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On Galois representations of local fields with imperfect residue fields

By

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Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$. Let V be a p-adic representation of the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ where we fix an algebraic closure $\overline{K}$ of K. When the residue field k is perfect (i.e. $e = 0$), Berger has proved a conjecture of Fontaine (Conjecture 1.1. below) which claims that, if V is a de Rham representation of G_K, V becomes a potentially semi-stable representation of G_K . (See Theorem 1.2.) Here, we generalize this result to the case when the residue field k is not necessarily perfect. For this, we prove some results on p-adic representations in the imperfect residue field case (see Theorem 1.3.) which are obtained by using the recent theory of p-adic differential modules and deduce this generalization of the result of Berger as a corollary. (See Theorem 1.4.)

In this survey article, we first state the results in Section 1. In Section 2, we review the property of the p-adic periods ring B_{dR}. Then, in Section 3 and Section 4, we give a sketch of the proof of Theorem 1.3.

§ 1. Results

Let K, k, G_K and V be as above. Fontaine, Hyodo, Kato and Tsuzuki define the p-adic periods rings (associated to K) which are equipped with the continuous action of G_K. (See [F1], [Ka1], [Ka2], [Tz3], [Br2] etc.)

$$(\mathbb{Q}_p \subset)\ B_{\mathrm{cris}} \subset B_{\mathrm{st}} \subset B_{\mathrm{dR}}.$$

With these rings, we classify the p-adic representation V of G_K as follows. We call the p-adic representation V of G_K

1. a de Rham representation of G_K if and only if we have the equality

$$\dim_{\mathbb{Q}_p} V = \dim_{(B_{\mathrm{dR}})^{G_K}} (B_{\mathrm{dR}} \otimes V)^{G_K} :$$

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2. a semi-stable representation of G_K if and only if we have the equality

$$\dim_{\mathbb{Q}_p} V = \dim_{(B_{\mathrm{st}})^{G_K}} (B_{\mathrm{st}} \otimes V)^{G_K} :$$

3. a crystalline representation of G_K if and only if we have the equality

$$\dim_{\mathbb{Q}_p} V = \dim_{(B_{\mathrm{cris}})^{G_K}} (B_{\mathrm{cris}} \otimes V)^{G_K}.$$

(In general, we have the inequality $\dim_{\mathbb{Q}_p} V \geq \dim_{(B_*)^{G_K}} (B_* \otimes V)^{G_K}$ for $* \in \{\mathrm{dR}, \mathrm{st}, \mathrm{cris}\}$.) It is well-known that we have the following implications (see [F1] etc.)

$$\text{cray. rep. of } G_K \Longrightarrow \text{st. rep. of } G_K \Longrightarrow \text{dR. rep. of } G_K.$$

Furthermore, we call the p-adic representation V of G_K a potentially de Rham (resp. semi-stable, crystalline) representation of G_K if V is a de Rham (resp. semi-stable, crystalline) representation of G_L where L/K is a finite extension. Then, it is well-known that a potentially de Rham representation of G_K is a de Rham representation of G_K. (See Section 2.) Thus, it is not difficult to see that a potentially semi-stable representation of G_K is a de Rham representation of G_K. Fontaine conjectured the converse.

Conjecture 1.1. *If the p-adic representation V is a de Rham representation of G_K, then V is a potentially semi-stable representation of G_K.*

Then, Berger has proved the following thing.

Theorem 1.2. *The conjecture of Fontaine is true if the residue field k is perfect.*

The aim of this note is to give a sketch of the proof of the generalization of this theorem to the imperfect residue field case. (Theorem 1.5.) For this, we state some results on p-adic representations in the imperfect residue field case. (Theorem 1.3.)

Let us fix some notations. Fix a lifting $(b_i)_{1 \leq i \leq e}$ of a p-basis of k in $\mathcal{O}_K$ (the ring of integers of K), and fix a p^m -th root b_i^{1/p^m} of b_i in $\overline{K}$ for each $m \geq 1$ satisfying $(b_i^{1/p^{m+1}})^p = b_i^{1/p^m}$. Put

$$K^{(\prime)} = \cup_{m \geq 1} K(b_i^{1/p^m}, 1 \leq i \leq e) \quad \text{and} \quad K' = \text{the } p\text{-adic completion of } K^{(\prime)}.$$

which depend on the choice of $\{b_i^{1/p^m}\}$. Then, K' is a complete discrete valuation field with perfect residue field, which is a canonical "**perfectzation**" of K. Furthermore, we can regard the Galois group $G_{K'} = \mathrm{Gal}(\overline{K'}/K')$ as a subgroup of G_K (see Section 2 for details) and think V as a p-adic representation of $G_{K'}$. Then, we obtain the following theorem ([Mo1] and [Mo2]).

Theorem 1.3. *Let K be a complete discrete valuation field of characteristic 0 with residue field of characteristic $p > 0$ such that $[k : k^p] < \infty$ and K' be as above. Let V denote a p-adic representation of G_K. Then, we have the following equivalences.*

1. *V is a de Rham representation of G_K if and only if V is a de Rham representation of $G_{K'}$.*
2. *V is a potentially semi-stable representation of G_K if and only if V is a potentially semi-stable representation of $G_{K'}$.*
3. *V is a potentially crystalline representation of G_K if and only if V is a potentially crystalline representation of $G_{K'}$.*

Remark 1.4. Though we don't introduce the definition of Hodge-Tate representations in this note, we also show that V is a Hodge-Tate representation of G_K if and only if V is a Hodge-Tate representation of $G_{K'}$. (For the definition of Hodge-Tate representations, see [F1] etc.)

With Theorem 1.2. and Theorem 1.3., we have the following equivalences:

$$\begin{array}{ccc} V : \text{dR. rep. of } G_K & \Longleftrightarrow & V : \text{dR. rep. of } G_{K'} \\ \Updownarrow & & \Updownarrow \\ V : \text{pst. rep. of } G_K & \Longleftrightarrow & V : \text{pst. rep. of } G_{K'} \end{array}$$

Thus, we obtain the generalization of Theorem 1.2. to the imperfect residue field case.

Theorem 1.5. *The conjecture of Fontaine is true even if the residue field k is not necessarily perfect.*

For simplicity, in this note, we shall consider only the de Rham representation case of Theorem 1.3..

§ 2. Preliminaries on the p-adic periods ring B_{dR}

§ 2.1. Definitions and properties of the ring B_{dR}

2.1.1. The case $e = 0$ (i.e. k is perfect)

Let K be as in Introduction and assume that the residue field k is perfect. Choose an algebraic closure $\overline{K}$ of K and put $\mathbb{C}_p =$ the p-adic completion of $\overline{K}$. Put

$$\widetilde{E} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, \dots) \mid x^{(i)} \in \mathbb{C}_p, (x^{(i+1)})^p = x^{(i)}\}.$$

Define a valuation v_E on $\widetilde{E}$ by $v_E(x) = v_p(x^{(0)})$ where v_p denotes the normalized valuation of $\mathbb{C}_p$ by $v_p(p) = 1$. Let $\epsilon = (\epsilon^{(n)})$ be an element of $\widetilde{E}$ such that $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$. The field $\widetilde{E}$ is the completion of an algebraic closure of $k((\epsilon - 1))$ for this valuation. Define $\widetilde{E}^+$ to be the ring of integers for this valuation. Put $\widetilde{A}^+ = W(\widetilde{E}^+)$ and

$$\widetilde{B}^+ = \widetilde{A}^+[1/p] = \{ \sum_{k \gg -\infty} p^k[x_k] \mid x_k \in \widetilde{E}^+ \}$$

where $[*]$ denotes the Teichmüller lift of $* \in \widetilde{E}^+$. This ring is equipped with a surjective homomorphism

$$\theta : \widetilde{B}^+ \twoheadrightarrow \mathbb{C}_p : \quad \sum p^k[x_k] \mapsto \sum p^k x_k^{(0)}.$$

The ring B_{dR}^+ is defined to be the completion by the Ker (θ)-adic topology of $\widetilde{B}^+$:

$$B_{\mathrm{dR}}^+ = \varprojlim_{n \geq 0} \widetilde{B}^+/(\mathrm{Ker}\,(\theta)^n).$$

This is a discrete valuation ring and $t = \log([\epsilon])$ (which converges in B_{dR}^+) is a generator of the maximal ideal. Put $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[1/t]$. This is a field and is equipped with an action of the Galois group G_K and a filtration defined by $\mathrm{Fil}^i B_{\mathrm{dR}} = t^i B_{\mathrm{dR}}^+$ $(i \in \mathbb{Z})$. The ring $(B_{\mathrm{dR}})^{G_K}$ is canonically isomorphic to K. If V is a p-adic representation of G_K, then $D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a de Rham representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{dR}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{dR}}(V)).$$

Furthermore, a potentially de Rham representation V of G_K is a de Rham representation of G_K. (See [F1].)

2.1.2. The case e is general (i.e. k is not necessarily perfect)

Let K be as in Introduction and assume that the residue field k is not necessarily perfect. If we construct B_{dR}^+, B_{dR} as in the perfect residue case (we denote $B_{\mathrm{dR}}^{+,\mathrm{naiv}}$, $B_{\mathrm{dR}}^{\mathrm{naiv}}$):

1. $\mathbb{C}_p =$ the p-adic completion of $\overline{K}$
2. $\widetilde{E} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p$ and $\widetilde{E}^+$
3. $\widetilde{A}^+ = W(\widetilde{E}^+)$, $\widetilde{B}^+ = \widetilde{A}^+[1/p]$ and $\theta : \widetilde{B}^+ \twoheadrightarrow \mathbb{C}_p$
4. $B_{\mathrm{dR}}^{+,\mathrm{naiv}} = \varprojlim_{n \geq 0} \widetilde{B}^+/(\mathrm{Ker}\,(\theta))^n$ and $B_{\mathrm{dR}}^{\mathrm{naiv}} = B_{\mathrm{dR}}^{+,\mathrm{naiv}}[1/t]$

then contrary to the perfect residue field case, we have $(B_{\mathrm{dR}}^{\mathrm{naiv}})^{G_K} \neq K$ in general. Now, we shall recall the imperfect residue field version of B_{dR}.

First, construct the ring $\widetilde{A}^+$ for K as above. Let $\alpha : \mathcal{O}_K \otimes_{\mathbb{Z}} \widetilde{A}^+ \twoheadrightarrow \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ be the natural surjection and define $\widetilde{A}^+_{(K)}$ to be

$$\widetilde{A}^+_{(K)} = \varprojlim_{n\geq 0}(\mathcal{O}_K \otimes_{\mathbb{Z}} \widetilde{A}^+)/(\mathrm{Ker}\,(\alpha))^n.$$

Let $\theta_K : \widetilde{A}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \twoheadrightarrow \mathbb{C}_p$ be the natural extension of $\theta : \widetilde{A}^+[1/p] \twoheadrightarrow \mathbb{C}_p$. Then, the imperfect residue field version of B^+_{dR} is defined to be the $\mathrm{Ker}\,(\theta_K)$-adic completion of $\widetilde{A}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$:

$$B^+_{\mathrm{dR}} = \varprojlim_{n\geq 0}(\widetilde{A}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)/(\mathrm{Ker}\,(\theta_K)^n).$$

Fix a lifting $(b_i)_{1\leq i\leq e}$ of a p-basis of k in $\mathcal{O}_K$ as in Introduction. Let $\widetilde{b_i} = (b_i^{(n)}) \in \widetilde{E}^+$ such that $b_i^{(0)} = b_i$, and then the series which defines $\log([\widetilde{b_i}]/b_i)$ converges in B^+_{dR} to an element t_i. This ring B^+_{dR} is endowed with an action of the Galois group G_K and a filtration defined by $\mathrm{Fil}^i B^+_{\mathrm{dR}} = m^i_{\mathrm{dR}}$ where the maximal ideal m_{dR} of B^+_{dR} is generated by $\{t, t_1, \ldots, t_e\}$. Put $B_{\mathrm{dR}} = B^+_{\mathrm{dR}}[1/t]$. Then, K is canonically embedded in B_{dR} and $(B_{\mathrm{dR}})^{G_K} = K$. If V is a p-adic representation of G_K, then $D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a de Rham representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{dR}}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{dR}}(V)).$$

Furthermore, we can show that a potentially de Rham representation V of G_K is a de Rham representation V of G_K in the same way as in the perfect residue field case.

§ 2.2. Comparison of the case 2.1.1 and 2.1.2

Fix the notations as in Section 2.1.2 and let $K^{(\prime)}$ and K' be as in Introduction. First, by the construction, we see that there exists a G_K-equivariant injection

$$f : B^{+,\mathrm{naiv}}_{\mathrm{dR}} \hookrightarrow B^+_{\mathrm{dR}}. \tag{2.1}$$

On the other hand, since K' is a complete discrete valuation field with perfect residue field, we can construct the ring $B^{+,\prime}_{\mathrm{dR}}$ for K' as in Section 2.1.1. We will see that there exists a morphism from the ring B^+_{dR} to the ring $B^{+,\prime}_{\mathrm{dR}}$. Since $K^{(\prime)}$ is a Henselian discrete valuation field, we have an isomorphism $G_{K'} \simeq G_{K^{(\prime)}} (\subset G_K)$. With this isomorphism, we identify $G_{K'}$ as a subgroup of G_K. Then, there exists a $G_{K'}$-equivariant surjection

$$g : B^+_{\mathrm{dR}} \twoheadrightarrow B^{+,\prime}_{\mathrm{dR}}. \tag{2.2}$$

Now, we will show that there exists a morphism between the ring $B^{+,\mathrm{naiv}}_{\mathrm{dR}}$ with the ring $B^{+,\prime}_{\mathrm{dR}}$. We have a bijective map from the set of finite extensions of $K^{(\prime)}$ contained in $\overline{K}$

to the set of finite extensions of K' contained in $\overline{K'}$ defined by $L \mapsto LK'$. Furthermore, LK' is the p-adic completion of L. Hence, we have an isomorphism of rings

$$\mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}} \simeq \mathcal{O}_{\overline{K'}}/p^n\mathcal{O}_{\overline{K'}}$$

where $\mathcal{O}_{\overline{K}}$ and $\mathcal{O}_{\overline{K'}}$ denote rings of integers of $\overline{K}$ and $\overline{K'}$. Thus, the fields $\mathbb{C}_p(K)$ (= the p-adic completion of $\overline{K}$) and $\mathbb{C}_p(K')$ (= the p-adic completion of $\overline{K'}$) are isomorphic (we will simply write $\mathbb{C}_p$). In the end, we have an isomorphism of rings

$$B_{\mathrm{dR}}^{+,\mathrm{naiv}} \simeq B_{\mathrm{dR}}^{+,\prime}$$

which coincides with the composition ((2.1) and (2.2))

$$g \circ f : B_{\mathrm{dR}}^{+,\mathrm{naiv}} \hookrightarrow B_{\mathrm{dR}}^{+} \twoheadrightarrow B_{\mathrm{dR}}^{+,\prime}.$$

From now on, we identify the ring $B_{\mathrm{dR}}^{+,\mathrm{naiv}}$ with the ring $B_{\mathrm{dR}}^{+,\prime}$. Then, it is well-known that the homomorphism

$$f : B_{\mathrm{dR}}^{+,\prime}[[t_1, \dots, t_e]] \to B_{\mathrm{dR}}^{+} \tag{2.3}$$

is an isomorphism of filtered algebras. (See [Br2] and [Ka1].) From this isomorphism, it follows easily that

$$i : B_{\mathrm{dR}}^{+,\prime} \hookrightarrow B_{\mathrm{dR}}^{+} \quad \text{and} \quad pr : B_{\mathrm{dR}}^{+} \twoheadrightarrow B_{\mathrm{dR}}^{+,\prime} : t_i \mapsto 0$$

are $G_{K'}$-equivariant homomorphisms and the composition

$$pr \circ i : B_{\mathrm{dR}}^{+,\prime} \hookrightarrow B_{\mathrm{dR}}^{+} \twoheadrightarrow B_{\mathrm{dR}}^{+,\prime}$$

is identity.

§ 3. Preliminaries on p-adic differential modules

In this section, we will introduce the recent theory of p-adic differential modules which plays an important role in this note. First, let us fix the notations. Put K, $K^{(\prime)}$ and K' as in Introduction. Put $K_\infty^{(\prime)} = \cup_{m\geq 0} K^{(\prime)}(\zeta_{p^m})$ and $K'_\infty = \cup_{m\geq 0} K'(\zeta_{p^m})$ where ζ_{p^m} denotes a primitive p^m-th root of unity in $\overline{K}$ such that $\zeta_{p^{m+1}}^p = \zeta_{p^m}$. Let $\hat{K}'_\infty$ denote the p-adic completion of K'_∞. These fields $K_\infty^{(\prime)}$ and $\hat{K}'_\infty$ are independent of the choice of $\{b_i^{1/p^m}\}$ (K'_∞ isn't). Then, we have

$$\hat{K}'_\infty \supset K'_\infty \supset K_\infty^{(\prime)}.$$

Let H_K denote the kernel of the cyclotomic character $\chi : G_{K'} \to \mathbb{Z}_p^*$. Note that, since we have $H_K \simeq G_{K_\infty^{(\prime)}}$, the subgroup H_K of G_K is independent of the choice of K'. Define

$\Gamma_K = G_K/H_K$. Let $\Gamma_0 = \mathrm{Gal}(K_\infty^{(\prime)}/K^{(\prime)})$ be the subgroup of Γ_K. Let Γ_i $(i \neq 0)$ be the subgroup of Γ_K such that actions of $\beta_i \in \Gamma_i$ $(i \neq 0)$ are given by

$$\beta_i(\epsilon^{(n)}) = \epsilon^{(n)} \quad \text{and} \quad \beta_i(b_j^{(n)}) = b_j^{(n)} \ (i \neq j).$$

Define the homomorphism $c_i : \Gamma_i \to \mathbb{Z}_p$ such that we have

$$\beta_i(b_i^{(n)}) = b_i^{(n)}(\epsilon^{(n)})^{c_i(\beta_i)}.$$

§ 3.1. Definitions of p-adic differential modules

We will give the definitions of p-adic differential modules $D_{\mathrm{Sen}}(V)$, $D_{\mathrm{Bri}}(V)$, $D^+_{\mathrm{dif}}(V)$ and $D^+_{e\text{-dif}}(V)$ which are obtained by Sen, Brinon, Fontaine and Andreatta-Brinon. We will have the following diagram:

$$\begin{array}{ccccc} (B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{H_K} & \supset & D^+_{\mathrm{dif}}(V) & \supset & D^+_{e\text{-dif}}(V) \\ \downarrow & & \downarrow & & \downarrow \\ (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K} & \supset & D_{\mathrm{Sen}}(V) & \supset & D_{\mathrm{Bri}}(V). \end{array}$$

The following results in Section 3.1.1 and 3.1.3 are obtained when V is a p-adic representation of $G_L = \mathrm{Gal}(\overline{L}/L)$ where L is a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p > 0$. However, in Section 3.1.1 and 3.1.3, for simplicity, we will state the results when V is a p-adic representation of $G_{K'}$.

3.1.1. The module $D_{Sen}(V)$

In the article [S3], Sen shows that the $\hat{K}'_\infty$-vector space $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ has dimension d $(= \dim_{\mathbb{Q}_p} V)$ and the union of the finite dimensional K'_∞-subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ stable under Γ_0 $(\simeq G_{K'}/H_K)$ is a K'_∞-vector space of dimension d stable under Γ_0 (called $D_{\mathrm{Sen}}(V)$). We have $\mathbb{C}_p \otimes_{K'_\infty} D_{\mathrm{Sen}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $\hat{K}'_\infty \otimes_{K'_\infty} D_{\mathrm{Sen}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_0$ is close enough to 1, then the series of operators on $D_{\mathrm{Sen}}(V)$:

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^k}{k}$$

converges to an operator $\nabla^{(0)} : D_{\mathrm{Sen}}(V) \to D_{\mathrm{Sen}}(V)$ and does not depend on the choice of γ.

3.1.2. The module $D_{Bri}(V)$

In the article [Br1], Brinon generalizes Sen's work above. He shows that the union of the finite dimensional $K_\infty^{(\prime)}$-subspaces of $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ stable under Γ_K is a $K_\infty^{(\prime)}$-vector space of dimension d stable under Γ_K (we call it $D_{\mathrm{Bri}}(V)$). We have $\mathbb{C}_p \otimes_{K_\infty^{(\prime)}} D_{\mathrm{Bri}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ and the natural map $\hat{K}'_\infty \otimes_{K_\infty^{(\prime)}} D_{\mathrm{Bri}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$ is an isomorphism. As in the case of $D_{\mathrm{Sen}}(V)$, the $K_\infty^{(\prime)}$-vector space $D_{\mathrm{Bri}}(V)$ is endowed with the action of the operator

$$\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^k}{k}$$

if $\gamma \in \Gamma_0$ is close enough to 1. In addition to this operator $\nabla^{(0)}$, if $\beta_i \in \Gamma_i$ is close enough to 1, then the series of operators on $D_{\mathrm{Bri}}(V)$:

$$\frac{\log(\beta_i)}{c_i(\beta_i)} = -\frac{1}{c_i(\beta_i)} \sum_{n \geq 1} \frac{(1-\beta_i)^n}{n}$$

converges to an operator $\nabla^{(i)} : D_{\mathrm{Bri}}(V) \to D_{\mathrm{Bri}}(V)$ and does not depend on the choice of β_i.

3.1.3. The module $D^+_{dif}(V)$

Let the ring B^+_{dR} be as in Section 2.1.2. In the article [F5], by using Sen's theory, Fontaine shows that the union of $K'_\infty[[t, t_1, \ldots, t_e]]$-submodules of finite type of $(B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{H_K}$ stable under Γ_0 ($\simeq G_{K'}/H_K$) is a $K'_\infty[[t, t_1, \ldots, t_e]]$-module of rank $d = \dim_{\mathbb{Q}_p} V$ stable under Γ_0 (called $D^+_{\mathrm{dif}}(V)$). We have $B^+_{\mathrm{dR}} \otimes_{K'_\infty[[t,t_1,\ldots,t_e]]} D^+_{\mathrm{dif}}(V) = B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ and the natural map $(B^+_{\mathrm{dR}})^{H_K} \otimes_{K'_\infty[[t,t_1,\ldots,t_e]]} D^+_{\mathrm{dif}}(V) \to (B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{H_K}$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_0$ is close enough to 1, then the series of operators on $D^+_{\mathrm{dif}}(V)$:

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^k}{k}$$

converges to an operator $\nabla^{(0)} : D^+_{\mathrm{dif}}(V) \to D^+_{\mathrm{dif}}(V)$ and does not depend on the choice of γ.

Remark 3.1. This $D^+_{\mathrm{dif}}(V)$ is a little different from the original one constructed by Fontaine in [F5].

3.1.4. The module $D^+_{e-dif}(V)$ Let the ring B^+_{dR} be as in Section 2.1.2. In the article [A-B], Andreatta and Brinon generalize Fontaine's work above. They show that the union of $K_\infty^{(\prime)}[[t, t_1, \ldots, t_e]]$-submodules of finite type of $(B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{H_K}$ stable under Γ_K is a $K_\infty^{(\prime)}[[t, t_1, \ldots, t_e]]$-module of rank d stable under Γ_K (we call it $D^+_{e-\mathrm{dif}}(V)$). We have $B^+_{\mathrm{dR}} \otimes_{K_\infty^{(\prime)}[[t,t_1,\ldots,t_e]]} D^+_{e-\mathrm{dif}}(V) = B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ and the natural map $(B^+_{\mathrm{dR}})^{H_K} \otimes_{K_\infty^{(\prime)}[[t,t_1,\ldots,t_e]]} D^+_{e-\mathrm{dif}}(V) \to (B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{H_K}$ is an isomorphism. As in the case of $D^+_{\mathrm{dif}}(V)$, the $K_\infty^{(\prime)}[[t, t_1, \ldots, t_e]]$-module $D^+_{e-\mathrm{dif}}(V)$ is endowed with the action of the operator

$$\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))}\sum_{k\geq 1}\frac{(1-\gamma)^k}{k}$$

if $\gamma \in \Gamma_0$ is close enough to 1. In addition to this operator $\nabla^{(0)}$, if $\beta_i \in \Gamma_i$ is close enough to 1, then the series of operators on $D^+_{e-\mathrm{dif}}(V)$:

$$\frac{\log(\beta_i)}{c_i(\beta_i)} = -\frac{1}{c_i(\beta_i)}\sum_{n\geq 1}\frac{(1-\beta_i)^n}{n}$$

converges to an operator $\nabla^{(i)} : D^+_{e-\mathrm{dif}}(V) \to D^+_{e-\mathrm{dif}}(V)$ and does not depend on the choice of β_i.

§ 3.2. Properties of differential operators

First, we consider the "meaning" of the equation $\nabla^{(j)}(F) = 0$. By definitions of differential operators, it follows easily that F is fixed by actions of an open subgroup of Γ_j. Thus, we can say that

"*Find solutions* $\{f_k\}_{k=1}^{d=\dim_{\mathbb{Q}_p} V}$ *(linearly independent over* K*) of* $\nabla^{(j)}(f_k) = 0$ *for* $0 \leq j \leq e$ *in* $D^+_{e-\mathrm{dif}}(V)[1/t]$"

$$\Downarrow$$

"*V is a potentially de Rham rep. of G_K, that is, a de Rham rep. of G_K*".

Thus, the theory of p-adic differential modules plays an important role in the proof of Theorem 1.3. Now, we will describe actions of operators $\nabla^{(j)}$ $(0 \leq j \leq e)$ on the module $D^+_{e-\mathrm{dif}}(V)$. First, by a standard argument, we can show that, if $x \in D^+_{e-\mathrm{dif}}(V)$, we have

$$\nabla^{(0)}(x) = \lim_{\gamma\to 1}\frac{\gamma(x)-x}{\chi(\gamma)-1} \quad \text{and} \quad \nabla^{(i)}(x) = \lim_{\beta_i\to 1}\frac{\beta_i(x)-x}{c_i(\beta_i)}.$$

With this presentation, we can easily describe actions of operators $\nabla^{(j)}$ $(0 \leq j \leq e)$ on the ring $K^{(\prime)}_\infty[[t, t_1, \ldots, t_e]]$ as follows.

Lemma 3.2. *We have*

$$\nabla^{(0)} = t\frac{d}{dt} \quad and \quad \nabla^{(i)} = t\frac{d}{dt_i} \quad (i \neq 0) \quad on\ K^{(\prime)}_\infty[[t, t_1, \ldots, t_e]].$$

We extend naturally actions of $K^{(\prime)}_\infty$-linear derivations $\nabla^{(0)}$ and $\nabla^{(i)}$ $(i \neq 0)$ on $D^+_{e-\mathrm{dif}}(V)$ to $D_{e-\mathrm{dif}}(V) = D^+_{e-\mathrm{dif}}(V)[1/t]$ by putting $\nabla^{(0)}(\frac{1}{t}) = -\frac{1}{t}$ and $\nabla^{(i)}(\frac{1}{t}) = 0$ $(i \neq 0)$. Now, compute the bracket $[\ ,\ ]$ of operators $\nabla^{(j)}$ $(0 \leq j \leq e)$.

Proposition 3.3. *On the* $K^{(\prime)}_\infty[[t, t_1, \ldots, t_e]][1/t]$*-module* $D_{e-\mathrm{dif}}(V)$ *as above, we have the following relation*

1. $\nabla^{(0)}\nabla^{(i)} - \nabla^{(i)}\nabla^{(0)} = \nabla^{(i)}$ *for all* $i \neq 0$*:*

2. $\nabla^{(j)}\nabla^{(i)} - \nabla^{(i)}\nabla^{(j)} = 0$ *for all* $i, j \neq 0$.

The following proposition describe actions of $\nabla^{(i)}$ $(i \neq 0)$ and plays a key role in the proof of Theorem 1.3..

Proposition 3.4. *Let M be a finite generated free $K_\infty^{(\prime)}[[t, t_1, \dots, t_e]][1/t]$-module endowed with $K_\infty^{(\prime)}$-linear operators $\{\nabla^{(j)}\}_{j=0}^{e}$ which satisfy Leibniz rule and relations in Proposition 3.3. Assume that M has a basis $\{g_j\}_{j=1}^{d}$ over $K_\infty^{(\prime)}[[t, t_1, \dots, t_e]]$ $[1/t]$ which satisfies $\nabla^{(0)}(g_j) = 0$. Then, the action of $\nabla^{(i)}$ $(i \neq 0)$ is given by*

$$\nabla^{(i)}(g_j) = t\sum_{k=1}^{d} c_k g_k, \ c_k \in K_\infty^{(\prime)}[[t, t_1, \dots, t_e]] \text{ and } \nabla^{(0)}(c_k) = 0.$$

Proof. Since $\{g_j\}_{j=1}^{d}$ forms a basis of M over $K_\infty^{(\prime)}[[t, t_1, \dots, t_e]][1/t]$, we have

$$\nabla^{(i)}(g_j) = \sum_{k=1}^{d} a_k g_k \qquad (a_k \in K_\infty^{(\prime)}[[t, t_1, \dots, t_e]][1/t]). \tag{3.1}$$

Then, by the relation of Proposition 3.3., we have

$$\sum_{k=1}^{d} \nabla^{(0)}(a_k) g_k = \sum_{k=1}^{d} a_k g_k$$

(note that we have $\nabla^{(0)}(g_j) = 0$ by hypothesis). Hence, we obtain the differential equation

$$\nabla^{(0)}(a_k) = a_k.$$

Define $c_k = a_k/t$, then it satisfies $\nabla^{(0)}(c_k) = a_k/t - a_k/t = 0$ and we see that c_k is contained in $K_\infty^{(\prime)}[[t, t_1, \dots, t_e]]$. Thus, the solutions of this differential equation have the following forms

$$a_k = c_k t \qquad \text{where } c_k \in K_\infty^{(\prime)}[[t, t_1, \dots, t_e]] \text{ and } \nabla^{(0)}(c_k) = 0. \tag{3.2}$$

Hence, we have, from (3.1) and (3.2),

$$\nabla^{(i)}(g_j) = t\sum_{k=1}^{d} c_k g_k \qquad \text{where } c_k \in K_\infty^{(\prime)}[[t, t_1, \dots, t_e]] \text{ and } \nabla^{(0)}(c_k) = 0.$$

□

Corollary 3.5. *With notations as in Proposition 3.4. above, we have*

$$(\nabla^{(1)})^{k_1}\cdots(\nabla^{(e)})^{k_e}(g_j)=t^{k_1+\cdots+k_e}\sum_{k=1}^{d}c_kg_k,\ c_k\in K_\infty^{(\prime)}[[t,t_1,\dots,t_e]]\ \textit{and}\ \nabla^{(0)}(c_k)=0.$$

§ 4. Proof of Theorem 1.3. in the de Rham representation case

Let us recall some notations

∘ K is a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p>0$ such that $[k:k^p]=p^e<\infty$:

∘ V is a p-adic representation of G_K of dimension d over $\mathbb{Q}_p$:

∘ $K'=$ the p-adic completion of $\cup_{m\geq 0}K(b_i^{1/p^m},1\leq i\leq e)$ is the complete discrete valuation field of characteristic 0 with perfect residue field:

∘ there exists a G_K-equivariant isomorphism

$$B_{\mathrm{dR}}=B_{\mathrm{dR}}^{+}[1/t]\simeq B_{\mathrm{dR}}^{+,\prime}[[t_1,\dots,t_e]][1/t].$$

(1) V: de Rham rep. of $G_K\implies V$: de Rham rep. of $G_{K'}$

Proof. Since V is a de Rham representation of G_K, there exists a G_K-equivariant isomorphism of B_{dR}-modules:

$$B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}V\simeq(B_{\mathrm{dR}})^d. \tag{4.1}$$

Now, by tensoring $B'_{\mathrm{dR}}\otimes_{B_{\mathrm{dR}}}$ (which is induced by the $G_{K'}$-equivariant surjection pr : $B_{\mathrm{dR}}\twoheadrightarrow B'_{\mathrm{dR}}$) over (4.1), we obtain a $G_{K'}$-equivariant isomorphism of B'_{dR}-modules:

$$B'_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}V\simeq(B'_{\mathrm{dR}})^d.$$

This means that V is a de Rham representation of $G_{K'}$. □

(2) V: de Rham rep. of $G_{K'}\implies V$: de Rham rep. of G_K

This is the difficult part of this note and the theory of p-adic differential modules plays a central role in the following proof. We have to bridge the gap between G_K and $G_{K'}$. Then, roughly speaking, since the differential operators $\{\nabla^{(i)}\}_{i=1}^{e}$ reflect this difference, it suffices to construct the solutions $\{f_k\}_{k=1}^{d=\dim_{\mathbb{Q}_p}V}$ of $\nabla^{(i)}(f_k)=0$ for $1\leq i\leq e$.

Lemma 4.1. *If V is a de Rham representation of $G_{K'}$, there exists a $G_{K'}$-equivariant isomorphism*

$$B_{\mathrm{dR}}\otimes D_{e\text{-dif}}(V)\simeq(B_{\mathrm{dR}})^d.$$

Proof. Since V is a de Rham representation of $G_{K'}$, there exists a $G_{K'}$-equivariant isomorphism of B'_{dR}-modules:

$$B'_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \simeq (B'_{\mathrm{dR}})^d. \tag{4.2}$$

Now, by tensoring $B_{\mathrm{dR}} \otimes_{B'_{\mathrm{dR}}}$ (which is induced by the $G_{K'}$-equivariant injection $i : B'_{\mathrm{dR}} \hookrightarrow B_{\mathrm{dR}}$) over (4.2), we obtain a $G_{K'}$-equivariant isomorphism of B_{dR}-modules:

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{dR}})^d.$$

On the other hand, we have a G_K-equivariant isomorphism

$$B_{\mathrm{dR}} \otimes D_{e-\mathrm{dif}}(V) \simeq B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V.$$

Thus, we obtain the desired isomorphism. □

Finally, we shall give the proof of (2).

Proof. We shall construct the $K_\infty^{(l)}$-linearly independent elements $\{f_j^{(*)}\}_{j=1}^d \in D_{e-\mathrm{dif}}(V)$ such that $\nabla^{(i)}(f_j^{(*)}) = 0$ for $0 \le i \le e$ and $1 \le j \le d$.

(A) Construction of $\{f_j^{(*)}\}_{j=1}^d \in D_{e-\mathrm{dif}}(V)$

Since V is a de Rham representation of $G_{K'}$, we have a basis $\{f_j\}_{j=1}^d$ of $D_{e-\mathrm{dif}}(V)$ over $K_\infty^{(l)}[[t, t_1, \dots, t_e]][1/t]$ such that, from Lemma 4.1.,

$$\nabla^{(0)}(f_j) = 0 \quad \text{for all } 1 \le j \le d.$$

Thus, we can apply Corollary 3.5. to the $K_\infty^{(l)}[[t, t_1, \dots, t_e]][1/t]$-module $D_{e-\mathrm{dif}}(V)$ generated by $\{f_j\}_{j=1}^d$ and then we can deduce

$$(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(g_j) = t^{k_1+\cdots+k_e} \sum_{k=1}^{d} c_k g_k, \quad c_k \in K_\infty^{(l)}[[t, t_1, \dots, t_e]] \text{ and } \nabla^{(0)}(c_k) = 0.$$

Then, if we define $f_j^{(*)} \in D_{e-\mathrm{dif}}(V)$ (converge for $(t, t_1, \dots, t_e)$-adic topology) by

$$f_j^{(*)} = \sum_{0 \le k_1, \dots, k_e} (-1)^{k_1+\cdots+k_e} \frac{t_1^{k_1} \cdots t_e^{k_e}}{k_1! \cdots k_e! t^{k_1+\cdots+k_e}} (\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(f_j),$$

it follows easily that we have $\nabla^{(i)}(f_j^{(*)}) = 0$ for $0 \le i \le e$.

(B) $\{f_j^{(*)}\}_{j=1}^d \in D_{e-\mathrm{dif}}(V)$ is linearly independent over $K_\infty^{(l)}$

By the presentation of $f_j^{(*)}$, we have

$$f_j^{(*)} = f_j + g_j \quad \text{where } f_j \notin, g_j \in (t_1, \dots, t_e) D_{e-\mathrm{dif}}(V).$$

Since $\{f_j\}_{j=1}^d$ forms a basis of $D_{e-\mathrm{dif}}(V)$ over $K_\infty^{(\prime)}[[t, t_1, \dots, t_e]][1/t]$, it is, in particular, linearly independent over $K_\infty^{(\prime)}$ $(\subset K_\infty^{(\prime)}[[t, t_1, \dots, t_e]][1/t])$. Then, it follows easily that $\{f_j^{(*)}\}_{j=1}^d$ is linearly independent over $K_\infty^{(\prime)}$ in $D_{e-\mathrm{dif}}(V)$.

(C) Conclusion

Therefore, on the K-vector space generated by $\{f_j^{(*)}\}_{j=1}^d$, $\log(\gamma)$ and $\log(\beta_i)$ act trivially ($\Leftrightarrow \nabla^{(0)}(f_j^{(*)}) = 0$ and $\nabla^{(i)}(f_j^{(*)}) = 0$ for all $1 \leq j \leq d$). Thus, this means that Γ_K acts on this K-vector space via finite quotient and there exists a finite extension L/K such that $\{f_j^{(*)}\}_{j=1}^d$ forms a basis of $D_{\mathrm{dR}}(V_L)$ over L $(\subset K_\infty^{(\prime)})$ where V_L denotes the restriction of V to G_L. Since a potentially de Rham representation of G_K is a de Rham representation of G_K, we complete the proof. □

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