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On several multiple zeta functions

By

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Abstract

In the introduction, a hypothetical relationship between new kind of multiple zeta functions and classical, basic zeta functions in number theory is stated. Some facts which may support validity of such relationship are outlined in the rest of the notes. Whereas the Riemann hypothesis is an assertion on the value distribution of Möbius function, the present work suggests that the value distribution of the power residue symbols are as intimately connected with the Riemann hypothesis as Möbius function.

Introduction.

We consider a multiple zeta function of the form

$$\sum_{m_1, m_2, m_3} \left(\frac{m_1}{m_2}\right)\left(\frac{m_2}{m_3}\right)\left(\frac{m_3}{m_1}\right)\frac{1}{|m_1m_2m_3|^{2\alpha}},$$

where $m_i \in \mathfrak{o} = \mathbb{Z}[\omega], \omega = e^{\frac{2\pi i}{3}} = \dfrac{-1+\sqrt{-3}}{2}$, $m_i \equiv 1(\mathrm{mod} 3)$, $(m_i, m_j) = 1$, and $\left(\frac{\alpha}{\beta}\right)$ stands for the cubic residue symbol of $F = \mathbb{Q}(\sqrt{-3})$.

It is expected that the convergence or holomorphy of this function in the region $\mathrm{Re}\alpha > \sigma_0$ implies the holomorphy of

$$\Lambda_3(s) = 16 \cdot 3^{3s}\zeta_F(3s - \frac{3}{2})L(s, \eta)^{-1}$$

in the region $\mathrm{Re}s > \sigma_0 + \frac{1}{6}$ except the zeros of $M(2s, K_{\frac{1}{3}}^3)$, where ζ_F is the Dedekind zeta function of F, $L(s, \eta)$ is the L-function of the Grössencharakter η determined by $\eta(c) = \frac{c}{|c|}$, $(c \equiv 1 \pmod 3)$, of F, $K_{\frac{1}{3}}$ is a Bessel function, and M means the Mellin transform defined by

$$M(s, f) = \int_0^\infty f(v)v^s \frac{dv}{v}.$$

We also consider a multiple zeta function of the form

$$\sum_{m_1, m_2, m_3} \left(\frac{m_1}{m_2}\right)\left(\frac{m_2}{m_3}\right)\left(\frac{m_3}{m_1}\right)\frac{\overline{\eta}(m_1m_2m_3)}{|m_1m_2m_3|^{2\alpha}}.$$

It is expected that that the convergence or holomophy of this function in the region $\mathrm{Re}\alpha > \sigma_0'$ implies the holomorphy of

$$\Lambda_{2,1}(s) = 8 \cdot 3^{3s+\frac{1}{2}}\left(1 - \frac{1}{3^s}\right)^{-1} \zeta_F(s)^{-1} L(3s - \frac{3}{2}, \overline{\eta}^3)$$

in the region $\mathrm{Re}s > \sigma_0' + \frac{1}{6}$ except zeros of $M(2s+1, K_{\frac{1}{3}}^3)$.

In the sequel, we outline some facts which may be evidences supporting validity of preceeding assertions.

§1. Cubic theta function.

On the upper half space $H = \{u = (z, v) | z \in \mathbb{C}, v > 0\}$, we define a function by the Fourier series

$$\theta(u) = \theta(z, v) = v^{\frac{2}{3}} + \sum_{m \neq 0} \tau(m) v K_{\frac{1}{3}}(4\pi |m| v) e(mz),$$

$(m \in \frac{1}{3\sqrt{-3}}\mathfrak{o}), e(z) = \exp(2\pi \mathrm{i}(z + \overline{z}))$, and call it cubic theta function. The Fourier coefficients are concretely givien by

$$\begin{aligned}
\tau(m) &= 2\left(\frac{3}{c}\right)^{-1} g(c) \left|\frac{d}{c}\right| 3^{\frac{N}{2}}, && \text{(case1)},\\
\tau(m) &= 2\zeta\left(\frac{3\omega}{c}\right)^{-1} g(c) \left|\frac{d}{c}\right| 3^{\frac{N}{2}}, && \text{(case2)},\\
\tau(m) &= 2\zeta^{-1}\left(\frac{3\omega^2}{c}\right)^{-1} g(c) \left|\frac{d}{c}\right| 3^{\frac{N}{2}}, && \text{(case3)},\\
\tau(m) &= 2g(c) \left|\frac{d}{c}\right| 3^{\frac{N}{2}}, && \text{(case4)},
\end{aligned}$$

$(\zeta = e^{\frac{2\pi \mathrm{i}}{9}})$, using cubic Gauss sums

$$g(c) = \sum_{\substack{\delta \bmod c \\ (\delta,\, c)=1}} \left(\frac{\delta}{c}\right) e\left(\frac{\delta}{c}\right), \quad (c \equiv 1 \pmod 3),$$

for four cases

$$\begin{aligned}
m &= \pm \sqrt{-3}^{3N-1} cd^3,\ (N \geqq 0), && \text{(case1)},\\
m &= \pm\omega \sqrt{-3}^{3N-1} cd^3,\ (N \geqq 0), && \text{(case2)},\\
m &= \pm\omega^2 \sqrt{-3}^{3N-1} cd^3,\ (N \geqq 0), && \text{(case3)},\\
m &= \pm \sqrt{-3}^{3N-3} cd^3,\ (N \geqq 0), && \text{(case4)},
\end{aligned}$$

where c are all quare free, and $c, d \equiv 1 \pmod 3$. Otherwise $\tau(m) = 0$.

An important property of the cubic theta function is that it is an automorphic function. Since $H = SL(2, \mathbb{C})/SU(2)$, $SL(2, \mathbb{C})$ operates on H. Put in particular

$$\Gamma = \left\{\sigma = \begin{pmatrix} a\, b \\ c\, d \end{pmatrix}; \sigma \equiv \begin{pmatrix} 1\, 0 \\ 0\, 1 \end{pmatrix} (\mathrm{mod} 3)\right\},$$

($\sigma \in SL(2, \mathfrak{o})$), then $\theta(u)$ satisfies

$$\theta(\sigma u) = \overline{\chi}(\sigma)\theta(u), \qquad (\sigma \in \Gamma).$$

Here, χ is given by $\chi(\sigma) = \left(\frac{c}{d}\right)$ or $\chi(\sigma) = 1$ according as $c \neq 0$ or $c = 0$, and is a character (representation) of Γ in the sense that $\chi(\sigma_1\sigma_2) = \chi(\sigma_1)\chi(\sigma_2)$, $(\sigma_i \in \Gamma)$, holds. We call χ a metaplectic character, and $\theta(u)$ a metaplectic automorphic function.

§2. Dirichlet series involving powers of cubic Gauss sums.

We now propose to investigate in general Dirichlet series of the form

$$\Lambda_n(s) = \sum_m \frac{\tau(m)^n}{|m|^{2s}},$$

where n is a natural number, because powers of cubic Gauss sums have various interesting properties. The series $\Lambda_1(s)$ can be treated in a usual way within the theory of automorphic forms, and, as for $\Lambda_2(s)$, Rankin's method is applicable to it. On the contrary, it is difficult to handle $\Lambda_3(s)$, but, by virtue of the important relation

$$\frac{g(c)^3}{|c|^3} = \mu(c)\frac{c}{|c|} = \mu(c)\eta(c)$$

containing Möbius function μ, a direct computation using explicit form of $\tau(m)$ in §1 gives rise to

$$\Lambda_3(s) = 16 \cdot 3^{3s}\zeta_F(3s - \frac{3}{2})L(s, \eta)^{-1},$$

as appeared in the former half part of introduction. This case is therefore remarkable.

An integral representation of routine form can be given to $\Lambda_n(s)$. Let $\theta_n(u) = \theta_n(z, v)$ be the n-ple convolution of $\theta(u)$ as a function on $\mathbb{C}/3\mathfrak{o}$. The convolution of functions on $\mathbb{C}/3\mathfrak{o}$ is defined by

$$\int_{\mathbb{C}/3\mathfrak{o}} f(z+t)g(-t)|dt|,$$

where $|dt|$ is the Euclidean measure on $\mathbb{C}$. Put $\theta_n(v) = \theta_n(0, v)$, let $\Xi_n(s)$ be

$$\Xi_n(s) = \int_0^\infty (m_0^{-(n-1)}\theta_n(v) - v^{\frac{2n}{3}})v^{2s}\frac{dv}{v},$$

$m_0 = \frac{9\sqrt{3}}{2}$ being the surface area of $\mathbb{C}/3\mathfrak{o}$. Then,

$$\Xi_n(s) = \int_0^\infty \sum_{m\neq 0} \tau(m)^n v^n K_{\frac{1}{3}}(4\pi|m|v)^n v^{2s}\frac{dv}{v}$$

$$= \sum_{m\neq 0} \frac{\tau(m)^n}{|m|^{2s+n}} \cdot (4\pi)^{-2s-n}M(2s+n, K_{\frac{1}{3}}^n),$$

and thus $\Lambda_n(s)$ takes place here. This integral representation itself is not very effective, but, to determine the region of holomorphy of $\Lambda_3(s)$, it is enough to reveal the behavior of the integrand

$$m_0^{-2}\theta_3(v) - v^2 = \sum_{m\neq 0} \tau(m)^3 v^3 K_{\frac{1}{3}}(4\pi|m|v)^3,$$

$(m \in \frac{1}{3\sqrt{-3}}\mathfrak{o})$, as $v \to 0$, since this integrand is rapidly decreasing as $v \to \infty$. So we go back to the origin from where the cubic theta function comes from.

§3. Metaplectic Eisenstein series.

We define an Eisenstein series

$$E(u,\alpha) = E(z,v,\alpha) = \sum_{\Gamma_\infty\backslash\Gamma} \chi(\sigma)v(\sigma u)^\alpha, \ (\mathrm{Re}\alpha > 2),$$

with the metaplectic character χ in §1. Here, $v(u)$ stands for the v-coordinate of $u = (z,v) \in H$. If $u = (z,v), \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$, then

$$v(\sigma u) = \frac{v}{|cz+d|^2 + c^2v^2}.$$

As a function of z with period $3\mathfrak{o}$, the Fourier expasion of $E(u,\alpha)$ is given by computing

$$\int_{\mathbb{C}/3\mathfrak{o}} E(u,\alpha)e(-mz)|dz|$$

$$= \int_{\mathbb{C}/3\mathfrak{o}} \sum_{c,\,d}\left(\frac{c}{d}\right)\frac{v^\alpha}{(|cz+d|^2+|c|^2v^2)^\alpha}e(-mz)|dz|,$$

$(c \equiv 0, d \equiv 1(\mathrm{mod}3), (c,\,d) = 1)$. The result is

$$E(u,\alpha) = v^\alpha + \varphi(\alpha)v^{2-\alpha}$$

$$+m_0^{-1}\sum_{m\neq 0}(2\pi)^\alpha\Gamma(\alpha)^{-1}|m|^{\alpha-1}\varphi_m(\alpha)\cdot$$

$$\cdot vK_{\alpha-1}(4\pi|m|v)e(mz),$$

$(m \in \frac{1}{3\sqrt{-3}}\mathfrak{o})$, where $\varphi_m(\alpha)$ is a Dirichlet series. In particular,

$$\varphi(\alpha) = m_0^{-1}\frac{\pi}{\alpha-1}\varphi_0(\alpha),$$

and

$$\varphi_0(\alpha) = 4\cdot 3^{-(3\alpha-2)}\left(1-\frac{1}{3^{3\alpha-2}}\right)^{-1}\frac{\zeta_F(3\alpha-3)}{\zeta_F(3\alpha-2)}.$$

Since $\varphi_0(\alpha)$ has a pole of first order at $\frac{4}{3}$, the general theory of Eisenstein series shows that $E(u,\alpha)$ itself has a pole of first order at $\frac{4}{3}$, and produces there the residual form

$$\rho(u) = \rho(z,v) = \lim_{\alpha\to\frac{4}{3}} (\alpha - \frac{4}{3})E(u,\alpha).$$

This is, together with $E(u,\alpha)$, a metaplectic automorphic function. The constant term of $\rho(u)$ is $v^{\frac{2}{3}}$ multiplied by a constant. Dividing $\rho(u)$ by the constant, there arises what we call cubic theta function.

Let $E_n(z,v,\alpha) = E_n(u,\alpha)$ be the n-ple convolution of $E(u,\alpha) = E(z,v,\alpha)$ as a function of z with period $3\mathfrak{o}$, and let

$$\Xi_n(s,\alpha)$$
$$= \int_0^\infty (m_0^{-(n-1)}E_n(0,v,\alpha) - (v^\alpha + \varphi(\alpha)v^{2-\alpha})^n)v^{2s}\frac{dv}{v}.$$

Then,

$$\Xi_n(s,\alpha) = m_0^{-n}2^{-4s+n\alpha-2n}\pi^{-2s+n\alpha-n}\Gamma(\alpha)^{-n}\cdot$$
$$\cdot\sum_{m\neq 0}\frac{\varphi_m(\alpha)^n}{|m|^{2s-n\alpha+2n}}M(2s+n,K^n_{\alpha-1}).$$

If enough information on the behavior of $E_n(0,v,\alpha)$ as $v\to 0$ is obtained, the region of holomorphy of the Dirichlet series contained in $\Xi_n(s,\alpha)$, and as a consequence, the region of holomorphy of $\Xi_3(s)$ or of $L(s,\eta)^{-1}$ will possibly be deduced. Toward this aim, we adopt termwise convolution of Eisenstein series.

§4. Termwise convolution.

To observe the behavior of $E_n(0,v,\alpha)$ as $v\to 0$, it suffices to observe the behavior of the value $E_3(v,\alpha)$ at $z=0$ of the triple convolution of $E(u,\alpha)-v^\alpha = E(z,v,\alpha)-v^\alpha$, since the behavior of v^α is very clear. The expression $E(z,v,\alpha)-v^\alpha$ means the sum of all terms in the Eisenstein series such that $c\neq 0$.

Now,

$$E_3(v,\alpha) = \int_{\mathbb{C}/3\mathfrak{o}}\int_{\mathbb{C}/3\mathfrak{o}}(E(z_1+z_2,v,\alpha)-v^\alpha)\cdot$$
$$\cdot(E(-z_1,v,\alpha)-v^\alpha)(E(-z_2,v,\alpha)-v^\alpha)|dz_1||dz_2|$$
$$= \int_{\mathbb{C}/3\mathfrak{o}}\int_{\mathbb{C}/3\mathfrak{o}}\sum_{c,\,d}\left(\frac{c_1}{d_1}\right)\frac{v^\alpha}{(|c_1(z_1+z_2)+d_1|^2+|c_1|^2v^2)^\alpha}\cdot$$
$$\cdot\left(\frac{c_2}{d_2}\right)\frac{v^\alpha}{(|-c_2z_1)+d_2|^2+|c_2|^2v^2)^\alpha}\cdot$$
$$\cdot\left(\frac{c_3}{d_3}\right)\frac{v^\alpha}{(|-c_3z_2)+d_3|^2+|c_3|^2v^2)^\alpha}|dz_1||dz_2|.$$

To proceed further, some notation should be prepared. We decompose c_i as

$$c_1 = c_0 n_2 n_3 c_1' = (\lambda_0 c_0')(\lambda_2' n_2')(\lambda_3' n_3')\lambda_1 \varepsilon_1 m_1,$$

$$c_2 = c_0 n_3 n_1 c_2' = (\lambda_0 c_0')(\lambda_3' n_3')(\lambda_1' n_1')\lambda_2 \varepsilon_2 m_2,$$

$$c_3 = c_0 n_1 n_2 c_3' = (\lambda_0 c_0')(\lambda_1' n_1')(\lambda_2' n_2')\lambda_3 \varepsilon_3 m_3,$$

where c_i' are mutually prime, so are n_i, too, and $(n_i, c_i') = 1$. Among the symbols, λ_i, λ_i' are powers of $\sqrt{-3}$, $\varepsilon_i^6 = 1$, and all others are $\equiv 1 \pmod 3$. Thus,

$$[c_1, c_2, c_3] = c_0 n_1 n_2 n_3 c_1' c_2' c_3'$$
$$= c_0 (\lambda_1' n_1')(\lambda_2' n_2')(\lambda_3' n_3') \cdot$$
$$\cdot (\lambda_1 \varepsilon_1 m_1)(\lambda_2 \varepsilon_2 m_2)(\lambda_3 \varepsilon_3 m_3)$$

is a l.c.m. of c_1, c_2, c_3, and λ_i, λ_i' are 1 except at most one, respectively. Furthemore we determine k by

$$\frac{d_1}{c_1} + \frac{d_2}{c_2} + \frac{d_3}{c_3} = \frac{k}{[c_1, c_2, c_3]}.$$

On the other hand, putting

$$\gamma_1(z) = \gamma_1(z, \alpha) = \frac{1}{(|z|^2 + 1)^\alpha}, \qquad (\mathrm{Re}\alpha > 1),$$

we denote by $\gamma_3(z)$ the triple convolution of $\gamma_1(z, \alpha)$ on $\mathbb{C}$, i.e.

$$\gamma_3(z) = \gamma_3(z, \alpha)$$
$$= \int_{\mathbb{C}} \int_{\mathbb{C}} \gamma_1(z + t_1 + t_2)\gamma_1(-t_1)\gamma_1(-t_2)|dt_1||dt_2|.$$

In addition, S will stand for a complete system of representatives of $\mathfrak{o}$ mod $c_0 n_1 n_2 n_3$, and μ_i, $(i = 1, 2, 3)$, will be parameters running through S.

A rather long computation in which cubic reciprocity law is repeatedly needed yields finally

$$E_3(v, \alpha) = \sum_{k \neq 0} \sum_{c_0, n} \frac{1}{|c_0|^{6\alpha} |n_1 n_2 n_3|^{4\alpha}} \cdot$$
$$\cdot \sum_{\lambda, \varepsilon, \mu} \left(\frac{\mu_1}{n_1'}\right)^{-1} \left(\frac{\mu_2}{n_2'}\right)^{-1} \left(\frac{\mu_3}{n_3'}\right)^{-1} \frac{j_k(c_0, n, \lambda, \varepsilon, \mu)}{|\lambda_1 \lambda_2 \lambda_3|^{2\alpha}} \cdot$$
$$\cdot \sum_{m \equiv \mu (\mathrm{mod}\, c_0 n_1 n_2 n_3)}$$
$$\left(\frac{\lambda_1' \lambda_2 \lambda_3 \varepsilon_2 \varepsilon_3}{m_1}\right)^{-1} \left(\frac{\lambda_2' \lambda_3 \lambda_1 \varepsilon_3 \varepsilon_1}{m_2}\right)^{-1} \left(\frac{\lambda_3' \lambda_1 \lambda_2 \varepsilon_1 \varepsilon_2}{m_3}\right)^{-1}.$$

$$\cdot\Big(\frac{k}{m_1m_2m_3}\Big)\Big(\frac{m_1}{m_2}\Big)\Big(\frac{m_2}{m_3}\Big)\Big(\frac{m_3}{m_1}\Big)\frac{1}{|m_1m_2m_3|^{2\alpha}}\cdot$$

$$\cdot v^{4-3\alpha}\gamma_3\Big(\frac{k}{vc_0n_1n_2n_3c_1'c_2'c_3'}\Big)$$

$$+\sum_{c_0,n}\sum_{\varepsilon}\frac{j_0(c_0,n,\varepsilon)}{|c_0|^{6\alpha}|n_1n_2n_3|^{4\alpha}}v^{4-3\alpha}\gamma_3(0).$$

Here, j_k, ($k \in \mathfrak{o}$), is a finite sum similar to the Jacobi sum, and symbols without index like n are abbreviations of triples like n_1, n_2, n_3.

Now, passing to the formal Mellin transform, the above formula turns finally into

$$\int_0^\infty \Big(E_3(v,\alpha)-\sum_{c_0,\,n}\sum_{\varepsilon}\frac{j_0(c_0,n,\varepsilon)}{|c_0|^{6\alpha}|n_1n_2n_3|^{4\alpha}}v^{4-3\alpha}\gamma_3(0)\Big)\cdot$$

$$\cdot v^{2s}\frac{dv}{v}$$

$$=\sum_{k\neq 0}\sum_{c_0,n}\frac{1}{|c_0|^{2s+3\alpha+4}|n_1n_2n_3|^{2s+\alpha+4}}\cdot$$

$$\cdot\sum_{\lambda,\,\varepsilon,\,\mu}\Big(\frac{\mu_1}{n_1'}\Big)^{-1}\Big(\frac{\mu_2}{n_2'}\Big)^{-1}\Big(\frac{\mu_3}{n_3'}\Big)^{-1}\frac{j_k(c_0,n,\lambda,\varepsilon,\mu)}{|\lambda_1\lambda_2\lambda_3|^{2s-\alpha+4}}\cdot$$

$$\cdot\sum_{m\equiv\mu(\mathrm{mod}\, c_0n_1n_2n_3)}$$

$$\Big(\frac{\lambda_1'\lambda_2\lambda_3\varepsilon_2\varepsilon_3}{m_1}\Big)^{-1}\Big(\frac{\lambda_2'\lambda_3\lambda_1\varepsilon_3\varepsilon_1}{m_2}\Big)^{-1}\Big(\frac{\lambda_3'\lambda_1\lambda_2\varepsilon_1\varepsilon_2}{m_3}\Big)^{-1}\cdot$$

$$\cdot\Big(\frac{k}{m_1m_2m_3}\Big)\Big(\frac{m_1}{m_2}\Big)\Big(\frac{m_2}{m_3}\Big)\Big(\frac{m_3}{m_1}\Big)\frac{|k|^{2s-3\alpha+4}}{|m_1m_2m_3|^{2s-\alpha+4}}\cdot$$

$$\cdot M(-2s+3\alpha-4,\gamma_3).$$

Actually this formula contains multiple zeta functions announced in the introduction, and so, we are lead to the hypotheses stated there.

It should be noted that all our arguments after §1 concern solely the former half part of the introduction. Corresponding arguments for the latter half are similar. The only difference is that we have to apply the differential operator $\frac{\partial}{\partial z}$ on the triple convolution of $E(z,v,\alpha)$.

Epilogue.

It seems to be fairly hard to investigate numbers σ_0 and σ_0' in the introduction, but an observation of the double convolution of Eisenstein series compared with results coming from Rankin's method, it is to a certain extent plausible that the double zeta function

$$\sum_{m_1, m_2} \left(\frac{m_1}{m_2}\right) \frac{1}{|m_1 m_2|^{2\alpha}},$$

$(m_1 \equiv m_2 \equiv 1(\text{mod}3),\ (m_1, m_2) = 1)$, is holomorphic in $\text{Re}\alpha > \frac{2}{3}$. So, for instance, the comparison of $\frac{2}{3}$ and σ_0' concerning

$$\sum_{m_1, m_2, m_3} \left(\frac{m_1}{m_2}\right)\left(\frac{m_2}{m_3}\right)\left(\frac{m_3}{m_1}\right) \frac{\overline{\eta}(m_1 m_2 m_3)}{|m_1 m_2 m_3|^{2\alpha}}$$

becomes a big question. Following hypotheses in the introduction, we can drive a vague imagination that, if $\sigma_0' = \frac{1}{3}$, then Riemann's hypothesis for $\zeta_F(s)$ can be valid, if $\sigma_0' = \frac{2}{3}$, then Riemann's hypothesis for $\zeta_F(s)$ is valid only up to $\text{Re}s > \frac{5}{6}$, and if $\sigma_0' = \frac{5}{6}$, then Riemann's hypothesis is false, that is, there exsists no region of the form $\text{Re}s > 1 - \epsilon, (\epsilon > 0)$, in which $\zeta_F(s)$ has no zero.

By means of the Fourier coefficients $\{a_m\}$ of an automorphic form, a Dirichlet series with the same coefficients is defined whose properties can be investigated fairly precisely. As for the coefficients $\{a_m\}$,$\{b_m\}$ of two automorphic forms, the properties of the Dirichlet series with coefficients $\{a_m b_m\}$ may also be studied by Rankin method, for instance. Nevertheless, for the coefficients $\{a_m\}, \{b_m\}, \{c_m\}$ of three automorphic forms, it is very difficult to consider the properties of the Dirichlet series with coefficients $\{a_m b_m c_m\}$, and, at least, no generally applicable, functional analytic method is expected presently. But, if such method were invented, we would not need complicated discussions as in the present notes.

The autor has not yet sufficiently determined zeros of $M(s, K^3_{\frac{1}{3}})$ and $M(s, \gamma_3)$.

Bibliography.

A suitable bibliography for the present notes is S.J.Patterson: The constant term of the cubic theta series, J. reine angew. Math., 336, (1982), 185-190, with its references.

In addition, more detailed versions of the present notes are available at

http://www.geocities.jp/okatayo119/

with file names Metap in Japanese and eMetap in English.