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Essential dimension of some twists of $\mu_{p^n}$

By

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Abstract

Fix an odd prime $p$ and a field $k$ of characteristic not $p$. Let $C_{p^n}$ be a twist of the $k$-group of $p^n$-th roots of the unity $\mu_{p^n}$. Then we give an upper bound for the essential dimension of a twisted $C_{p^n}$, which is expressed in terms of the degree of its minimal splitting field over $k$. We also compute the essential dimension of a twisted $C_8$ over fields of odd characteristic.

§1. INTRODUCTION

Fix a field $k$ and let $G$ be a finite $k$-group scheme. Then we have an interesting numerical invariant called the essential dimension, which depends on the group scheme $G$ and the base field $k$ (cf. Reichstein [4], Rost [5] and Berhuy-Favi [2] for the definition and basic facts).

Here we consider the case where the order of a $k$-group scheme $G$ is a power of a prime $p$, which is different from the characteristic of the base field $k$. There are two main results in this paper (Theorem 3.2 and Theorem 4.10).

First, we give an upper bound for the essential dimension of a twisted group $G$ of the group scheme $\mu_{p^n}$ over a field of characteristic $\neq p$, which depends on the minimal splitting field of $G$. Note that we have a trivial upper bound when $p = 2$.

Secondly, we show that

$$\text{ed}_k G = [K : k],$$

where $G$ is a twist of $\mu_8$ and $K$ is the minimal splitting field of $G$ over the base field $k$ with $\text{char}(k) \neq 2$. For the outline of this paper, in §2, after stating conjectures which are motivations of our investigation, we give precise formulation for the main results of this paper. In the following two sections we focus on the proof of Theorems. In §3, applying the section 5,6 in [2], we will discuss a proof of a generalization for Theorem 2.2. In §4, we will show that our first conjecture holds for the case where $G$ is a twist of the group scheme $\mu_8$. Our computation is based on an idea of Rost and used some facts in [6]. In fact, this result is a generalization of a result of Rost in [6] for $\mu_4$ (Theorem 2.1).
This is an announcement of the results of the author's master thesis and detailed proofs are not given.

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§ 2. Statement of results

Fix a field $k$ of characteristic is prime to $n$ and fix an algebraic closure $\overline{k}$ of $k$. Let $G$ be a $k$-form of the group $\mu_n$, the $k$-group scheme of the $n$-th roots of the unity. Denote by $K$ the minimal splitting field of the algebraic group $G$. The inclusion map $\mu_n \hookrightarrow G_m$ over $K$ induces a $k$-morphism $G \to T$, where $G_m$ is the multiplicative group scheme and $T := \text{Res}_{K/k}(G_m)$. Then we have that

$$1 \to G(\overline{k}) \to T \otimes_k \overline{k} \to (T/G)(\overline{k}) \to 1$$

where $T/G$ is a quotient, which exists as an affine group over $k$. By using a property of the scalar restriction functor $\text{Res}$ we see that the Hilbert 90 implies that the quotient map $\pi : T \to T/G$ is a classifying $G$-torsor. It shows that $\text{ed}_k G \leq [K : k]$.

Remark. The upper bound does not look like a good one. In particularly, in this paper we give a better upper bound for the case where $n$ is a power of an odd prime $p$ (Theorem 3.2). But we expect that it is the best possible value in the case below: Conjecture. Let $n$ be a power of 2 and $k$ be a field of odd characteristic. Then

$$\text{ed}_k G = [K : k].$$

In the case $p = 2$, our main result is to show that the above conjecture holds for $n = 8$. The main motivation of this paper is the following theorem, the case $n = 4$, proved in [6].

Theorem 2.1. (Rost) Let $k$ be a field of characteristic different from 2. Then

$$\text{ed}_k G = [K : k],$$

where $G$ is a twist of $\mu_4$ and $K$ is the minimal splitting field of $G$.

Now suppose that $p$ is an odd prime. Denote by $G_p$ the cyclic subgroup of order $p$ of the group $G$ and then it is a twisted form of the $\mu_p$. Moreover, we have that $\text{Gal}(K_p/k)$ is a subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ and $\text{Gal}(K/K_p)$ is a $p$-group, where $K_p$ is the corresponding field to the group $G_p$.

Conjecture. We expect that

$$\text{ed}_k G = \varphi([K_p : k])[K : K_p].$$
In fact it is motivated by the following theorem which is also deduced as a corollary of a general result in [3].

**Theorem 2.2.** (Buhler-Reichstein) The essential dimension for the cyclic group of order $\mathbb{Z}/p^n\mathbb{Z}$ over $\mathbb{Q}$ is at most $\varphi(p-1)p^{n-1}$, where $\varphi$ is the Euler function.

§ 3. AN UPPER BOUND

In this section we discuss a generalization of Theorem 2.2 to the twisted groups. We keep the notations of the previous section and our main reference is [2].

Denote by $F$ the subfield of $K/k$ such that $Gal(K/F) \cong Gal(K_p/k)$. Note that $Gal(K/k)$ is a cyclic group because $p$ is an odd prime. Put $s = [K : F]$. Hence for the character group of the torus $\text{Res}_{K/F}(G_m)$ one can conclude that

$$X^*(\text{Res}_{K/F}(G_m)) \cong \mathbb{Z}[x]/(x^s - 1),$$

where the action of $Gal(\bar{k}/F)$ is identified with multiplication by $x$.

For any subextension $L/F$ of $K$, the norm map $N_{K/L} : K^* \rightarrow L^*$ induces a map

$$N_{K/L} : \text{Res}_{K/F}(G_m) \rightarrow \text{Res}_{L/F}(G_m).$$

**Definition 3.1.** We now define algebraic groups $T_{K/F}$ and $\Lambda_{K/F}$ to be the kernel and the image of the map $\bigoplus N_{K/L}$, respectively, where

$$\bigoplus N_{K/L} : \text{Res}_{K/F}(G_m) \rightarrow \bigoplus_{F \subseteq L \subseteq K} \text{Res}_{L/F}(G_m).$$

Then we show that the algebraic group $T_{K/F}$ is a torus containing $G_p$. On applying the functor $\text{Res}_{F/k}$ to the torus we obtain a new $G$-torsor. It implies the following theorem since $\pi$ is a classifying one.

**Theorem 3.2.**

$$\text{ed}_k G \leq \varphi([K_p : k])[K : K_p].$$

§ 4. TWIST OF $\mu_8$

In this section, in order to give a proof of the main result we confirm the fundamental facts on the twists of $\mu_8$. The following notational conventions will be used throughout this chapter.

[a] - the image of $a \in k$ in $k/k^2$, $k_a := k(\bar{a})$, $k_{a,b} := k(\bar{a}, \bar{b})$, $H_a = \langle \bar{a} \otimes \frac{1}{\bar{a}} \rangle$, $T_a := \text{Res}_{k_a/k}(G_m)$, $T := \text{Res}_{k_{a,b}/k}(G_m)$, $\Gamma$ - the absolute Galois group of $k$. 
Note that for any field extension $L/k$ one has $T(L) = (L \otimes_k k_{a,b})^*$. Therefore we may assume that any element of $T(\overline{k})$ has the form

$$1 \otimes x + \overline{a} \otimes y + \overline{b} \otimes z + \overline{ab} \otimes t,$$

where $x, y, z, t \in \overline{k}$. Now assume that $k$ is a field of characteristic $\not= 2$. Then we have

$$H^1(k, \text{Aut} \, \mu_8) = H^1(k, (\mathbb{Z}/8\mathbb{Z})^*) \cong k^*/(k^*)^2 \times k^*/(k^*)^2.$$

We fix two generators of $\text{Aut} \, \mu_8$, namely $\sigma : \zeta_8 \mapsto \zeta_8^3$ and $\tau : \zeta_8 \mapsto \zeta_8^7$.

**Lemma 4.1.** Let $G$ be a $k$-form with respect to an element $[a] \times [b] \in H^1(k, \text{Aut} \, \mu_8)$. Then $G$ can be identified with a cyclic subgroup of $T(k)$ generated by

$$\langle \overline{a} \otimes \frac{1}{2} \overline{a} + \overline{b} \otimes \frac{-1}{2} \overline{b} \rangle.$$

Note that the image of $[a] \times [b]$ is $[ab]$ under the map $H^1(k, \text{Aut} \, \mu_8) \rightarrow H^1(k, \text{Aut} \, \mu_4)$, which is induced by the canonical injection $\mu_4 \rightarrow \mu_8$. Equivalently, the subgroup of $G$ of order 4 (denote by $G_{ab}$) is a $k$-form of $\mu_4$ with respect to $[ab]$ in $H^1(k, \text{Aut} \, \mu_4)$.

§ 4.1. The case of degenerate twists of $\mu_8$

If $[k_{a,b} : k] = 1$, then $\text{ed}_k G = 1$ because $G$ is isomorphic to $\mu_8$.

**Proposition 4.2.** If $[k_{a,b} : k] = 2$, then $\text{ed}_k G = 2$.

If $ab$ is a square, then $G_{ab} \cong \mu_4$ and so we need some observations on the functor $H^1(-, G)$. A classifying space of the functor $H^1(-, G)$ is induced by the following exact sequence.

**Lemma 4.3.** We have

$$1 \rightarrow G \times H_a \rightarrow T \otimes_k \overline{k} \overset{\pi_a}{\rightarrow} \overline{k}^* \times \frac{(T \otimes_k \overline{k})}{\overline{k}^*} \rightarrow 1,$$

where $\pi_a(x) = (N_{k_{a}/k}^4(x), [x^2])$.

Now to estimate the essential dimension of versal elements for the functor $H^1(-, G)$, we need some invariants. In fact, these invariants are cohomological invariants as follows

$$\eta_k : H^1(k, G) \rightarrow H^1(k, \mu_4),$$

$$\text{Res}_a : H^1(k, G) \rightarrow H^1(k_{a}, G) \cong k_{a}^*/(k_{a}^*)^8.$$

The invariant $\eta_k$ is a map induced by the norm map $N_{k_{a}/k} : G \rightarrow \mu_4$ and $\text{Res}_a$ is the canonical restriction map induced by the inclusion $Gal(\overline{k}/k_{a}) \hookrightarrow Gal(\overline{k}/k)$. 
Lemma 4.4.  
(1) One has for $t \in k^*$ and $\lambda \in k_a$:
\[
\eta_k(\delta_k(t, [\lambda])) = (t),
\]
\[
\text{Res}_a(\delta_k(t, [\lambda])) = (t\lambda^4/N_a^2(\lambda)).
\]
(2) The pair $(\delta_k(x, 1 + y^a), k(x, y))$ is a versal element for the functor $H^1(-, G)$.

Lemma 4.5.  If $(\alpha, F)$ is a versal element for $H^1(-, G)$, then $\text{ed}(\alpha) \geq 2$.

§4.2. The case of the generic $C_8$

We may assume that $a, b$ and $ab$ are all non-square's in $k$, since other cases are discussed. In this case, let us start to introduce some properties of the functor $H^1(-, G)$, which are necessary to prove our main result.

Some generalities to describe the functor $H^1(-, G)$.

Let $T/G$ denote the algebraic group $G_m \times T_a/G_m \times T_b/G_m \times T_{ab}/G_m$. Consider a morphism $\pi_{a,b}$ from $T$ to $T/G$, for $x \in T$, defined as
\[
\pi_{a,b}(x) = (N_{k^a}/k(x), [N_{k_a/b}/k_b(x)], [N_{k_a/b}/k_a(x)], [N_{k_{a,b}/k_{ab}}^2(x)]).
\]
Thus we can state the following useful proposition.

Proposition 4.6. There is an exact sequence of $\Gamma$-modules
\[
1 \to G \times H_a \times H_b \to T(\overline{k}) \xrightarrow{\pi_{a,b}} T/G(\overline{k}) \to 1.
\]

Invariants.

Applying the cohomology to the exact sequence we obtain an exact sequence
\[
k^* \times k^*_a/k^* \times k^*_b/k^* \times k^*/k^* \xrightarrow{\delta_k} H^1(k, G) \to 1,
\]
i.e.
\[
\delta_k(q) \in H^1(k, G)
\]
where $q = (c, [\alpha], [\beta], [\gamma])$ for $c \in k^*$, $\alpha \in k^*_a$, $\beta \in k^*_b$, $\gamma \in k^*_{ab}$.

Cohomological invariants for $G_{ab}$, which can be found in Rost [6], are given as:
\[
\eta_1 : H^1(k, G_{ab}) \to H^1(k, \mu_2),
\]
\[
\eta_2 : H^1(k, G_{ab}) \to H^2(k, \mu_2).
\]

Recall that the invariant $\eta_1$ is the map induced from the projection $G_{ab} \to \mu_2$. The invariant $\mu_2$ is the composition of $H^1(k, G) \to H^1(k, PGL(2))$ and the standard map $H^1(k, PGL(2)) \to H^2(k, \mu_2)$ arises from the embedding
\[
G_{ab} \to PGL(2)
\]
\[
\zeta \mapsto \begin{pmatrix} 1 & \overline{ab}^{-1}i \\ \overline{ab}i & 1 \end{pmatrix}
\]
where $\zeta \in G_{ab}(\bar{k})$ is a generator and $i = -1$. One has for $t \in k^*$ and $\lambda \in k_{ab}$:

\[
\begin{align*}
\eta_1(\delta_k(t, [\lambda])) &= (N_{k_{ab}/k}(\lambda)), \\
\eta_2(\delta_k(t, [\lambda])) &= (ab) \cup (t).
\end{align*}
\]

Consider the map $H^1(k, G) \rightarrow H^1(k, G_{ab})$, which is induced from $G \rightarrow^{[2]} G_{ab}$, we get cohomological invariants for $G$ as follows

\[
\begin{align*}
\eta_1 : H^1(k, G) &\rightarrow H^1(k, \mu_2), \\
\eta_2 : H^1(k, G) &\rightarrow H^2(k, \mu_2).
\end{align*}
\]

Here $\eta_1$ and $\eta_2$ are new notations and these are induced by old ones.

**Proposition 4.7.** One has

\[
\begin{align*}
\eta_1(\delta_k(q)) &= (N_{k_{ab}/k}(\gamma)), \\
\eta_2(\delta_k(q)) &= (ab) \cup (c).
\end{align*}
\]

Note that the group $G$ can be identified with $\langle 1 \otimes \frac{1}{2} + b \otimes \frac{i}{2}, b \rangle$ over $k_a$. Thus, for the canonical restriction map $\text{Res}_a : H^1(k, G) \rightarrow H^1(k_a, G)$, we get

\[
\text{Res}_a(\delta_k(q)) = \delta_{k_a} \left( \frac{\alpha^2}{N_a(\alpha)} c, [\beta^2 \gamma] \right).
\]

Similarly, for the restriction map $\text{Res}_b : H^1(k, G) \rightarrow H^1(k_b, G)$, we also have

\[
\text{Res}_b(\delta_k(q)) = \delta_{k_b} \left( \frac{\beta^2}{N_b(\beta)} c, [\alpha^2 \gamma] \right).
\]

Thus we obtain another invariants which are defined as an invariant $\eta_2$ for $G$:

\[
\begin{align*}
\eta_a : H^1(k, G) &\rightarrow H^2(k_a, \mu_2), \\
\eta_b : H^1(k, G) &\rightarrow H^2(k_b, \mu_2).
\end{align*}
\]

More precisely, for instance, the map $\eta_a$ is the composition of the map $H^1(k, G) \rightarrow H^1(k_a, G_a)$ and the map $H^1(k_a, G_a) \rightarrow H^2(k_a, \mu_2)$. The above considerations allow us to write the following proposition.

**Proposition 4.8.** We have

\[
\begin{align*}
\eta_a(\delta_k(q)) &= (b) \cup \left( \frac{\alpha^2}{N_a(\alpha)} c \right), \\
\eta_b(\delta_k(q)) &= (a) \cup \left( \frac{\beta^2}{N_b(\beta)} c \right).
\end{align*}
\]
Remark. It would not be useful to consider the map $H^1(k, G) \to H^1(k_a, \mu_2)$, because it gives the same result as $\eta_1$ in Proposition 4.7.

**Proposition 4.9.** Assume that $t_1, t_2, t_3$ and $t_4$ are independent variables over $k$. Set $E := k(t_1, t_2, t_3, t_4)$ and $\omega := (t_1, [1 + t_2 \overline{a}], [1 + t_3 \overline{b}], [1 + t_4 \overline{ab}])$. Then the pair 

$$(\delta_E(\omega), E)$$

is a versal element for $H^1(-, G)$.

We are now ready to state the Main Theorem.

**Theorem 4.10.** $\text{ed}_k G = 4$ if $a, b$ and $ab$ are nonsquare elements of $k$.

For the proof, we show that the smallest possible value of the essential dimension of any element for $G$ is 4 by using invariants $\eta_1, \eta_2, \eta_a$ and $\eta_b$.

**Corollary 4.11.** Let $k$ be a field of characteristic $\neq 2$. Then

$$\text{ed}_k(\mathbb{Z}/8\mathbb{Z}) = \begin{cases} 
1, & \text{if } 2 \text{ and } -1 \text{ are both squares in } k \\
4, & \text{if } 2, -2 \text{ and } -1 \text{ are not squares in } k \\
2, & \text{otherwise}
\end{cases}$$

**References**


