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On the equivariant Tamagawa number conjecture for CM elliptic curves

By

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Abstract

In this article we introduce the formulation of equivariant Tamagawa number conjecture for motives and a result on the case of CM elliptic curves.

§ 1. Introduction

The theory of L-functions attached to motives over number fields has been studied by many authors for a long time. In particular, it is well-known that the study of special values of L-functions and ζ-functions plays an important role in number theory. In general, it is conjectured that the special values of L-functions are related to informations of certain arithmetic groups. For example, Dirichlet's class number formula explains a relation between the residue of Dedekind zeta function for a number field K at $s = 1$ and the order of ideal class group of K. Moreover, Birch and Swinnerton-Dyer conjecture gives the relation between the leading term of the Taylor expansion of L-function for an elliptic curve over $\mathbb{Q}$ at $s = 1$ and the order of Tate-Shafarevich group for E.

The Tamagawa number conjecture formulated by Bloch and Kato [3] describes the leading terms of Taylor expansions at zero of L-functions of pure Chow motives with negative weight over number fields in terms of regulator maps of motivic cohomologies into Deligne's and étale cohomologies. Fontaine and Perrin-Riou [13] and Kato reformulated and extended the Tamagawa number conjecture to motives with commutative coefficients using the determinant functor of Knudsen and Mumford [21]. This conjecture is a generalization of Dirichlet's class number formula and Birch and Swinnerton-Dyer conjecture. Moreover Burns and Flach [5] formulated the equivariant version of Tamagawa number conjecture for a Chow motives over number fields with the action of a semisimple finite dimensional $\mathbb{Q}$-algebra. For the (equivariant) Tamagawa number

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conjecture there are some results. Burns and Greither [6] proved the equivariant Tamagawa number conjecture for all abelian extension of $\mathbb{Q}$ for the L-values at any integer points. Huber and Kings [15] gave independently a proof of a weaker version in the same case. Recently Bley [2] and Johnson [16] considered the case of abelian extension of imaginary quadratic fields. For another case, Kings proved a weak version of Tamagawa number conjecture for CM elliptic curves in the non-critical situation. In this article we introduce the formulation of the Tamagawa number conjecture for general pure Chow motives and consider the equivariant version of the result of Kings. For the generalization of the result of this article to Hecke characters, see [7].

§ 2. Determinant functor

In this section we review some facts on the determinant functor of Knudsen and Mumford [21]. For any commutative ring R, let $P(R)$ denote the category of finitely generated projective R-modules and $(P(R), is)$ its subcategory of isomorphisms. A graded invertible R-module is a pair (L, α) consisting of an invertible (that is, projective rank 1) R-module L and a locally constant function $\alpha : \mathrm{Spec}\,(R) \to \mathbb{Z}$.

A homomorphism $h : (L, \alpha) \to (M, \beta)$ of graded invertible modules is a homomorphism of R-modules $h : L \to M$ such that the localization $h_{\mathfrak{p}}$ is equal to 0 for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$ satisfying $\alpha(\mathfrak{p}) \neq \beta(\mathfrak{p})$. Let $Inv(R)$ denote the category of graded invertible modules and isomorphisms. Then the category $Inv(R)$ is a symmetric monoidal category with tensor product

$$(L, \alpha) \otimes (M, \beta) = (L \otimes_R M, \alpha + \beta),$$

the associativity constraint, the unit object $(R, 0)$, and the commutativity constraint

$$(L, \alpha) \otimes (M, \beta) \cong (M, \beta) \otimes (L, \alpha).$$

We define

$$(L, \alpha)^{-1} = (\mathrm{Hom}(L, R), -\alpha).$$

For a finitely generated projective R-module P we define

$$\mathrm{Det}_R P = \left(\bigwedge_R^{\mathrm{rank}_R P} P, \mathrm{rank}_R P \right).$$

This is a graded invertible R-module, so Det_R gives a functor $(P(R), is) \to Inv(R)$. For a bounded complex of finitely generated R-modules $P^\bullet$ we define

$$\mathrm{Det}_R(P^\bullet) = \bigotimes_{i \in \mathbb{Z}} \mathrm{Det}_R^{(-1)^{i+1}}(P^i).$$

Furthermore we set $\mathrm{Det}_R^{-1}(P^\bullet) = \mathrm{Det}_R(P^\bullet)^{-1}$.

We write $\mathcal{D}(R)$ for the derived category of the homotopy category of bounded complexes of R-modules, and $\mathcal{D}^p(R)$ for the full triangulated subcategory of perfect complexes of R-modules. Let $\mathcal{D}^{pis}(R)$ be the subcategory of $\mathcal{D}^p(R)$ in which the objects are the same but the morphisms are restricted to quasi-isomorphisms. We assume that R is reduced. Then the functor Det_R can be extended to a functor $\mathcal{D}^{pis}(R) \to Inv(R)$ in such way that for every distinguished triangle $C_1 \to C_2 \to C_3$ in $\mathcal{D}(R)$ there is a functorial isomorphism in $Inv(R)$

$$(\mathrm{Det}_R C_1)^{-1} \otimes \mathrm{Det}_R C_2 \overset{\cong}{\to} \mathrm{Det}_R C_3.$$

For R-module X, if $X[-1]$ belong to $\mathcal{D}^p(R)$, then we say that X is perfect. And we define $\mathrm{Det}_R(X) := \mathrm{Det}_R(X[-1])$ for any such X. If a complex C is bounded and each cohomology module is perfect, then C belongs to $\mathcal{D}^p(R)$ and there is a canonical isomorphism

$$\mathrm{Det}_R C \overset{\cong}{\to} \bigotimes_{i \in \mathbb{Z}} \mathrm{Det}_R^{(-1)^{i+1}}(H^i(C)).$$

If C is acyclic, then there is a canonical isomorphism

$$\mathrm{Det}_R C \overset{\cong}{\to} (R, 0).$$

Let G be any finite abelian group. For any commutative ring Z we write $x \mapsto x^\#$ for the Z-linear involution of the group ring $Z[G]$ which satisfies $g^\# = g^{-1}$ for each $g \in G$. If X is any complex of $Z[G]$-modules, then we write $X^\#$ for the scalar extension with respect to the morphism $x \mapsto x^\#$.

For any finitely generated projective $Z[G]$-module X (resp. object X of $\mathcal{D}^p(Z[G])$), we set $X^* := \mathrm{Hom}_Z(X, Z)$ (resp. $X^* := R\mathrm{Hom}_Z(X, Z)$), which we regard as endowed with the contragradient G-action. We see that if X is a finitely generated projective $Z[G]$-module (resp. object of $\mathcal{D}^p(Z[G])$), then so is X^*. We recall that for any $Z[G]$-module X one has a canonical isomorphism $X^* \cong \mathrm{Hom}_{Z[G]}(X, Z[G])^\#$, and that this induces that for each object X of $\mathcal{D}^p(Z[G])$ a canonical isomorphism in $Inv(Z[G])$

$$\mathrm{Det}_{Z[G]} X^* \cong \mathrm{Det}_{Z[G]}^{-1}(X^\#).$$

§ 3. L-functions and Motivic cohomology

Let $M = (X, \pi, r)$ be a pure Chow motive over a number field K, where X is a smooth projective variety over K, π is a projector in $CH^{\dim X}(X \times X)$ and r is a rational integer. For a prime ℓ let M_ℓ be the ℓ-adic realization of the motive M. We consider the ℓ-adic Galois representation

$$\rho_{M,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(M_\ell) \cong \mathrm{GL}_{\dim_{\mathbb{Q}_\ell} M_\ell}(\mathbb{Q}_\ell).$$

Let A be a semisimple, commutative, finite dimensional $\mathbb{Q}$-algebra. Assume that A acts on M. For a finite place v satisfying that $v \nmid \ell$, we define

$$P_v(M,s) := \mathrm{Det}_A(1 - \mathrm{Frob}_v \cdot \mathrm{N}_{K/\mathbb{Q}} v^{-s} | M_\ell^{I_v}),$$

where I_v is the inertia group. If $v|p = \ell$, we define

$$P_v(M,s) := \mathrm{Det}_A(1 - \varphi^{[K_{v,0}:\mathbb{Q}_p]} \cdot \mathrm{N}_{K/\mathbb{Q}} v^{-s} | (M_p \otimes B_{\mathrm{cris}})^{D_v}),$$

where D_v is the decomposition group, φ is the absolute Frobenius and $K_{v,0}$ is the unramified closure of $\mathbb{Q}_p$ in K_v. It is conjectured that $P_v(M,s)$ is independent of ℓ. This is known for if M has good reduction at v and if $A = \mathbb{Q}$. We define the A-equivariant L-function $L({}_AM, s)$ by

$$L({}_AM, s) := \prod_v P_v(M,s).$$

Then it is conjectured that the L-function $L({}_AM,s)$ has a meromorphic continuation to all $s \in \mathbb{C}$. The Taylor expansion

$$L({}_AM, s) = L^*({}_AM) s^{r({}_AM)} + \cdots$$

at $s = 0$ defines $L({}_AM) \in (A \otimes \mathbb{R})^\times$ and $r({}_AM) \in H^0(\mathrm{Spec}\,(A), \mathbb{Z})$. The Tamagawa number conjecture describes these quantities.

To formulate the Tamagawa number conjecture, we recall some notations related to motivic cohomology. For a motive $M = (X, \pi, r)$ we define the motivic cohomology $H^i_{\mathcal{M}}(M)$ by

$$\begin{aligned} H^0_{\mathcal{M}}(M) &:= \pi_*(CH^r(X)/CH^r_{\mathrm{hom}}(X)) \otimes \mathbb{Q}, \\ H^1_{\mathcal{M}}(M) &:= \pi_*(CH^r_{\mathrm{hom}}(X)) \otimes \mathbb{Q} \oplus \bigoplus_{i \in \mathbb{Z}, i \neq 0} \pi_*(K_i(X) \otimes \mathbb{Q})^{(r)}. \end{aligned}$$

It is conjectured that if $\mathcal{MM}_K$ is the category of mixed motives over K, the motivic cohomologies can be written as

$$\begin{aligned} H^0_{\mathcal{M}}(M) &= \mathrm{Hom}_{\mathcal{MM}_K}(\mathbb{Q}, M) \\ H^1_{\mathcal{M}}(M) &= \mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbb{Q}, M). \end{aligned}$$

Denote the weight of M by w. Furthermore, we define the finite (or integral) part of motivic cohomology $H^i_f(M)$ as follows: If $w \neq 1$, we put

$$\begin{aligned} H^0_f(M) &:= 0, \\ H^1_f(M) &:= \pi_* \mathrm{Im}(H^{w+2r+1}_{\mathcal{M}}(\mathcal{X}, \mathbb{Q}(r)) \to H^{w+2r+1}_{\mathcal{M}}(X, \mathbb{Q}(r))), \end{aligned}$$

where $\mathcal{X}$ is a regular, proper flat model of X (the existence of such model is still conjectural in general.) and $H^i_{\mathcal{M}}(X, \mathbb{Q}(r)) = (K_{-w}(X) \otimes \mathbb{Q})^{(r)}$

If $w = 1$, we put

$$H^0_f(M) := \pi_*(CH^r(X)/CH^r_{\text{hom}}(X)) \otimes \mathbb{Q},$$
$$H^1_f(M) := \pi_*(CH^r_{\text{hom}}(X)) \otimes \mathbb{Q}.$$

It is predicted that these spaces are finite dimensional.

Example 3.1. (1) For a number field L we set $M = h^0(\text{Spec}\,(L))$. Then we have

$$H^0_f(M) = \mathbb{Q},\ H^1_f(M) = 0,$$

For $M = h^0(\text{Spec}\,(L))(1)$ we have

$$H^0_f(M) = 0,\ H^1_f(M) = O_L^\times \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In general, for $M = h^0(\text{Spec}\,(L))(j)(j \neq 0, 1)$ we have

$$H^1_f(M) = K_{2j-1}(L) \otimes_{\mathbb{Z}} \mathbb{Q},$$

therefore these spaces are finite dimensional.

(2) Let X be a smooth projective variety. For $M = h^1(X)(1)$ we have

$$H^0_f(M) = \mathbb{Q},\ H^1_f(M) = \text{Pic}^0(X).$$

These spaces are finite dimensional by Mordell-Weil's theorem.

Conjecture 3.2 (Order of vanishing, Beilinson-Deligne).

$$r({}_AM) = \dim_A H^1_f(M^*(1)) - \dim_A H^0_f(M^*(1)).$$

This is a generalization of the Beilinson-Bloch conjecture.

§ 4. The Tamagawa number conjecture

The comparison isomorphism $M_{\text{B}} \otimes \mathbb{C} \cong M_{\text{dR}} \otimes \mathbb{C}$ induces the period map

$$\alpha_M : M_{\text{B}}^+ \otimes \mathbb{R} \to (M_{\text{dR}}/F^0 M_{\text{dR}}) \otimes \mathbb{R}.$$

We say that M is critical if this happens to be an isomorphism.

Conjecture 4.1 (Mot$_\infty$-conjecture, Fontaine/Perrin-Riou). *There is an exact sequence as $A_{\mathbb{R}} := A \otimes \mathbb{R}$-spaces*

$$0 \to H^0_f(M) \otimes \mathbb{R} \xrightarrow{\text{cl}} \ker(\alpha_M) \to (H^1_f(M^*(1)) \otimes \mathbb{R})^*$$
$$\xrightarrow{h} H^1_f(M) \otimes \mathbb{R} \xrightarrow{\text{reg}} \text{coker}(\alpha_M) \to (H^0_f(M^*(1)) \otimes \mathbb{R})^* \to 0.$$

Where cl *is the cycle map, h is a height paring, and reg is the Beilinson's regulator map.*

We define the fundamental line $\Xi({}_AM)$ which is a one dimension $\mathbb{Q}$-vector space by

$$\begin{aligned}\Xi({}_AM) :=&\mathrm{Det}_A(H^0_f(M)) \otimes \mathrm{Det}_A^{-1}(H^1_f(M)) \otimes \mathrm{Det}_A(H^1_f(M^*(1))^*)\\ &\otimes \mathrm{Det}_A^{-1}(H^0_f(M^*(1))^*) \otimes \mathrm{Det}_A^{-1} M_{\mathrm{B}}^+ \otimes \mathrm{Det}_A(M_{\mathrm{dR}}/F^0M_{\mathrm{dR}}),\end{aligned}$$

where $X^* = \mathrm{Hom}(X, A)$ is the dual of X as A-module.

Assuming Mot_∞-conjecture, we have an isomorphism

$$\theta_\infty : \mathrm{Det}_{A_{\mathbb{R}}}(0) \overset{\cong}{\to} \Xi(M) \otimes A_{\mathbb{R}}.$$

Conjecture 4.2 (Rationality, Beilinson-Deligne). $\theta_\infty(L^*({}_AM)^{-1}) \in \Xi({}_AM) \otimes 1.$

The conjecture on rationality is equivalent to the statement that the composite isomorphism

$$\zeta_{A_{\mathbb{R}}}(M) : \mathrm{Det}_{A_{\mathbb{R}}}(0) \overset{L^*({}_AM)^{-1}}{\longrightarrow} \mathrm{Det}_{A_{\mathbb{R}}}(0) \overset{\theta_\infty}{\longrightarrow} \Xi({}_AM) \otimes A_{\mathbb{R}}$$

is the scalar extension from A to $A_{\mathbb{R}}$ of an isomorphism

$$\zeta_A(M) : \mathrm{Det}_A(0) \overset{\cong}{\to} \Xi({}_AM).$$

Now we consider the ℓ-adic regulator map. For a prime p we define the complex $R\Gamma_f(\mathbb{Q}_p, M_\ell)$ by

$$R\Gamma_f(\mathbb{Q}_p, M_\ell) := \begin{cases} M_\ell^{I_p} \overset{1-\mathrm{Frob}_p}{\longrightarrow} M_\ell^{I_p} & \ell \neq p,\\ D_{\mathrm{cris}}(M_p) \overset{(1-\varphi,1)}{\longrightarrow} D_{\mathrm{cris}}(M_p) \oplus (D_{\mathrm{dR}}(M_p)/\operatorname{Fil}^0 D_{\mathrm{dR}}(M_p)) & \ell = p,\end{cases}$$

where

$$\begin{aligned} D_{\mathrm{cris}}(M_p) &:= H^0(D_p, M_p \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}) = (M_p \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{D_p},\\ D_{\mathrm{dR}}(M_p) &:= H^0(D_p, M_p \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) = (M_p \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{D_p},\\ \operatorname{Fil}^i D_{\mathrm{dR}}(M_p) &:= H^0(D_p, M_p \otimes_{\mathbb{Q}_p} B^i_{\mathrm{dR}}) = (M_p \otimes_{\mathbb{Q}_p} B^i_{\mathrm{dR}})^{D_p}.\end{aligned}$$

Let $R\Gamma(\mathbb{Q}_p, M_\ell)$ be the complex corresponding to the Galois cohomology

$$H^*_{\text{ét}}(\mathrm{Spec}\,(\mathbb{Z}[1/p]), M_\ell) = H^*(G_p, M_\ell),$$

where G_p is the Galois group of the maximal unramified extension outside p of $\mathbb{Q}$. This cohomology is concentrated in degrees $0, 1, 2, 3$. Then we have a canonical map $R\Gamma_f(\mathbb{Q}_p, M_\ell) \to R\Gamma(\mathbb{Q}_p, M_\ell)$, so define the complex by the mapping cone

$$R\Gamma_{/f}(\mathbb{Q}_p, M_\ell) := \mathrm{Cone}[R\Gamma_f(\mathbb{Q}_p, M_\ell) \to R\Gamma(\mathbb{Q}_p, M_\ell)][-1].$$

Therefore we have a distinguished triangle

$$R\Gamma_f(\mathbb{Q}_p, M_\ell) \to R\Gamma(\mathbb{Q}_p, M_\ell) \to R\Gamma_{/f}(\mathbb{Q}_p, M_\ell).$$

Let S be a finite set of places including ∞ and all bad primes. Furthermore we put

$$R\Gamma_c(\mathbb{Z}[1/S], M_\ell) := \mathrm{Cone}[R\Gamma(\mathbb{Z}[1/S], M_\ell) \to \bigoplus_{p\in S} R\Gamma(\mathbb{Q}_p, M_\ell)][-1].$$

Finally, we set

$$R\Gamma_f(\mathbb{Q}, M_\ell) := \mathrm{Cone}[R\Gamma(\mathbb{Z}[1/S], M_\ell) \to \bigoplus_{p\in S} R\Gamma_{/f}(\mathbb{Q}_p, M_\ell)][-1]$$

With this notation the cohomology groups $H^i_f(\mathbb{Q}, M_\ell) := H^i R\Gamma_f(\mathbb{Q}, M_\ell)$ is given by

$$H^0_f(\mathbb{Q}, M_\ell) = H^0(\mathbb{Z}[1/S], M_\ell)$$

and

$$H^1_f(\mathbb{Q}, M_\ell) = \mathrm{Ker}\left[H^1(\mathbb{Z}[1/S], M_\ell) \to \bigoplus_{p\in S} \frac{H^1(\mathbb{Q}_p, M_\ell)}{H^1_f(\mathbb{Q}_p, M_\ell)}\right].$$

By the octahedral axiom, we have a distinguished triangle

$$R\Gamma_c(\mathbb{Z}[1/S], M_\ell) \to R\Gamma_f(\mathbb{Q}, M_\ell) \to \bigoplus_{p\in S} R\Gamma_f(\mathbb{Q}_p, M_\ell). \tag{4.1}$$

Conjecture 4.3 (Mot$_\ell$-conjecture, Bloch-Kato). *There are a natural isomorphism $H^0_f(M) \otimes \mathbb{Q}_\ell \cong H^0_f(\mathbb{Q}, M_\ell)$ (which is given by the cycle class map) and $H^1_f(M) \otimes \mathbb{Q}_\ell \cong H^1_f(\mathbb{Q}, M_\ell)$ (which is given by the Chern class map).*

Since there is a duality $H^i_f(\mathbb{Q}, M_\ell) \cong H^{3-i}_f(\mathbb{Q}, M^*_\ell(1))^*$ for all i, this conjecture determines all cohomology groups of $R\Gamma_f(\mathbb{Q}, M_\ell)$. Assuming Mot$_\ell$-conjecture, we have an isomorphism

$$\theta_\ell : \Xi(M)\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \xrightarrow{\cong} \mathrm{Det}_{\mathbb{Q}_\ell} R\Gamma_c(\mathbb{Z}[1/S], M_\ell)$$

using the distinguished triangle (4.1). Let $\mathfrak{A} \subseteq$ be any $\mathbb{Z}$-order such that for any prime ℓ there exists a $G_{\mathbb{Q}}$-stable projective $\mathfrak{A}_\ell := \mathfrak{A} \otimes \mathbb{Z}_\ell$-lattice T_ℓ in M_ℓ. If $M = M' \otimes h^0(\mathrm{Spec}\,(L))$ arises by base change of a motive M' to a finite Galois extension $L/\mathbb{Q}$ with Galois group G then there exists a $G_{\mathbb{Q}}$-stable projective $\mathbb{Z}_\ell[G]$-lattice in M_ℓ. Now we can formulate the Tamagawa number conjecture.

Conjecture 4.4 (Integrality, equivariant Tamagawa number conjecture).

$$\mathfrak{A}_\ell \cdot \theta_\ell \circ \theta_\infty(L^*({}_AM)^{-1}) = \mathrm{Det}_{\mathfrak{A}_\ell} R\Gamma_c(\mathbb{Z}[1/S], T_\ell).$$

This conjecture is equivalent to the statement that the composite isomorphism

$$\zeta_{A_\ell}(M_\ell) : \mathrm{Det}_{A_\ell}(0) \overset{\zeta_A(M)\otimes A_\ell}{\longrightarrow} \Xi({}_AM) \otimes A_\ell \overset{\theta_\ell}{\to} \mathrm{Det}_{A_\ell} R\Gamma_c(\mathbb{Z}[1/S], M_\ell)$$

is the scalar extension from $\mathfrak{A}_\ell$ to A_ℓ of an isomorphism

$$\zeta_{\mathfrak{A}_\ell}(T_\ell) : \mathrm{Det}_{\mathfrak{A}_\ell}(0) \overset{\cong}{\to} \mathrm{Det}_{\mathfrak{A}_\ell} R\Gamma_c(\mathbb{Z}[1/S], T_\ell)$$

The Tamagawa number conjecture determines the value $L^*({}_AM) \in A_{\mathbb{R}}$ up to $\mathfrak{A}^\times$, and this conjecture is independent of the choice of S and T_ℓ.

For $M = h^0(\mathrm{Spec}\,(L))$ this conjecture is equivalent to the ℓ-part of Dirichlet's class number formula, for $M = h^1(X)(1)$ this conjecture is equivalent to the ℓ-part of the Birch and Swinnerton-Dyer conjecture.

§ 5. Known results

§ 5.1. Abelian extensions of $\mathbb{Q}$

A fundamental example of the Tamagawa number conjecture is Dirichlet's class number formula.

Theorem 5.1 (Class number formula). *Let K be a number field and*

$$\zeta_K(s) := \sum_{\mathfrak{a}\subset \mathcal{O}_K} \frac{1}{\mathrm{N}_{K/\mathbb{Q}}\,\mathfrak{a}^s} = \prod_{\mathfrak{p}:\text{prime}} \frac{1}{1 - \mathrm{N}_{K/\mathbb{Q}}\,\mathfrak{p}^{-s}}$$

the Dedekind zeta function of K. Then we have

$$\lim_{s\to 0} \frac{\zeta_K(s)}{s^{r_1+r_2-1}} = -\frac{h_K R_K}{w_K},$$

where r_1 (resp. r_2) is the number of real (resp. complex) places, h_K is the class number of K, R_K is the regulator of K and w_K is the number of roots of unit in K.

The class number formula is equivalent to the Tamagawa number conjecture for the motive $M = h^0(\mathrm{Spec}\,(K))$ and $\mathfrak{A} = \mathbb{Z}$. For Tate motives of an abelian extension of $\mathbb{Q}$, Burns and Greither proved the equivariant Tamagawa number conjecture unconditionally.

Theorem 5.2 (Burns-Greither [6], Flach [11]). *Let $F/\mathbb{Q}$ be an abelian extension with Galois group $G = \mathrm{Gal}(F/\mathbb{Q})$. We set $M = h^0(\mathrm{Spec}\,(F))(j)$ with $j \in \mathbb{Z}$, $\mathfrak{A} = \mathbb{Z}[G]$. Then for any prime number ℓ the equivariant Tamagawa number conjecture for M holds.*

From this result, we have the following formula.

Theorem 5.3 (Huber-Kings [15], Cohomological Lichtenbaum conjecture). *Let K be an abelian extension of $\mathbb{Q}$ and $\zeta_K^*(r)$ the leading term of the Taylor expansion of the Dedekind zeta function at $s = r$. Also we denote by $R_K(r)$ the Beilinson's regulator. Then, for any negative integer r we have the formula*

$$\zeta_K^*(r) \doteq R_K(r) \prod_{p:\text{prime}} \frac{\#H^2(\mathcal{O}_K[1/p], \mathbb{Z}_p(-r+1))}{\#H^1(\mathcal{O}_K[1/p], \mathbb{Z}_p(-r+1))_{\text{tors}}},$$

where $\doteq$ means that the both sides are the same up to powers of 2 and -1.

To prove Theorem 5.2, it is enough to show the case for $j \leq 0$ by the compatibility of the Tamagawa number conjecture with the functional equation of the L-function. For $j \leq 0$ Burns and Greither deduced this theorem from the equivariant main conjecture and the result concerning the image of Beilinson's element in the K-group under the étale Chern map which was computed by Huber and Wildeshaus ($j < 0$).

§ 5.2. Abelian extensions of imaginary quadratic fields

Recently Bley and Johnson investigate the case of Tate motives of abelian extensions of an imaginary quadratic fields.

Theorem 5.4 (Bley [2]). *Let K be an imaginary quadratic field and F/K an abelian extension. Let ℓ be an odd prime number satisfying $\ell \nmid h_K$ and ℓ splits in K, We set $M = h^0(\mathrm{Spec}\,(L))$, $\mathfrak{A} = \mathbb{Z}[\mathrm{Gal}(F/K)]$. Then the ℓ-part of the equivariant Tamagawa number conjecture for M holds.*

Theorem 5.5 (Johnson [16]). *Let K be an imaginary quadratic field and F/K an abelian extension and r an integer with $r < 0$. Let ℓ be a prime number satisfying $\ell \nmid 6[F : K]$ and ℓ splits in K, We set $M = h^0(\mathrm{Spec}\,(L))(r)$, $\mathfrak{A} = \mathbb{Z}[\mathrm{Gal}(F/K)]$. Then the ℓ-part of the equivariant Tamagawa number conjecture for M holds.*

§ 5.3. CM elliptic curves and Hecke characters

For CM elliptic curves, the non-critical case of (non-equivariant version of) the Tamagawa number conjecture was proved by Kings.

Theorem 5.6 (Kings [20]). *Let K be an imaginary quadratic field with class number one, E/K an elliptic curve with complex multiplication by $\mathcal{O}_K$, r an integer with $r \geq 2$. We denote the conductor of E by $\mathfrak{f}$ and put $M = h^1(E)(r)$, $\mathfrak{A} = \mathcal{O}_K$. Let ℓ be a prime number satisfying $(\ell, 6\mathfrak{f}) = 1$. Assuming $H^2(\mathcal{O}_K[1/\mathfrak{f}\ell], M_\ell) = 0$. The ℓ-part of (a weak version of) the Tamagawa number conjecture for M holds.*

We will consider the equivariant version of this result from the next section. We remark that Bars [1] extended Kings' result to some cases of Hecke characters of higher weights. For critical case of the equivariant Tamagawa number conjecture for CM elliptic curves, Flach showed the following result.

Theorem 5.7 (Flach [12]). *Let F be an abelian extension of a imaginary quadratic field K, E/F an elliptic curve with complex multiplication by $\mathcal{O}_K$. Let $\mathcal{A} = \mathrm{Res}_{F/K}$ the Weil restriction of E to K, $\varphi = \varphi_E$ the grössencharacter associated to E. Set $M = h^1(\varphi)$ and $\mathfrak{A} = \mathrm{End}_K(\mathcal{A})$. Suppose $L(E/F, 1) \neq 0$ and the equivariant main conjecture for the imaginary quadratic field K at ℓ (We will formulate this conjecture in section 10). Then the ℓ-part of the equivariant Tamagawa number conjecture for M holds.*

For general Hecke characters over imaginary quadratic fields with class number one, Tsuji [24] proved the Tamagawa number conjecture in critical cases.

§ 5.4. Modular forms

Recently Gealy showed the following result for the cases of motives associated to newforms which was constructed by Scholl.

Theorem 5.8 (Gealy [14]). *Let $f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_k(\Gamma_1(N))$, $(k \geq 2, N \geq 5)$ be a newform, r a negative integer and ℓ an odd prime. We set $M = M(f)(r)$, $\mathfrak{A} = \mathcal{O}_{K_f}$, where $K_f := \mathbb{Q}(\{a_n(f)\})$. Assume that the local representation at 2 is not supercuspidal and $H^2(\mathbb{Z}[1/N\ell], T_\ell)$ is finite. Furthermore if $k = 2$ (resp. $k = 4$) we suppose $L(f, 1) \neq 0$ (resp. $L(f, 2) \neq 0$). Then Kato's Iwasawa main conjecture for $\check{f}$ (where $\check{f}$ is the complex conjugate form of f) implies the ℓ-part of (a weak version of) the Tamagawa number conjecture for M.*

He carried out the computation of the image of the Beilinson-Kato element under the regulator map and the ℓ-adic Chern class map.

For the adjoint motives of modular forms, Diamond-Flach-Guo proved the following case.

Theorem 5.9 (Diamond-Flach-Guo [10]). *Let $f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_k(\Gamma_1(N))$, $(k \geq 2)$ be a newform and λ a finite place above ℓ does not divide $Nk!$. We set $M = \mathrm{Ad}^0 M(f)$ or $\mathrm{Ad}^0 M(f)(1)$, $\mathfrak{A} = \mathcal{O}_{K_f}$. Assume that the mod λ G_F-representation*

$$(\rho_{f,\lambda} \mod \lambda)|_F : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}_\ell})$$

is absolute irreducible, where F is the quadratic subfield of $\mathbb{Q}(\zeta_\ell)$. Then the λ-part of the Tamagawa number conjecture for M holds.

We remark that their proof is different from the previous examples as it is not based on either Euler systems or an Iwasawa main conjecture. They use the Taylor-Wiles method developed to show the modularity of elliptic curves over $\mathbb{Q}$.

§ 6. Tamagawa number conjecture for CM elliptic curves

Let K be an imaginary quadratic field with class number one. and E an elliptic curve over K with complex multiplication by $\mathcal{O}_K$. We denote by $\mathfrak{f}$ the conductor of E. Let $\varphi = \varphi_E : \mathbb{A}_K^\times \to K^\times$ be the grössencharacter associated to E, then we define the L-function φ_E by

$$L(\varphi_E, s) := \prod_{\mathfrak{p}:\text{prime}} (1 - \varphi_E(\mathfrak{p}) N\mathfrak{p}^{-s})^{-1}.$$

The L-function of $L(\varphi_E, s)$ satisfies

$$L(E/K, s) = L(\varphi_E, s) L(\overline{\varphi}_E, s),$$

where

$$L(E, s) := \prod_{\mathfrak{p}\nmid\mathfrak{f}} (1 - a_{\mathfrak{p}}(E) N\mathfrak{p}^{-s} + N\mathfrak{p}^{1-2s})^{-1} \prod_{\mathfrak{p}\mid\mathfrak{f}} (1 - a_{\mathfrak{p}}(E) N\mathfrak{p}^{-s})^{-1}$$

is the L-function of E/K. Let F be an abelian extension of K. We put $A = K[G] \supset \mathfrak{A} = \mathcal{O}_K[G]$, where $G = \mathrm{Gal}(F/K)$. For any Dirichlet character χ from G to $\mathbb{C}^\times$ we write $\varphi_\chi = \chi \cdot \varphi$. Let $M(\varphi)$ be the motive associated to grössencharacter φ over K. The A-equivariant L-function of $M(\varphi)$ is defined by

$$L({}_A M(\varphi), s) := \left(L(\varphi_\chi, s)\right)_{\chi \in \widehat{G}}.$$

For any positive integer r we have $\mathrm{ord}_{s=-r} L(\varphi_\chi, s) = 1$ by the functional equation. Then we consider the Tamagawa number conjecture for the L-value

$$L^*({}_A M) = \left(L'(\varphi_\chi, -r)\right)_{\chi \in \widehat{G}} \in \mathbb{R}[G],$$

where $M = M(\varphi)(-r)$. We put $E_F = E \times_K \mathrm{Spec}\, F$ and let $\mathcal{A}$ be the Neron model of $\mathrm{Res}_{F/K} E_F$. The finite part of motivic cohomology is given by

$$H_f^1(M) = (K_{2r+2}(\mathcal{A}) \otimes \mathbb{Q})^{(r+2)}.$$

Deninger constructed a rank one A-subspace $H_f^1(M)^{\mathrm{constr}}$ in $H_f^1(M)$. (The space $H_f^1(M)^{\mathrm{constr}}$ is generated by an element $\xi_{\mathfrak{f}}$ as an A-module. We will explain the definition of the element $\xi_{\mathfrak{f}}$ later.) We will use this subspace $H_f^1(M)^{\mathrm{constr}}$ instead of $H_f^1(M)$.

The Beilinson-Bloch conjecture predicts $H^1_f(M) = H^1_f(M)^{\mathrm{constr}}$, but this is still conjectural. Furthermore let

$$M_B := H^1_{\mathrm{B}}(E_F(\mathbb{C}), \mathbb{Q}(r+1))$$

be the Betti cohomology. For a prime ℓ let

$$M_\ell := H^1_{\text{ét}}(E_F \times_K \overline{K}, \mathbb{Q}_\ell(r+2))$$

be the ℓ-adic étale cohomology and we choose a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $\mathfrak{A}\otimes\mathbb{Z}_\ell$ lattice $T_\ell$in M_ℓ. Note that M_B and M_ℓ are realizations of the motive $M^*(1)$. We consider the Beilinson's regulator map

$$\rho_\infty : H^1_f(M)^{\mathrm{constr}} \otimes \mathbb{R} \overset{\cong}{\to} M_B \otimes \mathbb{R}$$

and the Soule's ℓ-adic regulator map

$$\rho_\ell : H^1_f(M)^{\mathrm{constr}} \otimes \mathbb{Q}_\ell \to H^1(\mathcal{O}_K[1/\mathfrak{m}\ell], M_\ell),$$

where $\mathfrak{m}$ is an ideal divisible by the conductor of extension F/K and the conductor of E. In this case, we will formulate a weaker version of equivariant Tamagawa number conjecture. We define the (modified) fundamental line $\Xi({}_AM)$ by

$$\Xi({}_AM) := \mathrm{Det}_A(H^1_f(M)^{\mathrm{constr}})^* \otimes_A \mathrm{Det}_A^{-1}(M_B)^* \in \mathit{Inv}(A).$$

Then ρ_∞ gives an A-equivariant isomorphism

$${}_A\vartheta_\infty : A \otimes \mathbb{R} \overset{\cong}{\to} \Xi(M).$$

Moreover Deninger proved the Beilinson's rationality conjecture as follows.

Theorem 6.1 (Deninger [8]). ${}_A\vartheta_\infty(L^*(M)^{-1}) \in \Xi(M) \otimes 1.$

Denote $A_\ell = A \otimes \mathbb{Q}_\ell$, $\mathfrak{A}_\ell = \mathfrak{A} \otimes \mathbb{Z}_\ell$. For this case, the Mot_ℓ-conjecture is equivalent to the following conjecture.

Conjecture 6.2 (Weak Leopoldt conjecture for M_ℓ). $H^2(\mathcal{O}_K[1/\mathfrak{m}\ell], M_\ell) = 0.$

Assuming the weak Leopoldt conjecture, ρ_ℓ gives an A_ℓ-equivariant isomorphism

$${}_A\vartheta_\ell : \Xi(M) \otimes \mathbb{Q}_\ell \overset{\cong}{\to} \mathrm{Det}_{A_\ell} R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], M_\ell).$$

Now we formulate the weak version of equivariant Tamagawa number conjecture for the Hecke character $\varphi = \varphi_E$ associated to an elliptic curve with complex multiplication by $\mathcal{O}_K$ over an imaginary quadratic field K.

Conjecture 6.3. *For any prime ℓ we have*

$$\mathfrak{A}_\ell \cdot {}_A\vartheta_\ell \circ {}_A\vartheta_\infty(L^*(M)^{-1}) = \mathrm{Det}_{\mathfrak{A}_\ell} R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], T_\ell).$$

as a lattice in $\mathrm{Det}_{A_\ell} R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], M_\ell)$.

We denote by $K(\mathfrak{m})$ the ray class field modulo $\mathfrak{m}$ of K. Our main result is the following.

Theorem 6.4 (Chida [7]). *Let ℓ be a prime satisfying $\ell \nmid 6[K(\mathfrak{m}) : K]$ and ℓ splits in K. Assume the weak Leopoldt conjecture for M_ℓ. Then we have*

$$\mathfrak{A}_\ell \cdot {}_A\vartheta_\ell \circ {}_A\vartheta_\infty(L^*(M)^{-1}) = \mathrm{Det}_{\mathfrak{A}_\ell} R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], T_\ell)$$

as a lattice in $\mathrm{Det}_{A_\ell} R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], M_\ell)$.

This is a generalization of Kings' result. In fact, we can generalize this result to Hecke character with infinite type (a, b), where $a, b \geq 0$(cf. [7]).

Remark 6.5.

1. By a functoriality of equivariant Tamagawa number conjecture, it suffices to give a proof for the case that $F = K(\mathfrak{m})$. So we will consider only this case.
2. If we assume the equivariant main conjecture for imaginary quadratic fields, then we can prove the weak version of the equivariant Tamagawa number conjecture for any prime $\ell > 3$.

§ 7. Computations on the regulator map

Let K be an imaginary quadratic field of class number one and E an elliptic curve over K with complex multiplication by $\mathcal{O}_K$. We denote $G_\mathfrak{m} = \mathrm{Gal}(K(\mathfrak{m})/K)$ for an ideal $\mathfrak{m}$ of O_K and $E' = E \times_K \mathrm{Spec}\, K(\mathfrak{m})$ for the base change of E. For χ a rational character of $G_\mathfrak{m}$ we consider a Hecke character

$$\varphi_\chi = \chi\varphi_E$$

with conductor $\mathfrak{f}$, where φ_E is the grössencharacter associated with E. We fix an isomorphism

$$\theta_{E'} : \mathcal{O}_K \xrightarrow{\cong} \mathrm{End}_{K(\mathfrak{m})}(E')$$

such that $\theta_{E'}^*(\alpha)\omega = \alpha\omega$ for any $\omega \in H^0(E, \Omega^1_{E'})$ and an embedding τ_0 of $K(\mathfrak{m})$ in $\mathbb{C}$ such that $j(E') = j(\mathcal{O}_K)$. Then $E'(\mathbb{C}) \cong \mathbb{C}/\Gamma$ where $\Gamma = \Omega\mathcal{O}_K$ for some $\Omega \in \mathbb{C}$. Let $\rho_\mathfrak{f} \in \mathbb{A}_K^\times$ be an idèle with the ideal $\mathfrak{f}$. By the theory of complex multiplication, there exists an unique element $f_\mathfrak{f} \in K^\times$ satisfying that the following two properties:

1. $f_{\mathfrak{f}}\mathcal{O}_K = (\rho_{\mathfrak{f}})$.

2. For any fractional ideal $\mathfrak{a} \subset K$ and any analytic isomorphism $\lambda : \mathbb{C}/\mathfrak{a} \to E(\mathbb{C})$, the following diagram commutes:

$$\begin{array}{ccc} K/\mathfrak{a} & \xrightarrow{f_{\mathfrak{f}}\cdot\rho_{\mathfrak{f}}^{-1}} & K/\mathfrak{a} \\ \big\downarrow\lambda & & \big\downarrow\lambda \\ E(K^{\mathrm{ab}}) & \xrightarrow{\left(\frac{K^{\mathrm{ab}}/K}{\rho_{\mathfrak{f}}}\right)} & E(K^{\mathrm{ab}}), \end{array}$$

where K^{ab} is the maximal abelian extension of K.

Then we define a divisor

$$_{\mathfrak{f}}\beta := ([\Omega f_{\mathfrak{f}}^{-1}]) \in E'[\mathfrak{f}](K(\mathfrak{m})).$$

For a choice of a square root of the discriminant d_K of K, we consider the map

$$\delta = (\mathrm{id}, \sqrt{d_K}) : E' \to E'^2 = E' \times_F E'$$

and let $\mathrm{pr} : E'^{r+1} = E'^r \times_F E' \to E'$ be the projection. Put $N = N_{K/\mathbb{Q}}\mathfrak{f}$.

Deninger constructed the Eisenstein symbol

$$\mathcal{E}_{\mathcal{M}} : \mathbb{Q}[E'[N]\backslash O]^0 \to H^{2r+2}_{\mathcal{M}}(E'^{2r+1}, \mathbb{Q}(2r+2))$$

and the Kronecker map

$$\mathcal{K}_{\mathcal{M}} : H^{2r+2}_{\mathcal{M}}(E'^{2r+1}, \mathbb{Q}(2r+2)) \to H^2_{\mathcal{M}}(M_{\varphi_\chi}, \mathbb{Q}(r+2)).$$

which is defined by composition of maps

$$\begin{aligned} H^{2r+2}_{\mathcal{M}}(E'^{2r+1}, \mathbb{Q}(2r+2)) &\xrightarrow{(\delta^r\times\mathrm{id})^*} H^{2r+2}_{\mathcal{M}}(E'^{r+1}, \mathbb{Q}(2r+2)) \\ &\xrightarrow{\mathrm{pr}_*} H^2_{\mathcal{M}}(E', \mathbb{Q}(r+2)) \\ &\longrightarrow H^2_{\mathcal{M}}(h^1(E'), \mathbb{Q}(r+2)) \\ &\longrightarrow H^2_{\mathcal{M}}(M_{\varphi_\chi}, \mathbb{Q}(r+2)). \end{aligned}$$

Now we need the result on the image of motivic element under the regulator map. Let

$$\rho_\infty : H^2_{\mathcal{M}}(M_{\varphi_\chi}, \mathbb{Q}(r+2)) \otimes \mathbb{R} \to H^1_B(M_{\varphi_\chi}, \mathbb{R}(r+1))$$

be the Beilinson's regulator map.

Theorem 7.1 (Deninger, [8][(2.11)]). *Let the notations as above. Then*

$$\rho_\infty(\mathcal{K}_{\mathcal{M}} \circ \mathcal{E}_{\mathcal{M}}({}_{\mathfrak{f}}\beta)) = t_{\varphi_\chi} L'(\overline{\varphi_\chi}, 0) e_\chi \eta,$$

where

$$t_{\varphi_\chi} = (-1)^{r-1} \frac{2^{r-1} N\mathfrak{f}^r \Phi(\mathfrak{m})}{(2r+1)!\Phi(\mathfrak{f})} \varphi_\chi(\rho_{\mathfrak{f}}),$$

$\Phi(\mathfrak{m}) = \mid (\mathcal{O}_K/\mathfrak{m})^\times \mid$, e_χ *is the projector associated to* χ *and* η *is a* A*-basis of* M_B *as in [8][Lemma 2.3].*

For any ideal $\mathfrak{f} \mid \mathfrak{m}$ we define the element

$$\xi_{\mathfrak{f}} := \mathcal{K}_{\mathcal{M}} \circ \mathcal{E}_{\mathcal{M}}({}_{\mathfrak{f}}\beta)$$

in the motivic cohomology $H^2_{\mathcal{M}}(M_{\varphi_\chi}, \mathbb{Q}(r+1))$. From the assumption that the class number of K is one, we have $\chi(\rho_{\mathfrak{f}}) = 1$. Also one can show that $\varphi_\chi(\rho_{\mathfrak{f}}) = f_{\mathfrak{f}}$. Therefore by the Deninger's theorem, we have

$$\rho_\infty(\xi_{\mathfrak{f}}) = (-1)^{r-1} \frac{2^{r-1} N\mathfrak{f}^r \Phi(\mathfrak{m})}{(2r+1)!\Phi(\mathfrak{f})} f_{\mathfrak{f}} L'(\overline{\varphi_\chi}, 0) e_\chi \eta.$$

Since

$$\Xi({}_AM)^\# = \mathrm{Det}_A^{-1}(H^1_f(M)^{\mathrm{constr}}) \otimes_A \mathrm{Det}_A(M_B),$$

we have

$$({}_A\vartheta^\#_\infty(L^*({}_AM)^{-1}))_\chi = (-1)^{r-1} \frac{2^{r-1} N\mathfrak{f}^r \Phi(\mathfrak{m})}{(2r+1)!\Phi(\mathfrak{f}) f_{\mathfrak{f}}} (\xi_{\mathfrak{f}})^{-1} \otimes e_\chi \eta$$

where $(\xi_{\mathfrak{f}})^{-1}$ is the χ-part of the dual basis of $H^1_f(M)^{\mathrm{constr}}$.

§ 8. Elliptic units

We will use elliptic units to describe the image of L-values under the ℓ-adic regulator map. Here we review some fundamental facts on elliptic units. For details, see de Shalit's book [9][Chapter 2]. Let $L = \mathbb{Z}\cdot\omega_1 + \mathbb{Z}\cdot\omega_2$ be a lattice in $\mathbb{C}$ with a basis (ω_1, ω_2) satisfying $\mathrm{Im}(\omega_1/\omega_2) > 0$. The Dedekind eta function is defined by

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - q_\tau^n); \quad q_\tau := e^{2\pi i \tau}$$

and we write

$$\eta^{(2)}(\omega_1, \omega_2) = \omega_2^{-1} 2\pi i \eta(\omega_1/\omega_2)^2.$$

This function depends on the choice of basis but the discriminant function

$$\Delta(L) = \Delta(\tau) = (2\pi i)^{12} \eta(\tau)^{24}$$

does not depend on the choice. We define a theta function by

$$\varphi(z,\tau) = e^{\pi iz\frac{z-\bar{z}}{\tau-\bar{\tau}}} q_\tau^{1/12}(q_z^{1/2} - q_z^{-1/2}) \prod_{n=1}^{\infty}(1 - q_z q_\tau^n)(1 - q_z^{-1} q_\tau^n)$$

where $q_z = e^{2\pi iz}$ and

$$\varphi(z;\omega_1,\omega_2) = \varphi(z/\omega_1, \omega_1/\omega_2).$$

For any pair of lattices $L \subseteq L'$ of index prime to 6 with bases $\omega = (\omega_1,\omega_2)$ and $\omega' = (\omega_1',\omega_2')$ satisfying $\mathrm{Im}(\omega_1/\omega_2) > 0$ and $\mathrm{Im}(\omega_1'/\omega_2') > 0$, it is shown by Robert in [22][Theorems 1,2] that there exists a unique choice of 12-th root of unity $C(\omega,\omega')$ so that the functions

$$\delta(L,L') := C(\omega,\omega')\eta^{(2)}(\omega)^{[L':L]}/\eta^{(2)}(\omega')$$

and

$$\psi(z,L,L') := \delta(L,L') \prod_{u\in T}(\wp(z;L) - \wp(u;L))^{-1}$$

only depends on the lattices L, L', where the set T is any set of representatives of $(L'\backslash\{O\})/(\pm 1 \ltimes L)$ and $\wp$ is the Weierstrass $\wp$-function associated to L, moreover ψ satisfies the distribution relation

$$\psi(z;K,K') = \prod_{i=1}^{[L:K]} \psi(z + t_i; L, L')$$

for any lattice $L \subseteq K$ so that $K \cap L' = L$ and where $K' = K + L'$ and $t_i \in K$ are a set of representatives of K/L. Then $\psi(z,L,L')$ is an elliptic function on the elliptic curve $E = \mathbb{C}/L$ with divisor $[L' : L](O) - \sum_{P\in L'/L}(P)$. Kato reproved Robert's result in a scheme theoretic context.

Lemma 8.1 (Kato [19] [Proposition 1.3]). *Let E/S be an elliptic curve over a base scheme S and $c : E \to \tilde{E}$ an S-isogeny of degree prime to 6. Then there is a unique function*

$${}_c\Theta_{E/S} \in \Gamma(E\backslash \ker(c), \mathcal{O}^\times)$$

satisfying that

1. $\mathrm{div}({}_c\Theta_{E/S}) = \deg(c)(0) - \sum_{P\in\ker(c)}(P)$.
2. *For any morphism $g : S' \to S$ we have $g_E^*({}_c\Theta_{E/S}) = {}_{c'}\Theta_{E'/S}$, where $g_E : E' := E\times_S S' \to E$ and c' is the base change of c.*
3. *For any S-isogeny $b : E \to E'$ of degree prime to $\deg(c)$ have $b_*({}_c\Theta_{E/S}) = {}_{c'}\Theta_{E'/S}$ where b_* is the norm map associated to the finite flat morphism $E\backslash\ker(c) \to E'\backslash\ker(c')$. Here c' is the isogeny $E' \to E'/b(\ker(c))$.*

4. For $S = \operatorname{Spec}\mathbb{C}, E = \mathbb{C}/L$ *and* $c : \mathbb{C}/L \to \mathbb{C}/\tilde{L}$ *for lattices* $L \subseteq \tilde{L}$ *we have*

$$_c\Theta_{E/S}(z) = \psi(z, L, \tilde{L}).$$

Let K be an imaginary quadratic field. For any integral ideal $\mathfrak{f} \neq 1$ and any (auxiliary) $\mathfrak{a}$ which is prime to $6\mathfrak{f}$ we define the elliptic unit by

$$_{\mathfrak{a}}z_{\mathfrak{f}} = \psi(1, \mathfrak{f}, \mathfrak{a}^{-1}\mathfrak{f}).$$

Lemma 8.2 ([9][Chapter II.2]). *The complex numbers* $_{\mathfrak{a}}z_{\mathfrak{f}}$ *satisfy the following properties:*

1. (Rationality) $_{\mathfrak{a}}z_{\mathfrak{f}} \in K(\mathfrak{f})$.

2. (Integrality)

$$_{\mathfrak{a}}z_{\mathfrak{f}} \in \begin{cases} \mathcal{O}^{\times}_{K(\mathfrak{f})} & \mathfrak{f} \text{ divisible by primes } \mathfrak{p} \neq \mathfrak{q} \\ \mathcal{O}^{\times}_{K(\mathfrak{f}),\{v|\mathfrak{f}\}} & \mathfrak{f} = \mathfrak{p}^n \text{ for some prime } \mathfrak{p}. \end{cases}$$

3. (Galois action) *For* $(\mathfrak{c}, \mathfrak{f}\mathfrak{a}) = 1$ *with Artin symbol* $\sigma_{\mathfrak{c}} \in \operatorname{Gal}(K(\mathfrak{f})/K)$ *we have*

$$_{\mathfrak{a}}z_{\mathfrak{f}}^{\sigma_{\mathfrak{c}}} = \psi(1; \mathfrak{c}^{-1}\mathfrak{f}, \mathfrak{c}^{-1}\mathfrak{a}^{-1}\mathfrak{f}).$$

4. (Norm compatibility) *For a prime ideal* $\mathfrak{p}$ *one has*

$$N_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})}({}_{\mathfrak{a}}z_{\mathfrak{p}\mathfrak{f}})^{w_{\mathfrak{f}}/w_{\mathfrak{p}\mathfrak{f}}} \begin{cases} {}_{\mathfrak{a}}z_{\mathfrak{f}} & \mathfrak{p} \mid \mathfrak{f} \neq 1 \\ {}_{\mathfrak{a}}z_{\mathfrak{f}}^{1-\sigma_{\mathfrak{p}}^{-1}} & \mathfrak{p} \nmid \mathfrak{f} \neq 1 \end{cases}$$

Remark 8.3. The relations in 2 of above lemma show the auxiliary nature of $\mathfrak{a}$. In the group ring $\mathbb{Q}[G_{\mathfrak{f}}]$ the element $N\mathfrak{a} - \sigma_{\mathfrak{a}}$ becomes invertible and

$$z_{\mathfrak{f}} = (N\mathfrak{a} - \sigma_{\mathfrak{a}})^{-1}{}_{\mathfrak{a}}z_{\mathfrak{f}} \in \mathcal{O}^{\times}_{K(\mathfrak{f}),\{v|\mathfrak{f}\}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

is independent of choice of $\mathfrak{a}$.

§ 9. Computations on the ℓ-adic regulator map

In this section, we compute the image of the element $\xi_{\mathfrak{f}}$ under ℓ-adic regulator map. Write $K_\ell = K \otimes \mathbb{Q}_\ell = \prod_{\mathfrak{l}|\ell} K_{\mathfrak{l}}$ and from now on, we fix a prime $\mathfrak{l}$ of $\mathcal{O}_K$ dividing ℓ. For an ideal $\mathfrak{m}$ of $\mathcal{O}_K$, we set

$$\mathfrak{m} = \mathfrak{m}_0\mathfrak{l}^m$$

with $\mathfrak{l} \nmid \mathfrak{m}_0$.

To compute the image under the ℓ-adic regulator map, we use the ℓ-adic realization of the elliptic polylogarithm. Here we give a brief review of result of Kings [20]. Let E be an elliptic curve over a base scheme T. Denote by $\pi : E \to T$ the structural morphism. Put $U = E \backslash e$, where e is the zero section. Let $Pol_{\mathbb{Q}_\ell}$ be the elliptic polylogarithm sheaf on U (lisse pro-sheaf on U). For any divisor

$$\beta = \sum_{t \in E[N](T)\backslash e} n_t t \in \mathbb{Q}[E[N](T)\backslash e]$$

we define the ℓ-adic Eisenstein class associated to β by

$$(\beta^* Pol_{\mathbb{Q}_\ell})^m := \sum_{t \in E[N]\backslash e} n_t(\sigma^m \mathrm{pr}_t t^* Pol_{\mathbb{Q}_\ell}) \in H^1(T, \mathrm{Sym}^m \mathcal{H}_{\mathbb{Q}_\ell}(1)),$$

where $\mathcal{H}_{\mathbb{Q}_\ell} := \underline{\mathrm{Hom}}_T(R^1\pi_*\mathbb{Q}_\ell, \mathbb{Q}_\ell)$ and σ^m, pr_t are suitable projections. For an ideal $\mathfrak{a}$ with $(N\mathfrak{a}, \ell N) = 1$, consider $[\mathfrak{a}] : E \to E$ any isogeny of degree $N\mathfrak{a}$. Kings gave an explicit description of the ℓ-adic Eisenstein class associated β using the elliptic units.

Now we assume $(\ell, N) = 1$.

Theorem 9.1 (Kings [20][Theorem 4.2.9]). *Let the notation be as above. For any $m > 0$, the ℓ-adic Eisenstein class*

$$N\mathfrak{a}([\mathfrak{a}]^{\otimes m} N\mathfrak{a} - 1)(\beta^* Pol_{\mathbb{Q}_\ell})^m)$$

is given by

$$\pm\frac{1}{m!}(\delta \sum_{t \in E[N](T)\backslash e} n_t \sum_{[\ell^n]t_n = t} {}_{\mathfrak{a}}\Theta_{E/T}(-t_n)\tilde{t_n}^{\otimes m})_n$$

in the cohomology group $H^1(T, \mathrm{Sym}^m \mathcal{H}_{\mathbb{Q}_\ell}(1))$, where δ is the boundary map, $\tilde{t_n}$ is the projection of t_n to $E[\ell^n]$.

The following result gives the relation between the image of $\mathcal{E}_{\mathcal{M}}(\beta)$ under ℓ-adic regulator map $\rho_{\text{ét},\ell}$ and ℓ-adic Eisenstein class associated to β.

Theorem 9.2 (Kings [20][Theorem 1.2.5]). *For a divisor*

$$\beta = \sum_{t \in E[N](T)\backslash e} n_t t \in \mathbb{Q}[E[N](T)\backslash e],$$

we have that

$$\rho_{\text{ét},\ell}(\mathcal{E}_{\mathcal{M}}(\beta)) = -N^{2m}(\beta^* Pol_{\mathbb{Q}_\ell})^m.$$

From these theorems, we have the following corollary.

Corollary 9.3. *For a divisor as in Theorem 9.2, we have that*

$$e_\chi \rho_{\text{ét},\ell}(\mathcal{K}_\mathcal{M} \circ \mathcal{E}_\mathcal{M}(\beta)) = \pm N^{2m} \frac{1}{m! N\mathfrak{a}([\mathfrak{a}]N\mathfrak{a} - 1)} \times \delta \left(\sum_{t \in E[N] \setminus e} n_t \sum_{[\ell^n] t_n = t} {}_{\mathfrak{a}}\Theta_{E/T}(-t_n) \otimes e_\chi(\tilde{t_n}^{\otimes m}) \right)_n .$$

We apply this result for the case $\beta = {}_\mathfrak{f}\beta$, $N = N\mathfrak{f}$, $m = 2r+1$, $T = \operatorname{Spec} \mathcal{O}_K \left[\frac{1}{\mathfrak{m}\ell}\right]$ and $\mathcal{H}_{\mathbb{Q}_\ell} = V_\ell E' := T_\ell E' \otimes \mathbb{Q}_\ell$. Now there is a commutative diagram

$$\begin{array}{ccc} H^{2r+2}_\mathcal{M}(E'^{2r+1}, \mathbb{Q}(2r+2)) & \xrightarrow{\mathcal{K}_\mathcal{M}} & H^2_\mathcal{M}(M_{\varphi_\chi}, \mathbb{Q}(r+2)) \\ \downarrow{\scriptstyle \rho_{\text{ét},\ell}} & & \downarrow{\scriptstyle \rho_{\text{ét},\ell}} \\ H^1(\mathcal{O}_K[1/\mathfrak{m}\ell], V_\ell E'(1)^{\otimes 2r+1}) & \xrightarrow{\mathcal{K}_\ell} & H^1(\mathcal{O}_K[1/\mathfrak{m}], M_\ell). \end{array}$$

Also we have

$$\mathcal{K}_\ell((\tilde{t_n}^{\otimes 2r+1})_n) = \delta(e_\chi((\tilde{t_n})_n) \otimes (\gamma(\tilde{t_n}^{\otimes r}))_n)$$

in the cohomology group $H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}], V_\ell E'(1))$, where $\gamma(\tilde{t_n}^{\otimes r}) := \langle \tilde{t_n}, \overline{d_K \tilde{t_n}} \rangle^{\otimes r}$ and $\langle \cdot, \cdot \rangle : E[\ell^n] \times E[\ell^n] \to \mu_{\ell^n}$ is the Weil pairing. Therefore we conclude that

$$e_\chi \rho_{\text{ét},\mathfrak{l}}(\mathcal{K}_\mathcal{M} \circ \mathcal{E}_\mathcal{M}({}_\mathfrak{f}\beta)) = \frac{\pm N\mathfrak{f}^{2r+1}}{(2r+1)! N\mathfrak{a}(\varphi_\chi(\mathfrak{a}) N\mathfrak{a} - 1)} \times \delta \left(\sum_{[\mathfrak{l}^n] t_n = \beta} {}_{\mathfrak{a}}\Theta_{E/\operatorname{Spec} \mathcal{O}_K[1/\mathfrak{m}\ell]}(-t_n) \otimes e_\chi((\tilde{t_n})_n \otimes \gamma(\tilde{t_n}^{\otimes r})_n) \right)_n .$$

in the cohomology $H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], V_\mathfrak{l} E(r))$, where

$$\rho_{\text{ét},\mathfrak{l}} : H^1_f(M) \otimes_K K_\mathfrak{l} \to H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], V_\mathfrak{l} E(r)).$$

is the $\mathfrak{l}$-part of the ℓ-adic regulator map. From now on, we denote

$${}_\mathfrak{a}\Theta_E(z) = {}_\mathfrak{a}\Theta_{E/\operatorname{Spec} \mathcal{O}_K[1/\mathfrak{m}\ell]}(z)$$

for simplicity.

From Corollary 9.3 and a computation on the action of the Frobenius endomorphism, one can get an explicit description of the image of $\xi_\mathfrak{f}$ by the ℓ-adic regulator map.

Theorem 9.4. *Setting $w_\mathfrak{f} = \#\{u \in \mathcal{O}_K^\times \mid u \equiv 1 \mod \mathfrak{f}\}$, we have that*

$$e_\chi \rho_{\text{ét},\mathfrak{l}}(\xi_\mathfrak{f}) = \pm \frac{w_\mathfrak{f} N\mathfrak{f}^{2r+1}}{(2r+1)!} (1 - \overline{\varphi_\chi}(\mathfrak{l}) N\mathfrak{p}^r) \delta(\operatorname{Tr}_{K(\mathfrak{l}^n\mathfrak{f})/K(\mathfrak{f})} y_{\mathfrak{l}^n\mathfrak{f}})_n$$

in the cohomology $H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], V_{\mathfrak{l}}E(r))$, *where*

$$y_{\mathfrak{l}^n\mathfrak{f}} := \frac{1}{N\mathfrak{a}(\varphi_\chi(\mathfrak{a})N\mathfrak{a}-1)}{}_{\mathfrak{a}}\Theta_E(-s_n)\otimes e_\chi(\tilde{s_n}\otimes\gamma(\tilde{s_n})^{\otimes r}),$$

where s_n *is a primitive* ℓ^n*-th root of* $t=\Omega f_{\mathfrak{f}}^{-1}$.

To prove the equivariant Tamagawa number conjecture, we need to compute the image of L-values in the cohomology with compact support. For convenience, we define the complex $\Delta_\varphi(K(\mathfrak{m}))$ by

$$\Delta_\varphi(K(\mathfrak{m})) := \mathrm{RHom}_{\mathbb{Z}_\ell}(R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], T_\ell(-1)^*), \mathbb{Z}_\ell)[-3],$$

then

$$\mathrm{Det}_{A_\ell}\Delta_\varphi(K(\mathfrak{m}))\otimes\mathbb{Q}_\ell \cong \mathrm{Det}_{A_\ell}(R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], M_\ell^*))^\#$$

by a property of determinant functor, where $A_\ell = A\otimes\mathbb{Q}_\ell$. By Shapiro's lemma and Artin-Verdier duality, we have

$$H^1(\Delta_\varphi(K(\mathfrak{m}))) \cong H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], T_\ell E(r))$$

and

$$H^2(\Delta_\varphi(K(\mathfrak{m}))) \cong H^2(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], T_\ell E(r))\oplus T_\ell(-1)$$

and $H^i(\Delta_\varphi(K(\mathfrak{m})))=0$ if $i\neq 1,2$. Assuming the weak Leopoldt conjecture, the ℓ-adic regulator map induces the isomorphism

$$\begin{aligned}
\Xi({}_AM)^\#\otimes\mathbb{Q}_\ell &\cong \mathrm{Det}_{A_\ell}^{-1}(H_f^1(M)^{\mathrm{constr}}\otimes\mathbb{Q}_\ell)\otimes\mathrm{Det}_{A_\ell}(M_B\otimes\mathbb{Q}_\ell)\\
&\xrightarrow{\cong} \mathrm{Det}_{A_\ell}^{-1}(H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], V_\ell E(r)))\otimes\mathrm{Det}_{A_\ell}M_\ell(-1)\\
&\overset{(*)}{\xrightarrow{\cong}} \mathrm{Det}_{A_\ell}^{-1}(H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], V_\ell E(r)))\otimes\mathrm{Det}_{A_\ell}M_\ell(-1)\\
&\xrightarrow{\cong} \mathrm{Det}_{A_\ell}(\Delta_\varphi(K(\mathfrak{m}))\otimes\mathbb{Q}_\ell),
\end{aligned}$$

where the map $(*)$ is multiplication with the Euler factors $\prod_{\mathfrak{p}|\mathfrak{m}\ell}(1-\mathrm{Frob}_\mathfrak{p})^{-1\#}$ which comes from the difference of trivializations. Using Theorem 9.4 and the relation $\rho_{\text{ét},\ell} = \bigoplus_{\mathfrak{l}|\ell}\rho_{\text{ét},\mathfrak{l}}^{\oplus\mathrm{ord}_\ell\mathfrak{l}}$, we have the following result for the image of L-value.

Theorem 9.5. *Assume the weak Leopoldt conjecture for* M_ℓ'. *The* χ*-part of* ${}_A\vartheta_\ell\circ{}_A\vartheta_\infty(L^*({}_AM)^{-1})^\#$ *is given by*

$$\pm\prod_{\mathfrak{p}|\mathfrak{m}_0}(1-\overline{\varphi_\chi}(\mathfrak{p})N\mathfrak{p}^r)^{-1}\cdot\frac{\Phi(\mathfrak{m})2^{r-1}f_\mathfrak{f}}{w_\mathfrak{f}\Phi(\mathfrak{f})N\mathfrak{f}^{r+1}}(\delta(\mathrm{Tr}_{K(\ell^n\mathfrak{f})/K(\mathfrak{f})}y_{\ell^n\mathfrak{f}}))_n^{-1}\otimes(s_n\otimes\zeta_{\ell^n}^{\otimes r}\cdot e_\chi\tau_0)_n,$$

where $\zeta_{\ell^n}=e^{2\pi i/\ell^n}$ *and* τ_0 *is the fixed embedding of* $K(\mathfrak{m})$ *into* $\mathbb{C}$ *as in Section 7.*

§ 10. The basis of integral lattices

We recall the equivariant main conjecture for imaginary quadratic fields which is formulated by Johnson [16]. Put

$$\Lambda = \varprojlim_n \mathbb{Z}_\ell[G_{\mathfrak{m}\ell^n}] \cong \mathbb{Z}_\ell[G^{\mathrm{tor}}_{\mathfrak{m}\ell^\infty}][[S_1, S_2]]$$

where $G^{\mathrm{tor}}_{\mathfrak{m}\ell^\infty}$ is the torsion subgroup of $G_{\mathfrak{m}\ell^\infty} = \varprojlim_n G_{\mathfrak{m}\ell^n}$. Then Λ is a finite product of complete local 3-dimensional Cohen-Macauley ring. Λ is regular if and only if $\ell \nmid G^{\mathrm{tor}}_{\mathfrak{m}\ell^\infty}$. Define a rank one free Λ-module

$$\mathbb{T} = \varprojlim_n H^0(\mathrm{Spec}\,(K(\mathfrak{m}\ell^n))\otimes_K\overline{\mathbb{Q}}, \mathbb{Z}_\ell).$$

and a perfect complex of Λ-modules

$$\Delta^\infty := R\mathrm{Hom}_\Lambda(R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], \mathbb{T}), \Lambda)^{\#}[-3]$$

Then $H^i(\Delta^\infty) = 0$ for $i \neq 1, 2$ and there is a canonical isomorphism

$$H^1(\Delta^\infty) \cong U^\infty_{\{v|\mathfrak{m}\ell\}} := \varprojlim_n \mathcal{O}_{K(\mathfrak{m}_0\ell^n)}[1/\mathfrak{m}\ell]^\times$$

and a short exact sequence

$$0 \to P^\infty_{\{v|\mathfrak{m}\ell\}} \to H^2(\Delta^\infty) \to X^\infty_{\{v|\mathfrak{m}\ell\infty\}} \to 0$$

where

$$P^\infty_{\{v|\mathfrak{m}\ell\}} := \varprojlim_n \mathrm{Pic}(\mathcal{O}_{K(\mathfrak{m}\ell^n)}[1/\mathfrak{m}\ell])\otimes_{\mathbb{Z}}\mathbb{Z}_\ell$$

$$X^\infty_{\{v|\mathfrak{m}\ell\infty\}} := \varprojlim_n \mathrm{Ker}\left[\bigoplus_{v|\mathfrak{m}\ell\infty \text{ in } K(\mathfrak{m}\ell^n)} \mathbb{Z} \xrightarrow{\Sigma} \mathbb{Z}\right] \otimes_{\mathbb{Z}}\mathbb{Z}_\ell.$$

These limits are taken with respect to Norm maps.

We recall that elliptic units discussed in section 8 form a norm compatible system. Let $\mathfrak{m}_0$ be the prime to ℓ-part of $\mathfrak{m}$. We consider $K(\mathfrak{m}_0\ell^n)$ as a subfield of $\mathbb{C}$ and denote the corresponding archimedean place by $\tau_{\mathfrak{m}_0\ell^n}$. Set

$${}_{\mathfrak{a}}z_{\mathfrak{m}_0\ell^\infty} = ({}_{\mathfrak{a}}z_{\mathfrak{m}_0\ell^n})_n \in H^1(\Delta^\infty),$$

$$\tau = (\tau_{\mathfrak{m}_0\ell^n})_n \in Y^\infty_{\{v|\mathfrak{m}_0\ell\infty\}} := \varprojlim_n Y_{\{v|\mathfrak{m}_0\ell\infty\}}(K(\mathfrak{m}_0\ell^n)) \otimes \mathbb{Z}_\ell,$$

where ${}_{\mathfrak{a}}z_{\mathfrak{f}}$ is the elliptic unit defined by

$${}_{\mathfrak{a}}z_{\mathfrak{f}} = \psi(1; \mathfrak{f}, \mathfrak{a}^{-1}\mathfrak{f})$$

and

$$Y_{\{v|\mathfrak{m}_0\ell\infty\}}(K(\mathfrak{m})) := \bigoplus_{v|\mathfrak{m}\ell\infty \text{ in } K(\mathfrak{m})} \mathbb{Z}.$$

We fix an embedding $\overline{\mathbb{Q}}_\ell \to \mathbb{C}$ and identify $\widehat{G}$ with the set of $\overline{\mathbb{Q}}_\ell$-valued characters. The total ring of fractions

$$Q(\Lambda) \cong \prod_{\psi \in (\widehat{G}^{\mathrm{tor}}_{\mathfrak{m}\ell^\infty})^{\mathbb{Q}_\ell}} Q(\psi)$$

of Λ is a product of fields indexed by the $\mathbb{Q}_\ell$-rational characters of $G^{\mathrm{tor}}_{\mathfrak{m}\ell^\infty}$. Since for any place w of K, the $\mathbb{Z}[G_{\mathfrak{m}\ell^n}]$-module $Y_{\{v|w\}}(K_{\mathfrak{m}\ell^n})$ is induced from the module $\mathbb{Z}$ on the decomposition group $D_w \subseteq G_{\mathfrak{m}\ell^n}$. For $w = \infty$ we have $[G_{\mathfrak{m}\ell^n} : D_w] = [K(\mathfrak{m}\ell^n) : K]$, and for non-archimedean place w the index $[G_{\mathfrak{m}\ell^n} : D_w]$ is bounded as $n \to \infty$. Then one can show

$$\dim_{Q(\psi)} U^\infty_{\{v|\mathfrak{m}\ell\}} \otimes_\Lambda Q(\psi) = \dim_{Q(\psi)} Y^\infty_{\{v|\mathfrak{m}_0\ell\infty\}} \otimes_\Lambda Q(\psi) = 1$$

for all characters ψ. Note that the inclusion $X^\infty_{\{v|\mathfrak{m}_0\ell\infty\}} \subseteq Y^\infty_{\{v|\mathfrak{m}_0\ell\infty\}}$ becomes an isomorphism after tensoring with $Q(\psi)$ and that $e_\psi({}_{\mathfrak{a}}z^{-1}_{\mathfrak{m}_0\ell^\infty} \otimes \tau)$ is a basis of

$$\mathrm{Det}^{-1}_{Q(\psi)}(U^\infty_{\{v|\mathfrak{m}\ell\}} \otimes_\Lambda Q(\psi)) \otimes \mathrm{Det}_{Q(\psi)}(X^\infty_{\{v|\mathfrak{m}_0\ell\infty\}} \otimes_\Lambda Q(\psi)) \cong \mathrm{Det}_{Q(\psi)}(\Delta^\infty \otimes_\Lambda Q(\psi)),$$

where e_ψ is the projector associated to ψ. Hence we get a $Q(\psi)$-basis

$$\mathcal{L} := (N\mathfrak{a} - \sigma_{\mathfrak{a}})_{\mathfrak{a}} z^{-1}_{\mathfrak{m}_0\ell^\infty} \otimes \tau$$

of $\mathrm{Det}_{Q(\psi)}(\Delta^\infty \otimes_\Lambda Q(\psi))$.

Conjecture 10.1 (Equivariant main conjecture, [16][Conjecture 3]). *There is an identity of invertible Λ-submodules*

$$\Lambda \cdot \mathcal{L} = \mathrm{Det}_\Lambda \Delta^\infty$$

of $\mathrm{Det}_{Q(\Lambda)}(\Delta^\infty \otimes_\Lambda Q(\Lambda))$.

Remark 10.2. Using computations analogous to [11][Theorem 5.2] one can show that the equivariant main conjecture for imaginary quadratic fields for primes ℓ satisfying that ℓ splits in K and $\ell \nmid G^{\mathrm{tor}}_{\mathfrak{m}\ell^\infty}$, which is implied by Rubin's work [23][Theorem 4.1].

To prove Theorem 6.4, we need to show that the identity in the equivariant main conjecture descends to

$$\mathbb{Z}[G_{\mathfrak{m}}] \cdot {}_A\theta_\ell \circ {}_A\theta_\infty(L^*({}_AM)^{-1})^\# = \mathrm{Det}_{\mathbb{Z}_\ell[G_{\mathfrak{m}}]} \Delta_\varphi(K(\mathfrak{m}))$$

in $\mathrm{Det}_{\mathbb{Q}_\ell[G_\mathfrak{m}]}\Delta_\varphi(K(\mathfrak{m})\otimes\mathbb{Q}_\ell)$. For this purpose, we introduce the twisting lemma for this case.

Now $T_\ell(-1)^*$ is a G_K-stable $\mathfrak{A}_\ell$-lattice in $M_\ell(-1)^*$. The action of G_K on $T_\ell(-1)^*$ factors through a character

$$\kappa : G_{\mathfrak{m}_0\ell^\infty} \to \mathfrak{A}_\ell^\times$$

and we also denote by $\kappa : \Lambda \to \mathfrak{A}_\ell$ the corresponding ring homomorphism. Let $(\xi_n)_{n\geq 0}$ be a $\mathfrak{A}_\ell$-basis of T_ℓ with the uniformization $E(\mathbb{C}) \cong \mathbb{C}/f_\mathfrak{f}\Omega\mathcal{O}_K$.

Lemma 10.3. (a) *There is a natural isomorphism*

$$\Delta^\infty\otimes^{\mathbb{L}}_{\Lambda,\kappa}\mathfrak{A}_\ell \cong \mathrm{R}\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], T_\ell(-1)^*)^*[-3] = \Delta_{\varphi_\chi}(K(\mathfrak{m})).$$

(b) *The image of an element*

$$u = (u_n)_{n\geq 0} \in \varprojlim_n H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], \mathbb{Z}/\ell^n\mathbb{Z}(1)) \cong U^\infty_{\{v|\mathfrak{m}\ell\}} = H^1(\Delta^\infty)$$

under the induced isomorphism

$$H^1(\Delta^\infty)\otimes^{\mathbb{L}}_{\Lambda,\kappa}\mathfrak{A}_\ell \cong H^1(\mathcal{O}_K[1/\mathfrak{m}\ell], T_\ell) \cong H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], T_\ell E(r))$$

is given by

$$\mathrm{Tr}_{K(\mathfrak{m}\ell^n)/K(\mathfrak{m})}(u_n \cup \xi_n)_{n\geq 0}.$$

(c) *The image of an element*

$$s = (s_n)_{n\geq 0} \in \varprojlim_n \mathbb{Z}/\ell^n\mathbb{Z}[G_{\mathfrak{m}_0\ell^n}]\cdot\tau = Y^\infty_{\{v|\infty\}}$$

under the isomorphism $Y^\infty_{\{v|\infty\}}\otimes_{\Lambda,\kappa}\mathfrak{A}_\ell \cong H^0(\mathrm{Spec}\,(K\otimes_\mathbb{Q}\mathbb{R}), M_\ell) = M_\ell(-1)$ *is given by*

$$(s_n \cup \xi_n)_{n\geq 0}.$$

Let R be a direct factor of A_ℓ which is a field, $\mathfrak{q}$ the kernel of the map $\kappa : \Lambda \to A_\ell \to R$ and $\Lambda_\mathfrak{q}$ the localization of Λ at $\mathfrak{q}$. Then $\mathcal{L}$ is a basis of $\mathrm{Det}_{\Lambda_\mathfrak{q}}\Delta^\infty_\mathfrak{q}$. Denote by $\mathcal{L}\otimes 1$ the image of $\mathcal{L}$ under the determinant

$$\mathrm{Det}_{\Lambda_\mathfrak{q}}\Delta^\infty_\mathfrak{q}\otimes_{\Lambda_\mathfrak{q},\kappa}A_\ell \cong \mathrm{Det}_{A_\ell}\mathrm{R}\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], T_\ell(-1)^*)^*[-3]$$

of the isomorphism Lemma 10.3 (a) .

The following theorem gives the image of $\mathcal{L}$ in the lattice $\mathrm{Det}_{\mathfrak{A}_\ell}\Delta_{\varphi_\chi}(K(\mathfrak{m}))$.

Theorem 10.4. *Assume the weak Leopoldt conjecture for M_ℓ. Then the χ-part of the image of $\mathcal{L}\otimes 1$ under*

$$\mathrm{Det}_{A_\ell}\mathrm{R}\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], T_\ell(-1)^*)^*[-3] \cong \mathrm{Det}_{A_\ell}\mathrm{R}\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], T_\ell)^\# \cong \Xi({}_AM)^\#\otimes_A A_\ell$$

is given by

$$\prod_{\mathfrak{p}|\mathfrak{m}_0}(1-\overline{\varphi_\chi}(\mathfrak{p})N\mathfrak{p}^r)^{-1}\frac{\Phi(\mathfrak{m})f_{\mathfrak{f}}}{w_{\mathfrak{f}}\Phi(\mathfrak{f})N\mathfrak{f}^{r+1}}(\delta(\mathrm{Tr}_{K(\ell^n\mathfrak{f})/K(\mathfrak{f})}y_{\ell^n\mathfrak{f}}))_n^{-1}\otimes(t_n\otimes\zeta_{\ell^n}^{\otimes r})_n\cdot e_\chi\tau_0.$$

Proof of Theorem 6.4

Assuming the equivariant main conjecture, one can show that $\mathcal{L}\otimes 1$ is a basis of

$$\mathrm{Det}_{A_\ell}\mathrm{R}\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell],T_\ell)^\#$$

by the previous descent argument. Since 2 is unit in $\Lambda_{\mathfrak{q}}$ we have proved

$$\mathcal{L}\otimes 1={}_A\theta_\ell\circ{}_A\theta_\infty(L^*({}_AM)^{-1})^\#$$

by using Theorem 9.5 and Theorem 10.4. This completes the proof of Theorem 6.4. □

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