Y_{\infty}, \rho) \to$$

Here is an example of $H^q(X_{\infty}, Y_{\infty}, \rho)$.

Let $X = D^2 \times S^1$, where $D^2$ is the two dimensional unit disk and $S^1$ is the unit circle. Let $Y = S^1 \times S^1$ be its boundary. Then the fundamental group of $X$ is an infinite cyclic group and let

$$\pi_1(X) \simeq \mathbb{Z} \xrightarrow{\rho} U(n)$$

be a unitary representation. By the homotopy invariance of cohomology groups and by the Gysin isomorphism we have

$$H^q(X_{\infty}, Y_{\infty}, \rho) = \begin{cases} V_\rho & q = 2 \\ 0 & q \neq 2. \end{cases}$$

$\gamma \in \pi_1(X)$ acts on $H^2(X_{\infty}, Y_{\infty}, \rho) \simeq V_\rho$ by $\rho(\gamma)$, which makes it a $\Lambda$-module.

In the following, we will always assume that the dimension of $X$ is three and that all $H.(X_{\infty}, \mathbb{C})$ and $H.(X_{\infty}, \rho)$ are finite dimensional vector spaces over $\mathbb{C}$. The arguments of §4 of [4] shows $X_{\infty}$ is a Riemann surface in the cohomological sense.
Fact 2.1. ([4])

1. For \( i \geq 3 \), \( H^i(X_\infty, \rho) \) vanishes.

2. For \( 0 \leq i \leq 2 \), \( H^i(X_\infty, \rho) \) is a finite dimensional vector space over \( \mathbb{C} \) and there is a perfect pairing:

\[
H^i(X_\infty, \rho) \times H^{2-i}(X_\infty, \rho) \to \mathbb{C}.
\]

The perfect pairing will be referred as the Milnor duality. It is easy to see that it is preserved by the action of \( \text{Gal}(X_\infty/X) \).

Thus each \( H^i(X_\infty, \rho) \) is a torsion \( \Lambda \)-module and its characteristic ideal \( \text{Char}_\Lambda(H^i(X_\infty, \rho)) \) is generated by

\[
A^i_\rho(t) = \det[t - \tau^* | H^i(X_\infty, \rho)],
\]

where \( \tau^* \) is the action of \( t \) on \( H^i(X_\infty, \rho) \).

Let \( h^q(\rho) \) be the dimension of \( H^q(X, \rho) \). Then in [6] we have shown the following results.

Theorem 2.1. Suppose that \( H^0(X_\infty, \rho) \) vanishes. Then we have

\[
h^1(\rho) \leq \text{ord}_{t=1} A^1_\rho(t),
\]

and the identity holds if the action of \( \tau^* \) on \( H^1(X_\infty, \rho) \) is semisimple. Moreover suppose that all \( h^q(\rho) \) vanish. Then we have

\[
|A^1_\rho(1)| = \delta |\tau^*_C(X, \rho)|^{-1},
\]

where \( \delta \) is an explicit positive constant. Here \( \tau^*_C(X, \rho) \) is the Frantz-Milnor-Reidemeister torsion, which is a geometric invariant of a representation. ([3])

When \( X \) is a mapping torus we can say more.

Theorem 2.2. ([6]) Let \( f \) be an automorphism of a connected finite CW-complex of dimension two \( S \) and \( X \) its mapping torus. Let \( \rho \) be a unitary representation of the fundamental group of \( X \) which satisfies \( H^0(S, \rho) = 0 \). Suppose that the surjective homomorphism

\[
\Gamma \to \mathbb{Z}
\]

is induced by the structure map

\[
X \to S^1,
\]
and that the action of $f^*$ on $H^1(S, \rho)$ is semisimple. Then the order of $A^1_\rho(t)$ at $t = 1$ is $h^1(\rho)$ and we have
\[
\lim_{t \to 1} |(t - 1)^{-h^1(\rho)}A^1_\rho(t)| = |\tau^*_\mathbb{C}(X, \rho)|^{-1}.
\]

Note that in Theorem 2.2 $X_\infty$ is $S \times \mathbb{R}$.

In order to introduce an analytic object—the Ruelle L-function—we need a geometric structure on $X$. Let $X$ be a connected hyperbolic threefold of finite volume. Thus its fundamental group may be considered as a torsion-free cofinite discrete subgroup $\Gamma_g$ of $PSL_2(\mathbb{C})$ and let $\rho$ be its unitary representation. By the one to one correspondence between the set of loxodromic conjugacy classes of $\Gamma_g$ and one of closed geodesics of $X$, the Ruelle L-function is defined to be a product of the inverse of the characteristic polynomials of $\rho(\gamma)$ over prime closed geodesics:
\[
R_\rho(s) = \prod_{\gamma} P_\gamma(s)^{-1}, \quad P_\gamma(s) = \det[1 - \rho(\gamma)e^{-sl(\gamma)}].
\]
Here $s$ is a complex number and $l(\gamma)$ is the length of $\gamma$. It absolutely convergents for $s$ whose real part is sufficiently large.

Let $X$ be a compact hyperbolic threefold satisfying $H^0(X, \rho) = 0$. Due to Fried([1]), it is known that $R_\rho(s)$ is meromorphically continued in the whole plane and that its order at $s = 0$ is $2h^1(\rho)$. Moreover he has shown its absolute value of the leading constant is equal to $|\tau^*_\mathbb{C}(X, \rho)|^{-2}$.

In the following we always assume that $X$ admits an infinite cyclic covering $X_\infty$. (i.e. the first Betti number of $X$ is positive.) Thus combining Fried's results and Theorem 2.1 and Theorem 2.2 we obtain the following theorem.

**Theorem 2.3.** Let $X$ be a compact hyperbolic threefold and $\rho$ a unitary representation of the fundamental group.

1. Suppose that $H^0(X_\infty, \rho)$ vanishes. Then
\[
2h^1(\rho) = \text{ord}_{s=0} R_\rho(s) \leq 2\text{ord}_{t=1} A^1_\rho(t),
\]
and the identity holds if the action of $\tau^*$ on $H^1(X_\infty, \rho)$ is semisimple. Moreover if all $h^q(\rho)$ vanish, we have
\[
|R_\rho(0)| = \delta_\rho |A^1_\rho(1)|^2,
\]
where $\delta_\rho$ is an explicit constant.
2. Suppose that $X$ is homeomorphic to a mapping torus of an automorphism $f$ of a compact surface $S$ and that the surjective homomorphism $\epsilon$ is induced by the structure map:

$$X \to S^1.$$ 

If $H^0(S, \rho)$ vanishes, we have

$$2h^1(\rho) = \text{ord}_{s=0} R_\rho(s) \leq 2\text{ord}_{t=1} A^1_\rho(t),$$

and the identity holds if the action of $f^*$ on $H^1(S, \rho)$ is semisimple. Moreover if this condition is satisfied, we have

$$\lim_{s \to 0} |s^{-2h^1(\rho)} R_\rho(s)| = \lim_{t \to 1} |(t-1)^{-h^1(\rho)} A^1_\rho(t)|^2.$$

Next we will consider a non-compact case. Let $X$ be a hyperbolic threefold of finite volume with one cusp and $\rho$ a unitary character of the fundamental group. The fundamental group at the cusp will be denoted by $\Gamma_\infty$. Here is a generalization of Fried's results.

**Theorem 2.4.** ([7] [8]) $R_\rho(s)$ is meromorphically continued on the whole plane and satisfies an analog of the Riemann hypothesis. Moreover it satisfies the following properties at the origin.

1. Suppose $\rho|_{\Gamma_\infty}$ is trivial. Then we have

$$\text{ord}_{s=0} R_\rho(s) = 2(h^1(\rho) - 2h^0(\rho) - 1).$$

2. Suppose $\rho|_{\Gamma_\infty}$ is nontrivial, then

$$\text{ord}_{s=0} R_\rho(s) = 2h^1(\rho).$$

Moreover if $h^1(\rho)$ vanishes we have

$$|R_\rho(0)| = |\tau^*_C(X, \rho)|^{-2}.$$ 

We remark that the "error term" $-2$ in the RHS of the first identity is caused by a pathology of the Hodge theory. Note that in the second case the assumption automatically implies vanishing of $h^0(\rho)$. Thus we have

**Theorem 2.5.** Let $X$ be a hyperbolic threefold of finite volume with one cusp and $\rho$ a unitary character of the fundamental group such that $h^0(\rho)$ vanishes.
1. Suppose $\rho|_{\Gamma_{\infty}}$ is trivial. Then we have

$$\text{ord}_{s=0} R_{\rho}(s) + 2 \leq 2\text{ord}_{t=1} A_{\rho}^1(t),$$

and the identity holds if the action of $\tau^*$ on $H^1(X_{\infty}, \rho)$ is semisimple.

2. Suppose $\rho|_{\Gamma_{\infty}}$ is nontrivial. Then we have

$$\text{ord}_{s=0} R_{\rho}(s) \leq 2\text{ord}_{t=1} A_{\rho}^1(t),$$

and the identity holds if the action of $\tau^*$ on $H^1(X_{\infty}, \rho)$ is semisimple. Moreover if all $h^q(\rho)$ vanish we have

$$|R_{\rho}(0)| = \delta_{\rho}|A_{\rho}^1(1)|^2.$$

In either case if we make a change of variables:

$$t = s + 1,$$

under a suitable assumption, our theorem implies two ideals in $\mathbb{C}[[s]]$ generated by $R_{\rho}(s)$ and $A_{\rho}^1(s)^2$ coincide. Thus our theorem may be considered as a solution of a geometric analog of the Iwasawa main conjecture.

In particular we may say for each prime closed geodesic $\gamma$, $P_{\gamma}(s)^{-1}$ divides $A_{\rho}^1$. In the next section we will explain this phenomenon from a viewpoint of the arithmetic topology.

§ 3. An explanation from the arithmetic topology

Let $X$ be a hyperbolic threefold of finite volume and $\rho$ a unitary representation of the fundamental group. We assume that $H^0(X_{\infty}, \rho)$ vanishes.

Note that $\epsilon$ induces a map from a set of prime closed geodesics $\Sigma_{\text{prim}}$ to $\mathbb{Z}$. Thus it is decomposed into two subsets:

$$\Sigma_{\text{prim}}^\epsilon = \{ \gamma \in \Sigma_{\text{prim}} | \epsilon(\gamma) \neq 0 \}$$

and its complement $\Sigma_{\text{prim}}^{\ddagger}$. An element of $\Sigma_{\text{prim}}^\epsilon$ (resp. $\Sigma_{\text{prim}}^{\ddagger}$) will be referred as $\epsilon$-split (resp. $\epsilon$-inert). For $\gamma \in \Sigma_{\text{prim}}^\epsilon$ its $\epsilon$-inertia degree $m_\epsilon(\gamma)$ is defined to be the absolute value of $\epsilon(\gamma)$.
Let $\gamma \in \Sigma_{\text{prim}}^{\iota}$ be of $\epsilon$-inertia degree 1. We may regard it as a smooth imbedded $S^1$ and let $C_{\infty}$ be a connected component of $X_{\infty} \times X S^1$. Thus we have a diagram:

$$
\begin{array}{ccc}
C_{\infty} & \to & X_{\infty} \\
p \downarrow & \downarrow & \pi \\
S^1 & \xrightarrow{\gamma} & X.
\end{array}
$$

We claim that $C_{\infty}$ is the universal covering of $S^1$ and that the diagram induces an isomorphism:

$$\text{Gal}(C_{\infty}/S^1) \simeq \text{Gal}(X_{\infty}/X) \simeq \mathbb{Z}.$$ 

In fact (4) implies the diagram:

$$
\begin{array}{ccc}
\pi_1(C_{\infty}) & \to & \pi_1(X_{\infty}) \\
p_* \downarrow & \downarrow & \pi \\
\mathbb{Z} = \pi_1(S^1) & \xrightarrow{\gamma_*} & \pi_1(X) \\
\downarrow \epsilon & & \downarrow \\
\mathbb{Z}, & & \mathbb{Z},
\end{array}
$$

which satisfies

$$\gamma_*(1) = \gamma.$$

If $C_{\infty}$ were not $\mathbb{R}$, it should be a circle. In particular the image of $p_*$ becomes a nontrivial subgroup of $\pi_1(S^1)$. But the image of $\epsilon \cdot \gamma_*$ is a subgroup of $\mathbb{Z}$ which is torsion free, the above diagram shows that $\epsilon(\gamma)$ should be zero. This contradicts to the choice of $\gamma$. Moreover since $\epsilon$-inertia degree of $\gamma$ is one, $\gamma_*$ gives a splitting of $\epsilon$ and we have

$$\text{Gal}(C_{\infty}/S^1) \simeq \text{Gal}(X_{\infty}/X) \simeq \mathbb{Z}.$$ 

Let $N(\gamma)$ be a small tubular neighborhood of $\gamma$ and $N_{\infty}(\gamma)$ its lift to $X_{\infty}$ along $C_{\infty}$:

$$N_{\infty}(\gamma) = \pi^{-1}(N(\gamma)).$$

We set

$$X_{\infty}(\gamma) = X_{\infty} \setminus N_{\infty}(\gamma).$$

By the excision we have

$$H^q(X_{\infty}, X_{\infty}(\gamma), \rho) \simeq H^q(N_{\infty}, \partial N_{\infty}(\gamma), \rho),$$

and the computation of the previous section implies

$$H^q(N_{\infty}, \partial N_{\infty}(\gamma), \rho) = \begin{cases} 
\Lambda/(\det[t - \rho(\gamma)]) & q = 2 \\
0 & q \neq 2.
\end{cases}$$
Thus the exact sequence (3) and our assumption show the vanishing of $H^0(X_\infty(\gamma), \rho)$ and an exact sequence of $\Lambda$-modules:

$$0 \rightarrow H^1(X_\infty, \rho) \xrightarrow{\operatorname{Res}} H^1(X_\infty(\gamma), \rho) \rightarrow \Lambda/(\det[t - \rho(\gamma)]) \rightarrow 0.$$ 

In particular we know the dimension of $H^1(X_\infty(\gamma), \rho)$ is finite and we have an identity of fractional ideals of $\Lambda$:

$$\operatorname{Char}_{\Lambda}(H^1(X_\infty, \rho)) = \operatorname{Char}_{\Lambda}(H^1(X_\infty(\gamma), \rho)) \cdot (\det[1 - \rho(\gamma)t^{-1}])^{-1}.$$ 

More generally let $\gamma$ be an element of $\Sigma_{\text{prim}}^\iota$. Then the subgroup

$$m_\epsilon(\gamma)\mathbb{Z} \subseteq \mathbb{Z} = \operatorname{Gal}(X_\infty/X)$$

determines a cyclic covering $X_{m_\epsilon(\gamma)}$ of $X$ with degree $m_\epsilon(\gamma)$. Note that $X_\infty$ is its infinite cyclic covering satisfying

$$\operatorname{Gal}(X_\infty/X_{m_\epsilon(\gamma)}) = m_\epsilon(\gamma)\mathbb{Z},$$

and that $\gamma$ lifts to a smooth embedded $S^1$ in $X_{m_\epsilon(\gamma)}$ which is mapped to $\pm m_\epsilon(\gamma)$ by

$$\pi_1(X_{m_\epsilon(\gamma)}) \rightarrow \operatorname{Gal}(X_\infty/X_{m_\epsilon(\gamma)}) = m_\epsilon(\gamma)\mathbb{Z}.$$ 

Now the previous argument shows the vanishing of $H^0(X_\infty(\gamma), \rho)$ and an exact sequence:

$$0 \rightarrow H^1(X_\infty, \rho) \xrightarrow{\operatorname{Res}} H^1(X_\infty(\gamma), \rho) \rightarrow \Lambda/(\det[t^{m_\epsilon(\gamma)} - \rho(\gamma)]) \rightarrow 0$$

and

$$\operatorname{Char}_{\Lambda}(H^1(X_\infty, \rho)) = \operatorname{Char}_{\Lambda}(H^1(X_\infty(\gamma), \rho)) \cdot (\det[1 - \rho(\gamma)t^{-m_\epsilon(\gamma)}])^{-1}.$$ 

Thus we have proved the following theorem.

**Theorem 3.1.** Suppose $H^0(X_\infty, \rho)$ vanishes. Then for $\gamma \in \Sigma_{\text{prim}}^\iota$, $H^0(X_\infty(\gamma), \rho)$ also vanishes and we have an exact sequence of $\Lambda$-modules:

$$0 \rightarrow H^1(X_\infty, \rho) \xrightarrow{\operatorname{Res}} H^1(X_\infty(\gamma), \rho) \rightarrow \Lambda/(\det[t^{m_\epsilon(\gamma)} - \rho(\gamma)]) \rightarrow 0.$$ 

In particular the dimension of $H^1(X_\infty(\gamma), \rho)$ is finite and we have an identity of fractional ideals of $\Lambda$:

$$\operatorname{Char}_{\Lambda}(H^1(X_\infty, \rho)) = \operatorname{Char}_{\Lambda}(H^1(X_\infty(\gamma), \rho)) \cdot (\det[1 - \rho(\gamma)t^{-m_\epsilon(\gamma)}])^{-1}.$$
Note that the Euler factor \( P_{\gamma}(s) \) of the Ruelle L-function is given by

\[
P_{\gamma}(s) = \det[1 - \rho(\gamma)t^{-m_e(\gamma)}]_{t=\exp\left[\frac{l(\gamma)s}{m_e(\gamma)}\right]}.
\]

Since we have

\[
\exp\left(\frac{l(\gamma)}{m_e(\gamma)}s\right) - 1 = \frac{l(\gamma)s}{m_e(\gamma)} + O(s^2),
\]

localizing at \( s = t - 1 \), the fact that \((\det[1 - \rho(\gamma)t^{-m_e(\gamma)}])^{-1}\) divides \( \text{Char}_{\Lambda}(H^1(X_{\infty}, \rho)) \) implies the divisibility of \( A_{\rho}^1 \) by \( P_{\gamma}(s)^{-1} \) in \( \mathbb{C}[[s]] \) for \( \gamma \in \Sigma_{\text{prim}}^\iota \).

We can formulate this fact in terms of an analog of the Euler system([5]).

First of all we remark that using the homology exact sequence:

\[
\rightarrow H_q(X_{\infty}(\gamma), \rho) \rightarrow H_q(X_{\infty}, \rho) \rightarrow H_q(X_{\infty}, X_{\infty}(\gamma), \rho) \rightarrow H_{q-1}(X_{\infty}(\gamma), \rho) \rightarrow
\]

and by the isomorphism

\[
H_q(X_{\infty}, X_{\infty}(\gamma), \rho) \simeq H_q(D^2, S^1, \rho)
\]

derived from the excision and the homotopy invariance of the homology group one may check that the dimension of \( H_*(X_{\infty}(\gamma), \mathbb{C}) \) and \( H_*(X_{\infty}(\gamma), \rho) \) are finite. Taking the dual of (5) over \( \mathbb{C} \), the Milnor duality shows an exact sequence of \( \Lambda \)-modules:

\[
0 \rightarrow \Lambda/(\det[t^{rn_{\epsilon}(\gamma)} - \rho(\gamma)]) \rightarrow H^1(X_{\infty}(\gamma), \rho) \overset{\text{Cor}}{\rightarrow} H^1(X_{\infty}, \rho) \rightarrow 0.
\]

Let us fix a nonzero element \( c_\infty \) of \( H^1(X_{\infty}, \rho) \) and choose its any lift \( c'(\gamma)_\infty \) to \( H^1(X_{\infty}(\gamma), \rho) \). Then

\[
c(\gamma)_\infty = F_\gamma(t)c'(\gamma)_\infty, \quad F_\gamma(t) = \det[t^{m_e(\gamma)} - \rho(\gamma)]
\]

is independent of a choice of the lift and satisfies

\[
\text{Cor}(c(\gamma)_\infty) = F_\gamma(t)c_\infty.
\]

More generally, for elements \( \{\gamma_1, \cdots, \gamma_N\} \) of \( \Sigma_{\text{prim}}^\iota \), we set

\[
X_{\infty}(\gamma_1 \cdots \gamma_N) = X_{\infty} \setminus N_{\infty}(\gamma_1) \cup \cdots \cup N_{\infty}(\gamma_N).
\]

Using Theorem 3.1, an induction argument shows that we have an exact sequence of \( \Lambda \)-modules:

\[
0 \rightarrow \Lambda/(F_{\gamma_N}(t)) \rightarrow H^1(X_{\infty}(\gamma_1 \cdots \gamma_N), \rho) \overset{\text{Cor}}{\rightarrow} H^1(X_{\infty}(\gamma_1 \cdots \gamma_{N-1}), \rho) \rightarrow 0.
\]
Therefore we can successively choose an element $c_{\infty}(\gamma_1 \cdots \gamma_N)$ of $H^1(X_\infty(\gamma_1 \cdots \gamma_N), \rho)$ so that

$$\text{Cor}(c_{\infty}(\gamma_1 \cdots \gamma_N)) = F_{\gamma_N}(t)c_{\infty}(\gamma_1 \cdots \gamma_{N-1}).$$

Thus $\{c_{\infty}(\gamma_1 \cdots \gamma_N)\}$ has the same property as the Euler system [5] §2.1. If we apply the co-restriction map "Cor" $N$-times to $c_{\infty}(\gamma_1 \cdots \gamma_N)$, we obtain an element $d_{\infty}(\gamma_1 \cdots \gamma_N)$ of $H^1(X_\infty, \rho)$ which satisfies

$$d_{\infty}(\gamma_1 \cdots \gamma_N) = \prod_{i=1}^{N} F_{\gamma_i}(t) \cdot c_{\infty}.$$

Now our solution of the geometric Iwasawa conjecture is formally described in the following way.

If two elements $c$ and $c'$ of $H^1(X_\infty, \rho)$ have a relation:

$$c' = f \cdot c, \quad f \in \Lambda,$$

$f^{-1}$ will be denoted by $\text{ind}_\Lambda(c, c')$. (Note that in fact since $H^1(X_\infty, \rho)$ is a torsion $\Lambda$-module it is formally defined.) In particular our topological Euler system gives the $\epsilon$-inert part of the Euler product:

$$\text{ind}_\Lambda(c_{\infty}, d_{\infty}(\prod_{\gamma \in \Sigma_{\text{prim}}} \gamma)) = \prod_{\gamma \in \Sigma_{\text{prim}}} F_{\gamma_i}^{-1} \in \mathbb{C}[[s]].$$

The following statement is a formal reformulation of Theorem 3.1, which should be compared with [5] Theorem 2.3.3.

**Theorem 3.2.** (formal) In $\mathbb{C}[[s]]$, $\text{ind}_\Lambda(c_{\infty}, d_{\infty}(\prod_{\gamma \in \Sigma_{\text{prim}}} \gamma))$ divides $\text{Char}_\Lambda(H^1(X_\infty, \rho)).$

**References**


