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On Galois representations of local fields with imperfect residue fields

By

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Let $K$ be a complete discrete valuation field of characteristic 0 with residue field $k$ of characteristic $p \neq 0$ such that $[k : k^p] = p^e < +\infty$. Let $V$ be a $p$-adic representation of the absolute Galois group $G_K = \text{Gal} (\overline{K}/K)$ where we fix an algebraic closure $\overline{K}$ of $K$. When the residue field $k$ is perfect (i.e. $e = 0$), Berger has proved a conjecture of Fontaine (Conjecture 1.1. below) which claims that, if $V$ is a de Rham representation of $G_K$, $V$ becomes a potentially semi-stable representation of $G_K$. (See Theorem 1.2.) Here, we generalize this result to the case when the residue field $k$ is not necessarily perfect. For this, we prove some results on $p$-adic representations in the imperfect residue field case (see Theorem 1.3.) which are obtained by using the recent theory of $p$-adic differential modules and deduce this generalization of the result of Berger as a corollary. (See Theorem 1.4.)

In this survey article, we first state the results in Section 1. In Section 2, we review the property of the $p$-adic periods ring $B_{\text{dR}}$. Then, in Section 3 and Section 4, we give a sketch of the proof of Theorem 1.3.

§ 1. Results

Let $K$, $k$, $G_K$ and $V$ be as above. Fontaine, Hyodo, Kato and Tsuzuki define the $p$-adic periods rings (associated to $K$) which are equipped with the continuous action of $G_K$. (See [F1], [Ka1], [Ka2], [Tz3], [Br2] etc.)

$$(\mathbb{Q}_p \subset) B_{\text{cris}} \subset B_{\text{st}} \subset B_{\text{dR}}.$$ 

With these rings, we classify the $p$-adic representation $V$ of $G_K$ as follows. We call the $p$-adic representation $V$ of $G_K$

1. a de Rham representation of $G_K$ if and only if we have the equality

$$\dim_{\mathbb{Q}_p} V = \dim_{(B_{\text{dR}})^{G_K}} (B_{\text{dR}} \otimes V)^{G_K}.$$
2. a semi-stable representation of $G_K$ if and only if we have the equality
\[ \dim_{\mathbb{Q}_p} V = \dim_{(B_{\text{st}})^{G_K}} (B_{\text{st}} \otimes V)^{G_K} : \]

3. a crystalline representation of $G_K$ if and only if we have the equality
\[ \dim_{\mathbb{Q}_p} V = \dim_{(B_{\text{cris}})^{G_K}} (B_{\text{cris}} \otimes V)^{G_K}. \]

(In general, we have the inequality $\dim_{\mathbb{Q}_p} V \geq \dim_{(B_{*})^{G_K}} (B_{*} \otimes V)^{G_K}$ for $* \in \{\text{dR}, \text{st}, \text{cris}\}$.) It is well-known that we have the following implications (see [F1] etc.)

\[ \text{cray. rep. of } G_K \implies \text{st. rep. of } G_K \implies \text{dR. rep. of } G_K. \]

Furthermore, we call the $p$-adic representation $V$ of $G_K$ a potentially de Rham (resp. semi-stable, crystalline) representation of $G_K$ if $V$ is a de Rham (resp. semi-stable, crystalline) representation of $G_L$ where $L/K$ is a finite extension. Then, it is well-known that a potentially de Rham representation of $G_K$ is a de Rham representation of $G_K$. (See Section 2.) Thus, it is not difficult to see that a potentially semi-stable representation of $G_K$ is a de Rham representation of $G_K$. Fontaine conjectured the converse.

**Conjecture 1.1.** *If the $p$-adic representation $V$ is a de Rham representation of $G_K$, then $V$ is a potentially semi-stable representation of $G_K$.*

Then, Berger has proved the following thing.

**Theorem 1.2.** *The conjecture of Fontaine is true if the residue field $k$ is perfect.*

The aim of this note is to give a sketch of the proof of the generalization of this theorem to the imperfect residue field case. (Theorem 1.5.) For this, we state some results on $p$-adic representations in the imperfect residue field case. (Theorem 1.3.)

Let us fix some notations. Fix a lifting $(b_i)_{1 \leq i \leq e}$ of a $p$-basis of $k$ in $\mathcal{O}_K$ (the ring of integers of $K$), and fix a $p^m$-th root $b_i^{1/p^m}$ of $b_i$ in $\overline{K}$ for each $m \geq 1$ satisfying $(b_i^{1/p^{m+1}})^p = b_i^{1/p^m}$. Put

\[ K^{(i)} = \bigcup_{m \geq 1} K(b_i^{1/p^m}, 1 \leq i \leq e) \quad \text{and} \quad K' = \text{the } p\text{-adic completion of } K^{(i)}. \]

which depend on the choice of $\{b_i^{1/p^m}\}$. Then, $K'$ is a complete discrete valuation field with perfect residue field, which is a canonical "perfectization" of $K$. Furthermore, we can regard the Galois group $G_{K'} = \text{Gal}(\overline{K}/K')$ as a subgroup of $G_K$ (see Section 2 for details) and think $V$ as a $p$-adic representation of $G_{K'}$. Then, we obtain the following theorem ([Mo1] and [Mo2]).
Theorem 1.3. Let $K$ be a complete discrete valuation field of characteristic $0$ with residue field of characteristic $p > 0$ such that $[k : k^p] < \infty$ and $K'$ be as above. Let $V$ denote a $p$-adic representation of $G_K$. Then, we have the following equivalences.

1. $V$ is a de Rham representation of $G_K$ if and only if $V$ is a de Rham representation of $G_K'$.

2. $V$ is a potentially semi-stable representation of $G_K$ if and only if $V$ is a potentially semi-stable representation of $G_K'$.

3. $V$ is a potentially crystalline representation of $G_K$ if and only if $V$ is a potentially crystalline representation of $G_K'$.

Remark 1.4. Though we don’t introduce the definition of Hodge-Tate representations in this note, we also show that $V$ is a Hodge-Tate representation of $G_K$ if and only if $V$ is a Hodge-Tate representation of $G_K'$. (For the definition of Hodge-Tate representations, see [F1] etc.)

With Theorem 1.2 and Theorem 1.3, we have the following equivalences:

\[
\begin{align*}
V : \text{dR. rep. of } G_K & \iff V : \text{dR. rep. of } G_K' \\
\Downarrow & \quad \Downarrow \\
V : \text{pst. rep. of } G_K & \iff V : \text{pst. rep. of } G_K'
\end{align*}
\]

Thus, we obtain the generalization of Theorem 1.2 to the imperfect residue field case.

Theorem 1.5. The conjecture of Fontaine is true even if the residue field $k$ is not necessarily perfect.

For simplicity, in this note, we shall consider only the de Rham representation case of Theorem 1.3.

§ 2. Preliminaries on the $p$-adic periods ring $B_{dR}$

§ 2.1. Definitions and properties of the ring $B_{dR}$

2.1.1. The case $e = 0$ (i.e. $k$ is perfect)

Let $K$ be as in Introduction and assume that the residue field $k$ is perfect. Choose an algebraic closure $\bar{K}$ of $K$ and put $\mathbb{C}_p = \text{the } p\text{-adic completion of } \bar{K}$. Put

\[
\tilde{E} = \lim_{x \to x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, ...) \mid x^{(i)} \in \mathbb{C}_p, (x^{(i+1)})^p = x^{(i)}\}.
\]
Define a valuation $v_E$ on $\tilde{E}$ by $v_E(x) = v_p(x^{(0)})$ where $v_p$ denotes the normalized valuation of $\mathbb{C}_p$ by $v_p(p) = 1$. Let $\epsilon = (\epsilon^{(n)})$ be an element of $\tilde{E}$ such that $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$. The field $\tilde{E}$ is the completion of an algebraic closure of $k((\epsilon - 1))$ for this valuation. Define $\tilde{E}^+$ to be the ring of integers for this valuation. Put $\tilde{A}^+ = W(\tilde{E}^+)$ and

$$\tilde{B}^+ = \tilde{A}^+[1/p] = \{ \sum_{k \gg -\infty} p^k[x_k] \mid x_k \in \tilde{E}^+ \}$$

where $[*]$ denotes the Teichmüller lift of $* \in \tilde{E}^+$. This ring is equipped with a surjective homomorphism

$$\theta: \tilde{B}^+ \to \mathbb{C}_p: \sum p^k[x_k] \mapsto \sum p^k x_k^{(0)}.$$

The ring $B_{\text{dR}}^+$ is defined to be the completion by the Ker $(\theta)$-adic topology of $\tilde{B}^+$:

$$B_{\text{dR}}^+ = \lim_{n \to \infty} \tilde{B}^+/(\text{Ker } (\theta)^n).$$

This is a discrete valuation ring and $t = \log([\epsilon])$ (which converges in $B_{\text{dR}}^+$) is a generator of the maximal ideal. Put $B_{\text{dR}} = B_{\text{dR}}^+[1/t]$. This is a field and is equipped with an action of the Galois group $G_K$ and a filtration defined by $\text{Fil}^i B_{\text{dR}} = t^i B_{\text{dR}}^+ (i \in \mathbb{Z})$. The ring $(B_{\text{dR}})^{G_K}$ is canonically isomorphic to $K$. If $V$ is a p-adic representation of $G_K$, then $D_{\text{dR}}(V) = (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a $K$-vector space. We say that a p-adic representation $V$ of $G_K$ is a de Rham representation of $G_K$ if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\text{dR}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\text{dR}}(V)).$$

Furthermore, a potentially de Rham representation $V$ of $G_K$ is a de Rham representation of $G_K$. (See [F1].)

2.1.2. The case $\epsilon$ is general (i.e. $k$ is not necessarily perfect)

Let $K$ be as in Introduction and assume that the residue field $k$ is not necessarily perfect. If we construct $B_{\text{dR}}^+$, $B_{\text{dR}}$ as in the perfect residue case (we denote $B_{\text{dR}}^{+,\text{naiv}}$, $B_{\text{dR}}^{\text{naiv}}$):

1. $\mathbb{C}_p = \text{the } p\text{-adic completion of } K$
2. $\tilde{E} = \varprojlim_{x \to xp} \mathbb{C}_p$ and $\tilde{E}^+$
3. $\tilde{A}^+ = W(\tilde{E}^+)$, $\tilde{B}^+ = \tilde{A}^+[1/p]$ and $\theta: \tilde{B}^+ \to \mathbb{C}_p$
4. $B_{\text{dR}}^{+,\text{naiv}} = \lim_{n \to \infty} \tilde{B}^+/(\text{Ker } (\theta)^n)$ and $B_{\text{dR}}^{\text{naiv}} = B_{\text{dR}}^{+,\text{naiv}}[1/t]

then contrary to the perfect residue field case, we have $(B_{\text{dR}}^{\text{naiv}})^{G_K} \neq K$ in general. Now, we shall recall the imperfect residue field version of $B_{\text{dR}}$. 

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First, construct the ring $\tilde{\mathbb{A}}^+$ for $K$ as above. Let $\alpha : \mathcal{O}_K \otimes \mathbb{Z} \tilde{\mathbb{A}}^+ \to \mathcal{O}_K/p\mathcal{O}_K$ be the natural surjection and define $\tilde{\mathbb{A}}^+_{(K)}$ to be

$$\tilde{\mathbb{A}}^+_{(K)} = \lim_{n \geq 0}(\mathcal{O}_K \otimes \mathbb{Z} \tilde{\mathbb{A}}^+)/((\mathrm{K}\mathrm{e}\mathrm{r}(\alpha))^n).$$

Let $\theta_K : \tilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathbb{C}_p$ be the natural extension of $\theta : \tilde{\mathbb{A}}^+[1/p] \to \mathbb{C}_p$. Then, the imperfect residue field version of $B^+_{\mathrm{dR}}$ is defined to be the Ker $(\theta_K)$-adic completion of $\tilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$:

$$B^+_{\mathrm{dR}} = \lim_{n \geq 0}(\tilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)/((\mathrm{K}\mathrm{e}\mathrm{r}(\theta_K))^n).$$

Fix a lifting $(b_i)_{1 \leq i \leq e}$ of a $p$-basis of $k$ in $\mathcal{O}_K$ as in Introduction. Let $\tilde{b}_i = (b_i^{(n)}) \in \tilde{E}^+$ such that $b_i^{(0)} = b_i$, and then the series which defines $\log([\tilde{b}_i]/b_i)$ converges in $B^+_{\mathrm{dR}}$ to an element $t_i$. This ring $B^+_{\mathrm{dR}}$ is endowed with an action of the Galois group $G_K$ and a filtration defined by $\mathrm{Fil}^iB^+_{\mathrm{dR}} = m^+_i$ where the maximal ideal $m^+_i$ of $B^+_{\mathrm{dR}}$ is generated by $\{t, t_1, \ldots, t_e\}$. Put $B^+_{\mathrm{dR}} = B^+_{\mathrm{dR}[1/t]}$. Then, $K$ is canonically embedded in $B^+_{\mathrm{dR}}$ and $(B^+_{\mathrm{dR}})^{G_K} = K$. If $V$ is a $p$-adic representation of $G_K$, then $D_{\mathrm{dR}}(V) = (B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a $K$-vector space. We say that a $p$-adic representation $V$ of $G_K$ is a de Rham representation of $G_K$ if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{dR}}(V) \quad \text{(we always have $\dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{dR}}(V)$)}.$$ 

Furthermore, we can show that a potentially de Rham representation $V$ of $G_K$ is a de Rham representation $V$ of $G_K$ in the same way as in the perfect residue field case.

§2.2. Comparison of the case 2.1.1 and 2.1.2

Fix the notations as in Section 2.1.2 and let $K^{(i)}$ and $K'$ be as in Introduction. First, by the construction, we see that there exists a $G_K$-equivariant injection

$$f : B^+_{\mathrm{dR}}^{+, \text{naïve}} \hookrightarrow B^+_{\mathrm{dR}}. \tag{2.1}$$

On the other hand, since $K'$ is a complete discrete valuation field with perfect residue field, we can construct the ring $B^+_{\mathrm{dR}}^{+, \prime}$ for $K'$ as in Section 2.1.1. We will see that there exists a morphism from the ring $B^+_{\mathrm{dR}}^{+}$ to the ring $B^+_{\mathrm{dR}}^{+, \prime}$. Since $K^{(i)}$ is a Henselian discrete valuation field, we have an isomorphism $G_{K'} \simeq G_{K^{(i)}}( \subset G_K)$. With this isomorphism, we identify $G_{K'}$ as a subgroup of $G_K$. Then, there exists a $G_{K'}$-equivariant surjection

$$g : B^+_{\mathrm{dR}} \twoheadrightarrow B^+_{\mathrm{dR}}^{+, \prime}. \tag{2.2}$$

Now, we will show that there exists a morphism between the ring $B^+_{\mathrm{dR}}^{+, \text{naïve}}$ with the ring $B^+_{\mathrm{dR}}^{+, \prime}$. We have a bijective map from the set of finite extensions of $K^{(i)}$ contained in $\overline{K}$...
to the set of finite extensions of $K'$ contained in $\overline{K}$ defined by $L \mapsto LK'$. Furthermore, $LK'$ is the $p$-adic completion of $L$. Hence, we have an isomorphism of rings

$$\mathcal{O}_{\overline{K}} / p^n \mathcal{O}_{\overline{K}} \simeq \mathcal{O}_{\overline{K'}} / p^n \mathcal{O}_{\overline{K'}}$$

where $\mathcal{O}_{\overline{K}}$ and $\mathcal{O}_{\overline{K'}}$ denote rings of integers of $\overline{K}$ and $\overline{K'}$. Thus, the fields $\mathbb{C}_{p}(K)$ (= the $p$-adic completion of $K$) and $\mathbb{C}_{p}(K')$ (= the $p$-adic completion of $K'$) are isomorphic (we will simply write $\mathbb{C}_{p}$). In the end, we have an isomorphism of rings

$$B_{dR}^{+, \text{naïve}} \simeq B_{dR}^{+}$$

which coincides with the composition ((2.1) and (2.2))

$$g \circ f : B_{dR}^{+, \text{naïve}} \hookrightarrow B_{dR}^{+} \rightarrow B_{dR}^{+}.$$  

From now on, we identify the ring $B_{dR}^{+, \text{naïve}}$ with the ring $B_{dR}^{+}$. Then, it is well-known that the homomorphism

$$(2.3) \quad f : B_{dR}^{+, [t_1, \ldots, t_e]} \rightarrow B_{dR}^{+}$$

is an isomorphism of filtered algebras. (See [Br2] and [Kal]1.) From this isomorphism, it follows easily that

$$i : B_{dR}^{+, r} \hookrightarrow B_{dR}^{+} \quad \text{and} \quad pr : B_{dR}^{+} \rightarrow B_{dR}^{+} : t_i \mapsto 0$$

are $G_{K'}$-equivariant homomorphisms and the composition

$$pr \circ i : B_{dR}^{+, r} \hookrightarrow B_{dR}^{+} \rightarrow B_{dR}^{+}$$

is identity.

### § 3. Preliminaries on $p$-adic differential modules

In this section, we will introduce the recent theory of $p$-adic differential modules which plays an important role in this note. First, let us fix the notations. Put $K$, $K^{r}$ and $K^{l}$ as in Introduction. Put $K^{(l)} = \bigcup_{m \geq 0} K^{(l)}(\zeta_{p^m})$ and $K_{\infty}^{l} = \bigcup_{m \geq 0} K^{l}(\zeta_{p^m})$ where $\zeta_{p^m}$ denotes a primitive $p^m$-th root of unity in $\overline{K}$ such that $\zeta_{p^{m+1}} = \zeta_{p^m}^p$. Let $\hat{K}_{\infty}^{l}$ denote the $p$-adic completion of $K_{\infty}^{l}$. These fields $K_{\infty}^{r}$ and $\hat{K}_{\infty}^{l}$ are independent of the choice of $\{b_i^{1/p^m}\}$ ($K_{\infty}^{r}$ isn’t). Then, we have

$$\hat{K}_{\infty}^{l} \supset K_{\infty}^{l} \supset K_{\infty}^{(l)}.$$  

Let $H_K$ denote the kernel of the cyclotomic character $\chi : G_{K'} \rightarrow \mathbb{Z}_{p}^{*}$. Note that, since we have $H_K \simeq G_{K_{\infty}^{(l)}}$, the subgroup $H_K$ of $G_{K}$ is independent of the choice of $K'$. Define
\[ \Gamma_K = G_K/H_K. \]

Let \( \Gamma_0 = \text{Gal}(K_{\infty}^{(i)}/K^{(i)}) \) be the subgroup of \( \Gamma_K \). Let \( \Gamma_i \) \((i \neq 0)\) be the subgroup of \( \Gamma_K \) such that actions of \( \beta_i \in \Gamma_i \) \((i \neq 0)\) are given by

\[ \beta_i(e^{(n)}) = e^{(n)} \quad \text{and} \quad \beta_i(b_j^{(n)}) = b_j^{(n)} \quad \text{for} \quad i \neq j. \]

Define the homomorphism \( c_i : \Gamma_i \rightarrow \mathbb{Z}_p \) such that we have

\[ \beta_i(b_i^{(n)}) = b_i^{(n)}(e^{(n)})^{c_i(\beta_i)}. \]

\section{3.1. Definitions of \( p \)-adic differential modules}

We will give the definitions of \( p \)-adic differential modules \( D_{\text{Sen}}(V) \), \( D_{\text{Br}}(V) \), \( D_{\text{diff}}^+(V) \) and \( D_{e-diff}^+(V) \) which are obtained by Sen, Brinon, Fontaine and Andreatta-Brinon. We will have the following diagram:

\[ (B_{dR}^+ \otimes_{\mathbb{Q}_p} V)^{H_K} \supset D_{\text{diff}}^+(V) \supset D_{e-diff}^+(V) \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K} \supset D_{\text{Sen}}(V) \supset D_{\text{Br}}(V). \]

The following results in Section 3.1.1 and 3.1.3 are obtained when \( V \) is a \( p \)-adic representation of \( G_L = \text{Gal}(\overline{L}/L) \) where \( L \) is a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic \( p > 0 \). However, in Section 3.1.1 and 3.1.3, for simplicity, we will state the results when \( V \) is a \( p \)-adic representation of \( G_{K'} \).

\subsection{3.1.1. The module \( D_{\text{Sen}}(V) \)}

In the article [S3], Sen shows that the \( \hat{K}_{\infty}^\text{-vector space} \) \((C_p \otimes_{\mathbb{Q}_p} V)^{H_K}\) has dimension \( d \) \((= \text{dim}_{\mathbb{Q}_p} V)\) and the union of the finite dimensional \( K'_{\infty} \)-subspaces of \((C_p \otimes_{\mathbb{Q}_p} V)^{H_K}\) stable under \( \Gamma_0 \) \((\simeq G_{K'}/H_K)\) is a \( K'_{\infty} \)-vector space of dimension \( d \) stable under \( \Gamma_0 \) \((\text{called } D_{\text{Sen}}(V))\). We have \( C_p \otimes_{K_{\infty}} D_{\text{Sen}}(V) = C_p \otimes_{\mathbb{Q}_p} V \) and the natural map \( \hat{K}_{\infty} \otimes_{K_{\infty}} D_{\text{Sen}}(V) \rightarrow (C_p \otimes_{\mathbb{Q}_p} V)^{H_K} \) is an isomorphism. Furthermore, if \( \gamma \in \Gamma_0 \) is close enough to 1, then the series of operators on \( D_{\text{Sen}}(V) \):

\[ \frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^k}{k} \]

converges to an operator \( \nabla^{(0)} : D_{\text{Sen}}(V) \rightarrow D_{\text{Sen}}(V) \) and does not depend on the choice of \( \gamma \).

\subsection{3.1.2. The module \( D_{\text{Br}}(V) \)}

In the article [Br1], Brinon generalizes Sen's work above. He shows that the union of the finite dimensional \( K_{\infty}^{(i)} \)-subspaces of \((C_p \otimes_{\mathbb{Q}_p} V)^{H_K}\) stable under \( \Gamma_K \) is a \( K_{\infty}^{(i)} \)-vector
space of dimension $d$ stable under $\Gamma_K$ (we call it $D_{Br}(V)$). We have $\mathbb{C}_p \otimes_{K_{\infty}} D_{Br}(V) = \mathbb{C}_p \otimes_{K_{\infty}} V$ and the natural map $\hat{K}'_{\infty} \otimes_{K_{\infty}} D_{Br}(V) \rightarrow (\mathbb{C}_p \otimes_{K_{\infty}} V)^{H_K}$ is an isomorphism. As in the case of $D_{Sen}(V)$, the $K_{\infty}^{(l)}$-vector space $D_{Br}(V)$ is endowed with the action of the operator

$$\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1 - \gamma)^k}{k}$$

if $\gamma \in \Gamma_0$ is close enough to 1. In addition to this operator $\nabla^{(0)}$, if $\beta_i \in \Gamma_i$ is close enough to 1, then the series of operators on $D_{Br}(V)$:

$$\log(\beta_i) = \frac{1}{c_i(\beta_i)} \sum_{n \geq 1} \frac{(1 - \beta_i)^n}{n}$$

corporates to an operator $\nabla^{(i)} : D_{Br}(V) \rightarrow D_{Br}(V)$ and does not depend on the choice of $\beta_i$.

3.1.3. The module $D_{\text{dif}}^+(V)$

Let the ring $B_{\text{dr}}^+$ be as in Section 2.1.2. In the article [F5], by using Sen’s theory, Fontaine shows that the union of $K_{\infty}^{(l)}[[t, t_1, \ldots, t_e]]$-submodules of finite type of $(B_{\text{dr}}^+ \otimes_{\mathbb{Q}_p} V)^{H_K}$ stable under $\Gamma_0$ ($\simeq G_{K'}/H_K$) is a $K_{\infty}^{(l)}[[t, t_1, \ldots, t_e]]$-module of rank $d = \dim_{\mathbb{Q}_p} V$ stable under $\Gamma_0$ (called $D_{\text{dif}}^+(V)$). We have $B_{\text{dr}}^+ \otimes_{K_{\infty}^{(l)}} D_{\text{dif}}^+(V) = B_{\text{dr}}^+ \otimes_{\mathbb{Q}_p} V$ and the natural map $(B_{\text{dr}}^+)^{H_K} \otimes_{K_{\infty}^{(l)}} D_{\text{dif}}^+(V) \rightarrow (B_{\text{dr}}^+ \otimes_{\mathbb{Q}_p} V)^{H_K}$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_0$ is close enough to 1, then the series of operators on $D_{\text{dif}}^+(V)$:

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1 - \gamma)^k}{k}$$

corporates to an operator $\nabla^{(0)} : D_{\text{dif}}^+(V) \rightarrow D_{\text{dif}}^+(V)$ and does not depend on the choice of $\gamma$.

Remark 3.1. This $D_{\text{dif}}^+(V)$ is a little different from the original one constructed by Fontaine in [F5].

3.1.4. The module $D_{e-dif}^+(V)$ Let the ring $B_{\text{dr}}^+$ be as in Section 2.1.2. In the article [A-B], Andreatta and Brinon generalize Fontaine’s work above. They show that the union of $K_{\infty}^{(l)}[[t, t_1, \ldots, t_e]]$-submodules of finite type of $(B_{\text{dr}}^+ \otimes_{\mathbb{Q}_p} V)^{H_K}$ stable under $\Gamma_K$ is a $K_{\infty}^{(l)}[[t, t_1, \ldots, t_e]]$-module of rank $d$ stable under $\Gamma_K$ (we call it $D_{e-dif}^+(V)$). We have $B_{\text{dr}}^+ \otimes_{K_{\infty}^{(l)}} D_{e-dif}^+(V) = B_{\text{dr}}^+ \otimes_{\mathbb{Q}_p} V$ and the natural map $(B_{\text{dr}}^+)^{H_K} \otimes_{K_{\infty}^{(l)}} D_{e-dif}^+(V) \rightarrow (B_{\text{dr}}^+ \otimes_{\mathbb{Q}_p} V)^{H_K}$ is an isomorphism. As in the case of $D_{\text{dif}}^+(V)$, the $K_{\infty}^{(l)}[[t, t_1, \ldots, t_e]]$-module $D_{e-dif}^+(V)$ is endowed with the action of
the operator
\[
\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^k}{k}
\]
if \(\gamma \in \Gamma_0\) is close enough to 1. In addition to this operator \(\nabla^{(0)}\), if \(\beta_i \in \Gamma_i\) is close enough to 1, then the series of operators on \(D_{e-dif}^+(V)\):
\[
\frac{\log(\beta_i)}{c_i(\beta_i)} = \frac{1}{c_i(\beta_i)} \sum_{n \geq 1} \frac{(1-\beta_i)^n}{n}
\]
converges to an operator \(\nabla^{(i)} : D_{e-dif}^+(V) \rightarrow D_{e-dif}^+(V)\) and does not depend on the choice of \(\beta_i\).

§ 3.2. Properties of differential operators

First, we consider the “meaning” of the equation \(\nabla^{(j)}(F) = 0\). By definitions of differential operators, it follows easily that \(F\) is fixed by actions of an open subgroup of \(\Gamma_j\). Thus, we can say that

“Find solutions \(\{f_k\}_{k=1}^{d=\dim_{\mathbb{Q}_p}V}\) (linearly independent over \(K\)) of \(\nabla^{(j)}(f_k) = 0\) for \(0 \leq j \leq e\) in \(D_{e-dif}^+(V)[1/t]\)”

\(\downarrow\)

“\(V\) is a potentially de Rham rep. of \(G_K\), that is, a de Rham rep. of \(G_K\)”.

Thus, the theory of \(p\)-adic differential modules plays an important role in the proof of Theorem 1.3. Now, we will describe actions of operators \(\nabla^{(j)}\) \((0 \leq j \leq e)\) on the module \(D_{e-dif}^+(V)\). First, by a standard argument, we can show that, if \(x \in D_{e-dif}^+(V)\), we have
\[
\nabla^{(0)}(x) = \lim_{\gamma \to 1} \frac{\gamma(x) - x}{\chi(\gamma) - 1} \quad \text{and} \quad \nabla^{(i)}(x) = \lim_{\beta_i \to 1} \frac{\beta_i(x) - x}{c_i(\beta_i)}.
\]

With this presentation, we can easily describe actions of operators \(\nabla^{(j)}\) \((0 \leq j \leq e)\) on the ring \(K_{\infty}[[t, t_1, \ldots, t_e]]\) as follows.

**Lemma 3.2.** We have
\[
\nabla^{(0)} = t \frac{d}{dt} \quad \text{and} \quad \nabla^{(i)} = t \frac{d}{dt_i} \quad (i \neq 0) \quad \text{on} \quad K_{\infty}[[t, t_1, \ldots, t_e]].
\]

We extend naturally actions of \(K_{\infty}^{(i)}\)-linear derivations \(\nabla^{(0)}\) and \(\nabla^{(i)}\) \((i \neq 0)\) on \(D_{e-dif}^+(V)\) to \(D_{e-dif}(V) = D_{e-dif}^+(V)[1/t]\) by putting \(\nabla^{(0)}(\frac{1}{t}) = -\frac{1}{t}\) and \(\nabla^{(i)}(\frac{1}{t}) = 0\) \((i \neq 0)\). Now, compute the bracket \([\ , \ ]\) of operators \(\nabla^{(j)}\) \((0 \leq j \leq e)\).

**Proposition 3.3.** On the \(K_{\infty}^{(i)}[[t, t_1, \ldots, t_e]][1/t]\)-module \(D_{e-dif}(V)\) as above, we have the following relation
1. $\nabla^{(0)}\nabla^{(i)} - \nabla^{(i)}\nabla^{(0)} = \nabla^{(i)}$ for all $i \neq 0$;

2. $\nabla^{(j)}\nabla^{(i)} - \nabla^{(i)}\nabla^{(j)} = 0$ for all $i, j \neq 0$.

The following proposition describe actions of $\nabla^{(i)}$ ($i \neq 0$) and plays a key role in the proof of Theorem 1.3.

**Proposition 3.4.** Let $M$ be a finite generated free $K_{\infty}^{(j)}[[t, t_1, \ldots, t_e]][1/t]$-module endowed with $K_{\infty}^{(j)}$-linear operators $\{\nabla^{(i)}\}_{j=0}^{e}$ which satisfy Leibniz rule and relations in Proposition 3.3. Assume that $M$ has a basis $\{g_j\}_{j=1}^{d}$ over $K_{\infty}^{(j)}[[t, t_1, \ldots, t_e]][1/t]$ which satisfies $\nabla^{(0)}(g_j) = 0$. Then, the action of $\nabla^{(i)}$ ($i \neq 0$) is given by

$$
\nabla^{(i)}(g_j) = t \sum_{k=1}^{d} c_k g_k, \quad c_k \in K_{\infty}^{(j)}[[t, t_1, \ldots, t_e]] \quad \text{and} \quad \nabla^{(0)}(c_k) = 0.
$$

**Proof.** Since $\{g_j\}_{j=1}^{d}$ forms a basis of $M$ over $K_{\infty}^{(j)}[[t, t_1, \ldots, t_e]][1/t]$, we have

$$
\nabla^{(i)}(g_j) = \sum_{k=1}^{d} a_k g_k \quad (a_k \in K_{\infty}^{(j)}[[t, t_1, \ldots, t_e]][1/t]).
$$

Then, by the relation of Proposition 3.3, we have

$$
\sum_{k=1}^{d} \nabla^{(0)}(a_k) g_k = \sum_{k=1}^{d} a_k g_k
$$

(note that we have $\nabla^{(0)}(g_j) = 0$ by hypothesis). Hence, we obtain the differential equation

$$
\nabla^{(0)}(a_k) = a_k.
$$

Define $c_k = a_k/t$, then it satisfies $\nabla^{(0)}(c_k) = a_k/t - a_k/t = 0$ and we see that $c_k$ is contained in $K_{\infty}^{(j)}[[t, t_1, \ldots, t_e]]$. Thus, the solutions of this differential equation have the following forms

$$
a_k = c_k t \quad \text{where} \quad c_k \in K_{\infty}^{(j)}[[t, t_1, \ldots, t_e]] \quad \text{and} \quad \nabla^{(0)}(c_k) = 0.
$$

Hence, we have, from (3.1) and (3.2),

$$
\nabla^{(i)}(g_j) = t \sum_{k=1}^{d} c_k g_k \quad \text{where} \quad c_k \in K_{\infty}^{(j)}[[t, t_1, \ldots, t_e]] \quad \text{and} \quad \nabla^{(0)}(c_k) = 0.
$$
Corollary 3.5. With notations as in Proposition 3.4. above, we have

\[(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e} (g_j) = t^{k_1 + \cdots + k_e} \sum_{k=1}^{d} c_k g_k, \quad c_k \in K_\infty[[t, t_1, \ldots, t_e]] \text{ and } \nabla^{(0)}(c_k) = 0.\]

§ 4. Proof of Theorem 1.3, in the de Rham representation case

Let us recall some notations

\(K\) is a complete discrete valuation field of characteristic 0 with residue field \(k\) of characteristic \(p > 0\) such that \([k : k^p] = p^e < \infty\):

\(V\) is a \(p\)-adic representation of \(G_K\) of dimension \(d\) over \(\mathbb{Q}_p\):

\(K_\infty^\text{REJECT}\) is the \(p\)-adic completion of \(\bigcup_{m \geq 0} K(b_i^{1/p^m}), 1 \leq i \leq e\) is the complete discrete valuation field of characteristic 0 with perfect residue field:

there exists a \(G_K\)-equivariant isomorphism

\[B_{\text{dR}} = B_{\text{dR}}^+ [1/t] \simeq B_{\text{dR}}^+[t_1, \ldots, t_e][1/t].\]

(1) \(V: \text{ de Rham rep. of } G_K \Rightarrow V: \text{ de Rham rep. of } G_{K'}\)

Proof. Since \(V\) is a de Rham representation of \(G_K\), there exists a \(G_K\)-equivariant isomorphism of \(B_{\text{dR}}\)-modules:

\[B_{\text{dR}} \otimes_{\mathbb{Q}_p} V \simeq (B_{\text{dR}})^d.\]

Now, by tensoring \(B_{\text{dR}}' \otimes_{B_{\text{dR}}} (\text{which is induced by the } G_{K'}\text{-equivariant surjection } pr : B_{\text{dR}} \twoheadrightarrow B_{\text{dR}}')\) over (4.1), we obtain a \(G_{K'}\)-equivariant isomorphism of \(B_{\text{dR}}'\)-modules:

\[B_{\text{dR}}' \otimes_{\mathbb{Q}_p} V \simeq (B_{\text{dR}}')^d.\]

This means that \(V\) is a de Rham representation of \(G_{K'}\). \(\square\)

(2) \(V: \text{ de Rham rep. of } G_{K'} \Rightarrow V: \text{ de Rham rep. of } G_K\)

This is the difficult part of this note and the theory of \(p\)-adic differential modules plays a central role in the following proof. We have to bridge the gap between \(G_K\) and \(G_{K'}\). Then, roughly speaking, since the differential operators \(\{\nabla^{(i)}\}_{i=1}^{e}\) reflect this difference, it suffices to construct the solutions \(\{f_k\}_{k=1}^{d=\text{dim}_{\mathbb{Q}_p} V}\) of \(\nabla^{(i)}(f_k) = 0\) for \(1 \leq i \leq e\).

Lemma 4.1. If \(V\) is a de Rham representation of \(G_{K'}\), there exists a \(G_{K'}\)-equivariant isomorphism

\[B_{\text{dR}} \otimes D_{e\text{-dif}}(V) \simeq (B_{\text{dR}})^d.\]
**Proof.** Since $V$ is a de Rham representation of $G_{K'}$, there exists a $G_{K'}$-equivariant isomorphism of $B'_{dR}$-modules:

\[(4.2) \quad B'_{dR} \otimes_{\mathbb{Q}_p} V \simeq (B'_{dR})^d.\]

Now, by tensoring $B_{dR} \otimes B_{dR}'$ (which is induced by the $G_{K'}$-equivariant injection $i : B'_{dR} \hookrightarrow B_{dR}$) over (4.2), we obtain a $G_{K'}$-equivariant isomorphism of $B_{dR}$-modules:

\[B_{dR} \otimes_{\mathbb{Q}_p} V \simeq (B_{dR})^d.\]

On the other hand, we have a $G_K$-equivariant isomorphism

\[B_{dR} \otimes D_{e-dif}(V) \simeq B_{dR} \otimes_{\mathbb{Q}_p} V.\]

Thus, we obtain the desired isomorphism. \hfill \Box

Finally, we shall give the proof of (2).

**Proof.** We shall construct the $K_\infty^{(\text{REJECT})}$-linearly independent elements $\{f_j^{(*)}\}_{j=1}^d \in D_{e-dif}(V)$ such that $\nabla^{(i)}(f_j^{(*)}) = 0$ for $0 \leq i \leq e$ and $1 \leq j \leq d$.

(A) Construction of $\{f_j^{(*)}\}_{j=1}^d \in D_{e-dif}(V)$

Since $V$ is a de Rham representation of $G_{K'}$, we have a basis $\{f_j\}_{j=1}^d$ of $D_{e-dif}(V)$ over $K_\infty^{(l)}[[t, t_1, \ldots, t_e]] [1/t]$ such that, from Lemma 4.1,

\[\nabla^{(0)}(f_j) = 0 \quad \text{for all } 1 \leq j \leq d.\]

Thus, we can apply Corollary 3.5. to the $K_\infty^{(l)}[[t, t_1, \ldots, t_e]] [1/t]$-module $D_{e-dif}(V)$ generated by $\{f_j\}_{j=1}^d$ and then we can deduce

\[(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(g_j) = t^{k_1+\cdots+k_e} \sum_{k=1}^d c_k g_k, \quad c_k \in K_\infty^{(l)}[[t, t_1, \ldots, t_e]] \quad \text{and} \quad \nabla^{(0)}(c_k) = 0.\]

Then, if we define $f_j^{(*)} \in D_{e-dif}(V)$ (converge for $(t, t_1, \ldots, t_e)$-adic topology) by

\[f_j^{(*)} = \sum_{0 \leq k_1, \ldots, k_e} (-1)^{k_1+\cdots+k_e} \frac{t^{k_1} \cdots t^{k_e}}{k_1! \cdots k_e! t^{k_1+\cdots+k_e}} (\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(f_j),\]

it follows easily that we have $\nabla^{(i)}(f_j^{(*)}) = 0$ for $0 \leq i \leq e$.

(B) $\{f_j^{(*)}\}_{j=1}^d \in D_{e-dif}(V)$ is linearly independent over $K_\infty^{(l)}$

By the presentation of $f_j^{(*)}$, we have

\[f_j^{(*)} = f_j + g_j \quad \text{where } f_j \not\in, g_j \in (t_1, \ldots, t_e)D_{e-dif}(V).\]
Since \( \{f_j^{(*)}\}_{j=1}^{d} \) forms a basis of \( D_{\text{e-diff}}(V) \) over \( K_{\infty}^{(t)}[[t, t_1, \ldots, t_e]][1/t] \), it is, in particular, linearly independent over \( K_{\infty}^{(t)} (\subset K_{\infty}^{(t)}[[t, t_1, \ldots, t_e]][1/t]) \). Then, it follows easily that \( \{f_j^{(*)}\}_{j=1}^{d} \) is linearly independent over \( K_{\infty}^{(t)} \) in \( D_{\text{e-diff}}(V) \).

(C) Conclusion

Therefore, on the \( K \)-vector space generated by \( \{f_j^{(*)}\}_{j=1}^{d} \), \( \log(\gamma) \) and \( \log(\beta_i) \) act trivially \((\Leftrightarrow \nabla^{(0)}(f_j^{(*)}) = 0 \text{ and } \nabla^{(i)}(f_j^{(*)}) = 0 \text{ for all } 1 \leq j \leq d) \). Thus, this means that \( \Gamma_K \) acts on this \( K \)-vector space via finite quotient and there exists a finite extension \( L/K \) such that \( \{f_j^{(*)}\}_{j=1}^{d} \) forms a basis of \( D_{\text{dR}}(V_L) \) over \( L (\subset K_{\infty}^{(t)}) \) where \( V_L \) denotes the restriction of \( V \) to \( G_L \). Since a potentially de Rham representation of \( G_K \) is a de Rham representation of \( G_K \), we complete the proof.

\[ \square \]

References


