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Ramification and tame characters of a finite flat representation of rank two

By

Shin HATTORI*

Abstract

Let $K$ be a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field $F$ and uniformizer $\pi$. In this paper, we propose an example of the main theorem of the paper [10]. Namely, we calculate the conductor $c(\mathcal{G})$ in the sense of Abbes and Saito for a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ which is reducible, killed by $p$ and of rank $p^2$, and show that the $I_K$-module $\mathcal{G}(\bar{K})$ contains the fundamental character of level $c(\mathcal{G})$. For this purpose, we show that the Dieudonné functor of Breuil is compatible with the base extension $K(\pi^{1/p})/K$

§ 1. Introduction

Let $K$ be a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field $F$, $\pi = \pi_K$ be its uniformizer, $G_K$ be its absolute Galois group and $I_K$ be its inertia subgroup. For $j \in \mathbb{Q}_{>0}$, we define a tame character $\theta_j : I_K \to \bar{F}^\times$ to be $\theta_j^{k'}$, where $k'/l'$ is the prime-to-$p$-denominator part of $j$ mod $\mathbb{Z}$ ([12]). In other words, we set $\theta_j(\sigma) = (\sigma^{1/l'})^{k'} \mod \mathfrak{m}_K$, where $\mathfrak{m}_K$ is the maximal ideal of $\mathcal{O}_K$. We refer any of $\mathbb{F}_p$-conjugates of $\theta_j$ as the fundamental character of level $j$.

Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$. When $\mathcal{G}$ is killed by $p$ and monogenic, that is to say, when the affine algebra of $\mathcal{G}$ is generated over $\mathcal{O}_K$ by one element, it is well-known that the tame characters appearing in the $I_K$-module $\mathcal{G}(\bar{K})$ are determined by the slopes of the Newton polygon of a defining equation of $\mathcal{G}$, as follows.

**Proposition 1.1** ([12], Proposition 10). Let $\mathcal{G}$ be as above and write the affine algebra of $\mathcal{G}$ as $\mathcal{O}_K[T]/(f(T))$ with $f(0) = 0$. Let $s_1, \ldots, s_r$ be the negatives of the slopes of the Newton polygon of $f(T)$. Then the semi-simplification of the $I_K$-module $\mathcal{G}(\bar{K}) \otimes_{\mathbb{F}_p} \mathbb{F}_p$ is the direct sum of fundamental characters of level $s_i$.

On the other hand, for an elliptic modular form $f$ of level $N$ prime to $p$, we also have a description of the tame characters of the associated mod $p$ Galois representation

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$\rho_f$ ([9, Theorem 2.5, Theorem 2.6], [8, Section 4.3]). This is based on Raynaud's theory of prolongations of finite flat group schemes or the integral $p$-adic Hodge theory. However, for an analogous study of the associated mod $p$ Galois representation of a Hilbert modular form over a totally real number field, we encounter a local field of higher absolute ramification index. In this case, these two theories no longer work well and we need some other techniques to study the tame characters of a Galois representation.

In this paper, we show the following theorem, which suggests that the semi-simplification of a finite flat representation can be described by the ramification jumps of its finite flat model over $\mathcal{O}_K$.

**Theorem 1.2.** Let $\mathcal{G}$ be a finite flat group scheme which is reducible, killed by $p$ and of rank $p^2$. Let $c(\mathcal{G})$ be its conductor in the sense of [2], [3]. Then the $I_K$-module $\mathcal{G}(\overline{K})$ contains the fundamental character of level $c(\mathcal{G})$.

To prove the main theorem, firstly we show compatibility of the theory of Breuil ([5]) with the base extension from $K$ to $K_1 = K(\pi^{1/p})$ (Theorem 3.3). Using this theorem, we can write down a defining equation of $\mathcal{G}$ over $\mathcal{O}_{K_1}$ and calculate explicitly the tubular neighborhoods and conductor of $\mathcal{G}$ as in [10, Section 5].

In fact, we can show this more generally. In [10], we generalize Proposition 1.1 to the higher dimensional case (namely, the case where $\mathcal{G}$ is not monogenic) without any restriction on the absolute ramification index of $K$, on the residue field $F$ and on $\mathcal{G}$. There we show that we can, at least for the finite flat case, determine the semi-simplification of a Galois representation using the ramification theory of Abbes and Saito ([2], [3]). The main theorem of [10] is the following, whose proof is given there by totally different method from that of Theorem 1.2 in this paper.

**Theorem 1.3** ([10], Theorem 1.1).

Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$. Write $\{G^j\}_{j \in \mathbb{Q}_{>0}}$ for the ramification filtration of $\mathcal{G}$ in the sense of [2] and [3]. Then the graded piece $G^j(\overline{K})/G^{j+}(\overline{K})$ is killed by $p$ and the $I_K$-module $G^j(\overline{K})/G^{j+}(\overline{K}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ is the direct sum of fundamental characters of level $j$.

§ 2. Review of the ramification theory of Abbes and Saito

Let $K$ be a complete discrete valuation field with residue field $F$ which may be imperfect. Set $\pi = \pi_K$ to be a uniformizer of $K$. The separable closure of $K$ is denoted by $\overline{K}$ and the absolute Galois group of $K$ by $G_K$. In [2] and [3], Abbes and Saito defined the ramification theory of a finite flat $\mathcal{O}_K$-algebra of relative complete intersection. In this section, we gather the necessary definitions and briefly recall their theory.
Let $A$ be a finite flat $\mathcal{O}_K$-algebra and $\mathcal{A}$ be a complete Noetherian semi-local ring (with its topology defined by $\text{rad}(A)$) which is of formally smooth over $\mathcal{O}_K$ and whose quotient ring $A/\text{rad}(A)$ is of finite type over $F$. A surjection of $\mathcal{O}_K$-algebras $\mathcal{A} \to A$ is called an embedding if $A/\text{rad}(A) \to A/\text{rad}(A)$ is an isomorphism. For an embedding $(A \to A)$ and $j \in \mathbb{Q}_{>0}$, the $j$-th tubular neighborhood of $(A \to A)$ is the $K$-affinoid variety $X^j(A \to A)$ constructed as follows. Write $j = k/l$ with $k, l$ non-negative integers. Put $I = \text{Ker}(A \to A)$ and

$$A_0^{k,l} = A[I^l/\pi^k]^\wedge,$$

where $\wedge$ means the $\pi$-adic completion. Then $A_0^{k,l}$ is a quotient ring of the Tate algebra $\mathcal{O}_K\langle T_1, \ldots, T_r \rangle$ for some $r$. Its generic fiber $\mathcal{A}_K = A_0^{k,l} \otimes_{\mathcal{O}_K} K$ is independent of the choice of a representation $j = k/l$ ([3, Lemma 1.4]) and set

$$X^j(A \to A) = \text{Sp}(A_K^j).$$

We put $F(A) = \text{Hom}_{\mathcal{O}_K\text{-alg}}(A, \mathcal{O}_K)$ and

$$F^j(A) = \lim \pi_0(X^j(A \to A)_R).$$

Here $\pi_0(X^j_R)$ denotes the set of geometric connected components of a $K$-affinoid variety $X$ and the projective limit is taken in the category of embeddings of $A$. Note that the projective family $\pi_0(X^j(A \to A)_R)$ is constant ([3, Section 1.2]). These define contravariant functors $F$ and $F^j$ from the category of finite flat $\mathcal{O}_K$-algebras to the category of finite $G_K$-sets. Moreover, there are morphisms of functors $F \to F^j$ and $F^{j'} \to F^j$ for $j' \geq j > 0$.

Suppose that $A$ is of relative complete intersection over $\mathcal{O}_K$ and $A \otimes_{\mathcal{O}_K} K$ is etale over $K$. Then the natural map $F(A) \to F^j(A)$ is surjective. The family $\{F(A) \to F^j(A)\}_{j \in \mathbb{Q}_{>0}}$ is separated, exhaustive and its jumps are rational ([2, Proposition 6.4]). The conductor of $A$ is defined to be

$$c(A) = \inf\{j \in \mathbb{Q}_{>0} | F(A) \to F^j(A) \text{ is an isomorphism}\}.$$

If $B$ is the affine algebra of a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ which is generically etale, then $B$ is of relative complete intersection (for example, [5, Proposition 2.2.2]) and the theory above can all be applied to $B$. By the functoriality, $F^j(B)$ is endowed with a $G_K$-module structure ([1, Lemme 2.1.1]) and the natural map $\mathcal{G}(\overline{K}) = F(B) \to F^j(B)$ is a $G_K$-homomorphism. Let $\mathcal{G}^j$ denote the schematic closure ([11]) in $\mathcal{G}$ of the kernel of this homomorphism. It is called the $j$-th ramification filtration of $\mathcal{G}$. We refer $c(B)$ as the conductor of $\mathcal{G}$, which is denoted also by $c(\mathcal{G})$. We put

$$\mathcal{G}^{j+}(\overline{K}) = \bigcup_{j' > j} \mathcal{G}^{j'}(\overline{K}).$$
We write the $j$-th tubular neighborhood of $B$ with respect to some embedding as $X^j_B$ by abuse of notation.

**Example 2.1.** For integers $0 \leq s_1, \ldots, s_r \leq e$, let $\mathcal{G} = \mathcal{G}(s_1, \ldots, s_r)$ denote the Raynaud $\mathbb{F}_{p^r}$-vector space scheme ([11]) over $\mathcal{O}_K$ defined by the $r$ equations

$$T_1^p = \pi^{s_1}T_2, T_2^p = \pi^{s_2}T_3, \ldots, T_r^p = \pi^{s_r}T_1.$$ 

We set

$$j_k = (ps_k + p^2s_{k-1} + \cdots + p^k s_r + p^{k+1}s_{r-1} + \cdots + p^r s_k) / (p^r - 1).$$ 

Then we have ([10, Theorem 5.5])

$$c(\mathcal{G}) = \sup_k j_k.$$ 

In this case, we see that the $I_K$-module $\mathcal{G}(\bar{K})$ is given by the fundamental character of level $c(\mathcal{G})$. For the proof, we refer to [10], where we take an appropriate syntomic cover of the affine algebra of $\mathcal{G}$ and compare its $j$-th tubular neighborhood with $X^j_B$.

§3. **Proof of Theorem 1.2**

In this section, we assume that $K$ is as in Section 1 and write its residue field as $k$ in accordance with [5].

Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ which is reducible, killed by $p$ and of $\mathbb{F}_p$-rank two. Namely, we have an exact sequence

$$0 \rightarrow \mathcal{G}(e - r) \rightarrow \mathcal{G} \rightarrow \mathcal{G}(e - s) \rightarrow 0$$

for some integers $0 \leq r, s \leq e$.

To state our result, let us recall the theory of filtered $\phi_1$-modules of Breuil ([5]). In the following, we take the divided power envelope of a $W$-algebra only with respect to the compatibility condition with the natural divided power structure on $pW$.

Let $e$ be the absolute ramification index of $K$, $W = W(k)$ and $\sigma$ be the Frobenius of $W$. We fix once and for all a uniformizer $\pi$ of $K$. Let $E(u) = u^e - pF(u)$ be the Eisenstein polynomial of $\pi$ over $W$ and set $S = S_\pi = (W[u]^{\text{PD}})^\wedge$, where the divided power envelope of $W[u]$ is taken with respect to an ideal $(E(u))$ and $\wedge$ means the $\pi$-adic completion. The ring $S$ is endowed with a $\sigma$-semilinear map $\phi : u \mapsto u^p$, which we also call Frobenius, and the natural filtration induced by the divided power structure. We set $\phi_1 = p^{-1}\phi|_{\text{Fil}^1 S}$ and $c = \phi_1(E(u)) \in S^\times$. We define $\phi, \phi_1$ and a filtration on $S_n = S/p^n$ similarly.

In [5], the following categories of filtered $\phi_1$-modules are defined. Set $'\mathcal{M}$ to be the category consisting of following data;
• an $S$-module $M$ and its $S$-submodule $\text{Fil}^1 M$ containing $\text{Fil}^1 S.M$,

• a $\phi$-semilinear map $\phi_1 : \text{Fil}^1 M \rightarrow M$ satisfying

$$\phi_1(s_1 m) = \phi_1(s_1) \phi(m),$$

where $s_1 \in \text{Fil}^1 S$, $m \in M$ and $\phi(m) = c^{-1} \phi_1 (E(u)m)$.

Let $\mathcal{M}_1$ be the full subcategory of $\mathcal{M}$ consisting of $M$ satisfying

• the $S_1$-module $M$ is free of finite rank,

• $\phi_1(\text{Fil}^1 M)$ generates $M$ as an $S$-module.

and $\mathcal{M}$ be the minimal full subcategory of $\mathcal{M}$ which contains $\mathcal{M}_1$ and stable under extension.

The category $\mathcal{M}$ is shown to be categorically anti-equivalent to the category $(p\text{-Gr}/\mathcal{O}_K)$ of finite flat group schemes over $\mathcal{O}_K$ which is killed by some $p$-power ([5]). Let us recall the definition of this equivalence. Let $\text{Spf}(\mathcal{O}_K)_{\text{syn}}$ be the category of formally syntomic $p$-adic formal schemes, endowed with the Grothendieck topology generated by the surjective families of formally syntomic morphisms. Write $(\text{Ab}/\mathcal{O}_K)$ for the category of abelian sheaves on $\text{Spf}(\mathcal{O}_K)_{\text{syn}}$. The sheaves $\mathcal{O}_{n,\pi}$ and $\mathcal{J}_{n,\pi}$ are defined by the formula

$$\mathcal{O}_{n,\pi}(\mathfrak{X}) = H^0_{\text{crys}}((\mathfrak{X}_n/S_n)_{\text{crys}}, \mathcal{O}_{\mathfrak{X}_n/S_n})$$

and

$$\mathcal{J}_{n,\pi}(\mathfrak{X}) = H^0_{\text{crys}}((\mathfrak{X}_n/S_n)_{\text{crys}}, \mathcal{J}_{\mathfrak{X}_n/S_n}),$$

where $\mathfrak{X}_n = \mathfrak{X}/p^n$. We also set $\mathcal{O}_{\infty,\pi} = \varprojlim \mathcal{O}_{n,\pi}$ and $\mathcal{J}_{\infty,\pi} = \varprojlim \mathcal{J}_{n,\pi}$. We let the crystalline Frobenius map be denoted by $\phi : \mathcal{O}_{n,\pi} \rightarrow \mathcal{O}_{n,\pi}$. We can define the natural morphism $\phi_1 : \mathcal{J}_{n,\pi} \rightarrow \mathcal{O}_{n,\pi}$ which makes the following diagram commutative.

$$\begin{array}{ccc}
\mathcal{J}_{n,\pi} & \xrightarrow{\phi_1} & \mathcal{O}_{n,\pi} \\
\uparrow & & \downarrow \times p \\
\mathcal{J}_{n+1,\pi} & \xrightarrow{\phi} & \mathcal{O}_{n+1,\pi}
\end{array}$$

Let $\mathcal{G} \in (p\text{-Gr}/\mathcal{O}_K)$ and $M \in \mathcal{M}$. Define

$$\text{Mod}_K(\mathcal{G}) = \text{Hom}_{(\text{Ab}/\mathcal{O}_K)}(\mathcal{G}, \mathcal{O}_{\infty,\pi})$$

and

$$\text{Gr}_K(M) = \text{Hom}_{\mathcal{M}}(M, \mathcal{O}_{\infty,\pi}).$$

Then the main theorem of [5] is the following.
Theorem 3.1 ([5]). The functor $\text{Gr}_K$ defines an anti-equivalence of categories $\mathcal{M} \to (p\text{-Gr}/\mathcal{O}_K)$ and its quasi-inverse is $\text{Mod}_K$.

Now let us return to our $\mathcal{G}$. Let $M = \text{Mod}_K(\mathcal{G})$ be the filtered $\phi_1$-module of $\mathcal{G}$. Replacing $K$ with an unramified extension, we may assume that we have an exact sequence in $\mathcal{M}$

$$0 \to M(s) \to M \to M(r) \to 0,$$

where $M(s)$ is the filtered $\phi_1$-module defined by $M(s) = S_1 e$, Fil$^1 M(s) = u^s S_1 e$ and $\phi_1(u^s e) = e$. By [7, Lemma 5.2.2], we may assume that $\tilde{M} = M/\text{Fil}^pS.M$ is of the following type;

- $\tilde{M} = \tilde{S}_1 e_0 \oplus \tilde{S}_1 e_1$, where $\tilde{S}_1 = k[u]/(u^{ep})$
- Fil$^1 \tilde{M} = \langle u^s e_0, u^r e_1 + fe_0 \rangle$, where $f \in u^{\sup(0,r+s-e)} \tilde{S}_1$
- $\phi_1(u^s e_0) = e_0$ and $\phi_1(u^r e_1 + fe_0) = e_1$.

Put $m = v_u(f)$. Then we have the following theorem.

Theorem 3.2. If $s, m \geq r$, then $c(\mathcal{G}) = p(e - r)/(p - 1)$. Otherwise, $c(\mathcal{G})$ is equal to

$$\begin{cases} \sup(p(e - r)/(p - 1), p(e - s)/(p - 1)) & \text{if } m \geq (ps - r)/(p - 1), \\ p(e - r)/(p - 1) + (r - m) & \text{if } m < (ps - r)/(p - 1). \end{cases}$$

Moreover, the $I_K$-module $\mathcal{G}(\bar{\mathcal{O}})$ contains the fundamental character of level $c(\mathcal{G})$.

To prove this theorem, we first write down a defining equation of $\mathcal{G}$. This is possible after taking a base extension from $K$ to $K_1 = K(\pi_1)$, where $\pi_1 = \pi^{1/p}$ ([5, Proposition 3.1.2]) and using the theorem below.

Theorem 3.3. Let $S' = S_{\pi_1}$ be the $p$-adic completion of the divided power envelope constructed as $S$ starting from $E_1(v) = E(v^p) \in W[v]$ and consider a map $S \to S'$ defined by $u \mapsto v^p$. Then we have a canonical isomorphism of filtered $\phi_1$-modules

$$\text{Mod}_{K_1}(\mathcal{G} \otimes_{\mathcal{O}_K} \mathcal{O}_{K_1}) \simeq \text{Mod}_K(\mathcal{G}) \otimes_S S'.$$

Then we can calculate the tubular neighborhoods of $\mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K_1}$ and its conductor. As for the assertion on the tame character, we know from [6, Theorem 3.4.3] that it suffices to consider $\mathcal{G} \times_{\mathcal{O}_K} \mathcal{O}_{K_1}$. We can easily check from the shape of a defining equation of $\mathcal{G}$ over $\mathcal{O}_{K_1}$ that if $m > (ps - r)/(p - 1)$ and $s < r$, then the $I_{K_1}$-module $\mathcal{G}(\bar{\mathcal{O}})$ splits. Hence this assertion follows from the assertion on the conductor.

We omit the calculation here as it is just the same as in the proof of [10, Theorem 5.5]. In the rest of this paper, we prove Theorem 3.3.
Lemma 3.4. The $S$-module $S'$ is free of finite rank.

Proof. The $W[u]$-algebra $W[v]$ is free of finite rank. We have

$$ (E(u))W[v] = (E_1(v)). $$

Therefore $W[v]^{PD} = W[u]^{PD} \otimes_{W[u]} W[v]$ from [4, Proposition 3.21] and $W[u]^{PD} \rightarrow W[v]^{PD}$ is also free of finite rank. Thus

$$ (W[v]^{PD})^\wedge = (W[u]^{PD})^\wedge \otimes_{W[u]^{PD}} W[v]^{PD}. $$

This concludes the proof. \(\square\)

Let the categories of filtered $\phi_1$-modules over $S'$ be denoted by $'M'$ and $M'$. From the lemma above, we can define a filtered $\phi_1$-module structure on $M' = M \otimes_S S'$ for any $M \in 'M$ by $\text{Fil}^M M' = (\text{Fil}^M M) \otimes_S S'$ and $\phi_1, M' = \phi_1 \otimes \phi$. If $M \in M$, then we have $M' \in M'$.

For a presheaf $\mathcal{F}$ on $\text{Spf}(O_K)_{\text{syn}}$, we let $\mathcal{F}|_{O_K}$ denote the restriction of $\mathcal{F}$ to $\text{Spf}(O_K)_{\text{syn}}$. If $\mathcal{F}$ is a sheaf on $\text{Spf}(O_K)_{\text{syn}}$, then $\mathcal{F}|_{O_K}$ is also a sheaf on $\text{Spf}(O_K)_{\text{syn}}$. By [5, Corollaire 2.3.3], we have the following exact sequences in $(\text{Ab}/O_K)$.

(1) \[ 0 \rightarrow \mathcal{O}_{r, \pi}|_{O_K} \xrightarrow{\times_p} \mathcal{O}_{r+s, \pi}|_{O_K} \rightarrow \mathcal{O}_{s, \pi}|_{O_K} \rightarrow 0 \]

(2) \[ 0 \rightarrow \mathcal{J}_{r, \pi}|_{O_K} \xrightarrow{\times_p} \mathcal{J}_{r+s, \pi}|_{O_K} \rightarrow \mathcal{J}_{s, \pi}|_{O_K} \rightarrow 0 \]

Consider an $O_{K_1}$-algebra

$$ A' = O_{K_1}(X_1', \ldots, X_r')/(f_1, \ldots, f_s), $$

where $O_{K_1}(X_1', \ldots, X_r')$ is the $\pi$-adic completion $O_{K_1}[X_1', \ldots, X_r']^\wedge$ and $f_1, \ldots, f_s$ is a transversally regular sequence in that ring. Then $\text{Spf}(A') \in \text{Spf}(O_{K_1})_{\text{syn}}$. Put

$$ A_i' = O_{K_1}(X_0'^{p^{-i}}, \ldots, X_r'^{p^{-i}})/(X_0' - \pi, f_1, \ldots, f_s) $$

and $A_i' = \varprojlim A_i'$. Note that the formal scheme $\text{Spf}(A_i')$ is a covering of $\text{Spf}(A')$ in $\text{Spf}(O_{K_1})_{\text{syn}}$. The $W$-algebra $A_i'$ is isomorphic to

$$ O_K[T]/(T^p - \pi)(X_0'^{p^{-i}}, \ldots, X_r'^{p^{-i}})/(X_0' - T, f_1, \ldots, f_s) $$

$$ = W[u, T]/(E(u), T^p - u)(X_0'^{p^{-i}}, \ldots, X_r'^{p^{-i}})/(X_0' - T, f_1, \ldots, f_s) $$

$$ = W(X_0'^{p^{-i}}, \ldots, X_r'^{p^{-i}})/(E(X_0'^p), f_1, \ldots, f_s). $$

We also set

$$ A_{\infty}'' = A_{\infty}/p = k[X_0'^{p^{-\infty}}, \ldots, X_r'^{p^{-\infty}}]/(X_0'^{ep}, \tilde{f}_1, \ldots, \tilde{f}_s), $$

where $\tilde{f}_i$ is the image of $f_i$. Put $O_{r, \pi}(A_{\infty}') = \varinjlim_i O_{r, \pi}(A_i')$ and $J_{r, \pi}(A_{\infty}') = \varinjlim_i J_{r, \pi}(A_i')$. 

\[ \text{RAMIFICATION AND CHARACTERS OF A FLAT REPRESENTATIONS} \]
Lemma 3.5. There exists a canonical isomorphism
\[ O_{n, \pi}(\mathfrak{A}'_{\infty}) = H_{\text{crys}}^{0}(\mathfrak{A}'_{\infty}/p^{n}/S_{n}) \to (W_{n}(A'_{\infty}) \otimes_{W, \sigma^{n}} W_{n}[u])^{\text{PD}}. \]

Here the divided power envelope is taken with respect to the kernel of a surjection
\[ W_{n}(A'_{\infty}) \otimes_{W, \sigma^{n}} W_{n}[u] \to \mathfrak{A}'_{\infty}/p^{n} \]
\[ (x_{0}, \ldots, x_{n-1}) \otimes 1 \mapsto \sum_{k=0}^{n-1} p^{k} \hat{x}_{k}^{p^{n-k}} \]
\[ 1 \otimes u \mapsto X_{0}^{p} \]
where \( \hat{x}_{k} \) denotes a lifting of \( x_{k} \) in \( \mathfrak{A}'_{\infty}/p^{n} \).

Proof. We repeat exactly the same argument as in [5, Lemme 2.3.2]. Indeed, this surjection induces a PD-thickening
\[ (W_{n}(A'_{\infty}) \otimes_{W, \sigma^{n}} W_{n}[u])^{\text{PD}} \to \mathfrak{A}'_{\infty}/p^{n} \]
of \( \mathfrak{A}'_{\infty}/p^{n} \) over \( S_{n} \) and thus we have the natural projection
\[ H_{\text{crys}}^{0}(\mathfrak{A}'_{\infty}/p^{n}/S_{n}) \to (W_{n}(A'_{\infty}) \otimes_{W, \sigma^{n}} W_{n}[u])^{\text{PD}}. \]
Its inverse map is defined as follows. For any affine PD-thickening \( U \to T \) of \( \mathfrak{A}'_{\infty}/p^{n} \) over \( S_{n} \), we have a map
\[ (W_{n}(A'_{\infty}) \otimes_{W, \sigma^{n}} W_{n}[u])^{\text{PD}} \to \Gamma(U, O_{U}) \]
\[ (x_{0}, \ldots, x_{n-1}) \otimes 1 \mapsto \sum_{k=0}^{n-1} p^{k} \hat{t}_{k}^{p^{n-k}} \]
\[ 1 \otimes u \mapsto u, \]
where \( \hat{t}_{k} \) is a lifting of \( x_{k} \) in \( \Gamma(T, O_{T}) \). This is a well-defined ring homomorphism, patches in the non-affine case and induces the inverse map of the natural projection. \( \square \)

Let us define a morphism \( \Psi_{M} : \text{Gr}_{K}(M)|_{O_{K_{1}}} \to \text{Gr}_{K_{1}}(M') \) of \( (\text{Ab}/O_{K_{1}}) \) as follows. For any \( \mathfrak{X}' \in \text{Spf}(O_{K_{1}})_{\text{syn}} \), we want to set
\[ \Psi_{M, \mathfrak{X}'} : \text{Hom}_{\mathcal{M}}(M, O_{n, \pi}(\mathfrak{X}')) \to \text{Hom}_{\mathcal{M}'}(M \otimes_{S'} S', O_{n, \pi_{1}}(\mathfrak{X}')) \]
by \( f \mapsto (m \otimes s' \mapsto s'.pr^{*}_{\mathfrak{X}'}(f(m))) \), where
\[ pr^{*}_{\mathfrak{X}'} : O_{n, \pi}(\mathfrak{X}') = H_{\text{crys}}^{0}(\mathfrak{X}'_{n}/S_{n}) \to H_{\text{crys}}^{0}(\mathfrak{X}'_{n}/S_{n}') = O_{n, \pi_{1}}(\mathfrak{X}') \]
is the natural pull-back. The map \( \text{pr}^{*}_{X} \) respects the filtration. To show the compatibility with \( \phi_{1} \), note that we have a diagram

\[
\begin{align*}
\mathcal{J}_{n+1, \pi}|_{\mathcal{O}_{K_{1}}} & \longrightarrow \mathcal{J}_{n, \pi}|_{\mathcal{O}_{K_{1}}} & \longrightarrow \phi_{1} & \phi_{1} \quad \mathcal{O}_{n, \pi}|_{\mathcal{O}_{K_{1}}} & \xrightarrow{\times p} \mathcal{O}_{n+1, \pi}|_{\mathcal{O}_{K_{1}}} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{J}_{n+1, \pi_{1}} & \longrightarrow \mathcal{J}_{n, \pi_{1}} & \longrightarrow \phi_{1} & \phi_{1} \quad \mathcal{O}_{n, \pi_{1}} & \xrightarrow{\times p} \mathcal{O}_{n+1, \pi_{1}},
\end{align*}
\]

where the vertical arrows are the pull-backs and the left and right squares are commutative. The composites of the horizontal maps are \( \phi \). From the exact sequences (1) and (2), we see that the middle square is also commutative. In other words, the map \( \text{pr}^{*}_{X} \) is compatible with \( \phi_{1} \). Therefore, we get a morphism of \( \text{Ab}/\mathcal{O}_{K_{1}} \)

\[ \Psi_{M} : \text{Gr}_{K}(M)|_{\mathcal{O}_{K_{1}}} \rightarrow \text{Gr}_{K}(M') \).

**Theorem 3.6.** The canonical map \( \Psi_{M} \) is an isomorphism.

**Proof.**

As the functor \( \text{Gr}_{K} \) is exact, by devissage we may assume that \( pM = 0 \). The sheaves of both sides come from finite flat group schemes \( \text{Gr}_{K}(M) \times_{\mathcal{O}_{K}} \mathcal{O}_{K_{1}} \) and \( \text{Gr}_{K}(M') \). Thus the bijectivity can be checked after taking the functor \( \text{Mod}_{K_{1}} \). In other words, it suffices to show that

\[ \Phi_{M} : M' = M \otimes_{S} S' \rightarrow \text{Hom}_{\text{Ab}/\mathcal{O}_{K_{1}}}(\text{Hom}_{\mathcal{M}}(M, \mathcal{O}_{1, \pi}|_{\mathcal{O}_{K_{1}}}), \mathcal{O}_{1, \pi_{1}}), \]

defined by \( m \otimes s' \mapsto (f \mapsto s'.\text{pr}^{*}(f(m))) \) is an isomorphism of \( \mathcal{M}' \). Here \( \text{pr}^{*} \) denotes the pull-back map \( \mathcal{O}_{1, \pi}|_{\mathcal{O}_{K_{1}}} \rightarrow \mathcal{O}_{1, \pi_{1}} \).

We have \( \text{rank}_{S_{1}}(M \otimes_{S} S') = \text{rank}_{S_{1}}(M) \) and

\[ \text{rank}_{S_{1}}(\text{Hom}_{\text{Ab}/\mathcal{O}_{K_{1}}}(\text{Hom}_{\mathcal{M}}(M, \mathcal{O}_{1, \pi}|_{\mathcal{O}_{K_{1}}}), \mathcal{O}_{1, \pi_{1}})) = \text{rank}_{S_{1}}(\text{Mod}_{K_{1}}(\text{Gr}_{K}(M) \times_{\mathcal{O}_{K}} \mathcal{O}_{K_{1}})) = \text{rank}_{S_{1}}(M). \]

By [5, Lemme 3.3.2], it suffices to show \( \ker(\Phi_{M}) \subseteq \text{Fil}^{p}S'_{i}'M' \).

Take an adapted basis \( \{e_{1}, \ldots, e_{d}\} \) of \( M \). Let \( m = \sum_{i=1}^{d} s_{i}'e_{i} \) be an element of \( \ker(\Phi_{M}) \). Consider the affine algebra \( R_{M} \) of \( \text{Gr}_{K}(M) \) and the element \( f \in \text{Hom}_{\mathcal{M}}(M, \mathcal{O}_{1, \pi}(R_{M})) \simeq \text{Gr}_{K}(M)(R_{M}) \) corresponding to \( \text{id}_{R_{M}} \). Then, from the proof of [5, Proposition 3.1.5], we have

\[ f(e_{i}) = \bar{X}_{i,0} + u\bar{X}_{i,1} + \cdots + u^{p-1}\bar{X}_{i,p-1} \mod \mathcal{J}_{1, \mu}^{[p]}(R_{M}), \]

where \( X_{i,0}, \ldots, X_{i,p-1} \) are the generators of \( R_{M} \) as in [5, p.507] and \( \bar{X}_{i,k} \) is the image of \( X_{i,k} \) in \( R_{M}/p \). Here we regard \( \bar{X}_{i,k} \) as an element of \( \mathcal{O}_{1, \pi}(R_{M}) \) by the natural map \( (R_{M}/p) \otimes_{k, \sigma} k[u] \rightarrow \mathcal{O}_{1, \pi}(R_{M}) \). Let us write \( f_{1} \) for the image of \( f \) by the natural map

\[ \text{Hom}_{\mathcal{M}}(M, \mathcal{O}_{1, \pi}(R_{M})) \rightarrow \text{Hom}_{\mathcal{M}}(M, \mathcal{O}_{1, \pi}(R'_{M})), \]
where \( R'_M = R_M \otimes_{O_K} O_{K_1} \). As \( m \in \text{Ker}(\Phi_M) \), we have

\[
\sum s'_i \text{pr}_{R'_M}^*(f_i(e_i)) = 0.
\]

Let \( \bar{X}'_{i,k} \) be the image of \( \bar{X}_{i,k} \) by the natural map \( (R'_M/\mathfrak{p}) \otimes_{k, \sigma} k[v] \rightarrow O_{1, \pi_1}(R'_M) \). Now we claim that \( \text{pr}_{R'_M}^*(\bar{X}'_{i,k}) = \bar{X}'_{i,k} \). It is sufficient to show this equality on an appropriate syntomic cover of \( R'_M \). Thus we may consider \( \text{pr}_{R'_M}^*: O_{1, \pi_1}(R'_M) \rightarrow O_{1, \pi_1}(R'_M) \), where \( R'_M \) is the ring constructed from \( \mathfrak{A}' = R'_M \) as in the proof of Lemma 3.5. Then the composite

\[
((R'_M/\mathfrak{p}) \otimes_{k, \sigma} k[v])^{PD} \xrightarrow{\text{H}^D_{\text{crys}}(R'_M/\mathfrak{p})} \xrightarrow{\phi} ((R'_M/\mathfrak{p}) \otimes_{k, \sigma} k[v])^{PD}
\]

maps \( 1 \otimes u \rightarrow 1 \otimes v^p \) and \( r \otimes 1 \) to \( \hat{r}^{p} \otimes 1 \), where \( \hat{r} \) is a lifting of \( r \) by the canonical surjection \( ((R'_M/\mathfrak{p}) \otimes_{k, \sigma} k[v])^{PD} \rightarrow R'_M/\mathfrak{p} \). We may take \( \hat{r} \) to be \( r^{i/p} \otimes 1 \). Thus the claim follows.

Now we have

\[
\sum_{i=1}^{d} s'_i(\bar{X}'_{i,0} + v^p \bar{X}'_{i,1} + \cdots + v^{p(p-1)} \bar{X}'_{i,p-1}) = 0
\]

in \( O_{1, \pi_1}(R'_M)/J_{1, \pi_1}^{[p]}(R'_M) \). This equation also holds in

\[
O_{1, \pi_1}(R'_M)/J_{1, \pi_1}^{[p]}(R'_M)
\]

and its subring

\[
(R'_M/\mathfrak{p})[v]/(v^p - X'_0) = (R'_M/\mathfrak{p})[v]/(v^p - \pi_1)
\]

(see [5, Lemme 2.3.2]). As \( R'_M/\mathfrak{p} \) is a subring of \( (R'_M/\mathfrak{p})[v]/(v^p - \pi_1) \). Let us write \( \bar{s}'_i \) for \( s'_i \) mod \( v \in k \). Taking mod \( v \), we have

\[
\sum_{i=1}^{d} \bar{s}'_i \bar{X}'_{i,0} = 0
\]

in \( (R'_M/\mathfrak{p})[v]/(v, v^p - \pi_1) = R'_M/\pi = R_M/\pi \). From the proof of [5, Proposition 3.1.1], we know that \( X'_{1,0}, \ldots, X'_{d,0} \) are linearly independent over \( k \) in \( R_M/\pi \). Thus \( \bar{s}'_i = 0 \) and \( s'_i \in vS'_1 + \text{Fil}^p S'_1 \) for all \( i \). Take \( s'^{(1)}_i \in S'_1 \) satisfying \( s'_i - v s'^{(1)}_i \in \text{Fil}^p S'_1 \). Then we have

\[
v \sum_{i=1}^{d} s'^{(1)}_i (\bar{X}'_{i,0} + v^p \bar{X}'_{i,1} + \cdots + v^{p(p-1)} \bar{X}'_{i,p-1}) = 0
\]
in \((R'_M/p)[v]/(v^p - \pi_1)\). However,

\[
R'_M/p \simeq (O_{K_1}/p)^{\oplus N} \simeq (k[T]/(T^{ep}))^{\oplus N}
\]

for some \(N\) and

\[
(k[T]/(T^{ep}))[v]/(v^p - T) \simeq k[v]/(v^{ep^2}).
\]

Thus \((R'_M/p)[v]/(v^p - \pi_1)\) is finite flat over \(k[v]/(v^{ep^2})\), and we have

\[
\sum_{i=1}^{d} s_i^{(1)}( \tilde{X}_{i,0} + v^p \tilde{X}_{i,1} + \cdots + v^{p(p-1)} \tilde{X}_{i,p-1} ) \in v^{ep^2-1}(R'_M/p)[v]/(v^p - \pi_1).
\]

Taking mod \(v\) and repeating this procedure show \(s_i' \in v^{ep^2}S_1' + \text{Fil}^pS_1' = \text{Fil}^pS_1'.\) In other words, \(m \in \text{Fil}^pS_1'M'.\) This concludes the proof. \(\square\)

**Remark 3.7.** In general, let \(L\) be a totally ramified extension over \(K\) of degree \(e'\) whose uniformizer is denoted by \(\pi_L\). When we define \(S_L = S_{\pi_L}\) as above, there exists a morphism \(S \rightarrow S_L\) respecting the filtration and \(\phi_1\) if and only if \(\pi_L^{e'} = \pi \zeta_{p-1}^i\) for some \(i\).

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**References**


