<table>
<thead>
<tr>
<th>Title</th>
<th>On several multiple zeta functions (Proceedings of the Symposium on Algebraic Number theory and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>KUBOTA, Tomio</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2007), B4: 123-130</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2007-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/174168">http://hdl.handle.net/2433/174168</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
On several multiple zeta functions

By

Tomio KUBOTA

Abstract

In the introduction, a hypothetical relationship between new kind of multiple zeta functions and classical, basic zeta functions in number theory is stated. Some facts which may support validity of such relationship are outlined in the rest of the notes. Whereas the Riemann hypothesis is an assertion on the value distribution of Möbius function, the present work suggests that the value distribution of the power residue symbols are as intimately connected with the Riemann hypothesis as Möbius function.

Introduction.
We consider a multiple zeta function of the form

\[
\sum_{m_1, m_2, m_3} \left( \frac{m_1}{m_2} \right) \left( \frac{m_2}{m_3} \right) \left( \frac{m_3}{m_1} \right) \frac{1}{|m_1 m_2 m_3|^{2\alpha}},
\]

where \( m_i \in \mathbb{Z} \), \( \omega = -1 + \frac{-3}{2} \), \( m_i \equiv 1 \pmod{3} \), \( (m_i, m_j) = 1 \), and \( \left( \frac{\alpha}{\beta} \right) \) stands for the cubic residue symbol of \( F = \mathbb{Q}(\sqrt{-3}) \).

It is expected that the convergence or holomorphy of this function in the region \( \text{Re} \alpha > \sigma_0 \) implies the holomorphy of

\[
\Lambda_3(s) = 16 \cdot 3^{3s} \zeta_F(3s - \frac{3}{2}) L(s, \eta)^{-1}
\]

in the region \( \text{Re} s > \sigma_0 + \frac{1}{6} \) except the zeros of \( M(2s, K^3_1) \), where \( \zeta_F \) is the Dedekind zeta function of \( F \), \( L(s, \eta) \) is the L-function of the Grössencharakter \( \eta \) determined by \( \eta(c) = \frac{c}{|c|} \), \( (c \equiv 1 \pmod{3}) \), of \( F \), \( K^3_1 \) is a Bessel function, and \( M \) means the Mellin transform defined by

\[
M(s, f) = \int_0^\infty f(v)v^{s-1}dv.
\]

We also consider a multiple zeta function of the form

\[
\sum_{m_1, m_2, m_3} \left( \frac{m_1}{m_2} \right) \left( \frac{m_2}{m_3} \right) \left( \frac{m_3}{m_1} \right) \frac{\overline{\eta}(m_1 m_2 m_3)}{|m_1 m_2 m_3|^{2\alpha}}
\]
It is expected that the convergence or holomorphy of this function in the region \( \text{Re} \alpha > \sigma'_0 \) implies the holomorphy of

\[
\Lambda_{2,1}(s) = 8 \cdot 3^{3s+\frac{1}{2}} \left(1 - \frac{1}{3^s}\right)^{-1} \zeta_F(s)^{-1} L(3s - \frac{3}{2}, \eta^3)
\]

in the region \( \text{Re} s > \sigma'_0 + \frac{1}{6} \) except zeros of \( M(2s + 1, K_{\frac{3}{2}}) \).

In the sequel, we outline some facts which may be evidences supporting validity of preceding assertions.

§1. Cubic theta function.

On the upper half space \( H = \{u = (z, v) | z \in \mathbb{C}, v > 0\} \), we define a function by the Fourier series

\[
\theta(u) = \theta(z, v) = v^{\frac{2}{3}} + \sum_{m \neq 0} \tau(m)vK_{\frac{1}{3}}(4\pi|m|v)e(mz),
\]

\((m \in \frac{1}{3\sqrt{-3}}0), e(z) = \exp(2\pi i(z + \overline{z}))\), and call it cubic theta function. The Fourier coefficients are concretely given by

\[
\tau(m) = \begin{cases} 
2(\frac{3}{c})^{-1}g(c)|d|3^{\frac{N}{2}}, & \text{(case1)}, \\
2\zeta(\frac{3\omega}{c})^{-1}g(c)|d|3^{\frac{N}{2}}, & \text{(case2)}, \\
2\zeta^{-1}(\frac{3\omega^2}{c})^{-1}g(c)|d|3^{\frac{N}{2}}, & \text{(case3)}, \\
2g(c)|d|3^{\frac{N}{2}}, & \text{(case4)},
\end{cases}
\]

\((\zeta = e^{\frac{2\pi i}{9}})\), using cubic Gauss sums

\[
g(c) = \sum_{\delta \mod c} \left(\frac{\delta}{c}\right), \quad (c \equiv 1 \pmod{3}),
\]

for four cases

\[
m = \pm \overline{-3}^{3N-1}cd^3, \quad (N \geq 0), \quad \text{(case1)},
\]

\[
m = \pm \omega \overline{-3}^{3N-1}cd^3, \quad (N \geq 0), \quad \text{(case2)},
\]

\[
m = \pm \omega^2 \overline{-3}^{3N-1}cd^3, \quad (N \geq 0), \quad \text{(case3)},
\]

\[
m = \pm \overline{-3}^{3N-3}cd^3, \quad (N \geq 0), \quad \text{(case4)},
\]

where \(c\) are all square free, and \(c, d \equiv 1 \pmod{3}\). Otherwise \(\tau(m) = 0\).

An important property of the cubic theta function is that it is an automorphic function. Since \(H = SL(2, \mathbb{C})/SU(2)\), \(SL(2, \mathbb{C})\) operates on \(H\). Put in particular

\[
\Gamma = \{\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \sigma \equiv \begin{pmatrix} 10 \\ 01 \end{pmatrix} \pmod{3}\},
\]
(σ ∈ SL(2, ℵ)), then θ(u) satisfies

$$\theta(σu) = \overline{χ}(σ)θ(u), \quad (σ ∈ Γ).$$

Here, χ is given by $χ(σ) = \begin{pmatrix} c \\ d \end{pmatrix}$ or $χ(σ) = 1$ according as $c ≠ 0$ or $c = 0$, and is a character (representation) of Γ in the sense that $χ(σ_1σ_2) = χ(σ_1)χ(σ_2)$, $(σ_i ∈ Γ)$, holds. We call χ a metaplectic character, and θ(u) a metaplectic automorphic function.

§2. Dirichlet series involving powers of cubic Gauss sums.

We now propose to investigate in general Dirichlet series of the form

$$Λ_n(s) = \sum_{m} \frac{τ(m)^n}{|m|^{2s}},$$

where $n$ is a natural number, because powers of cubic Gauss sums have various interesting properties. The series $Λ_1(s)$ can be treated in a usual way within the theory of automorphic forms, and, as for $Λ_2(s)$, Rankin’s method is applicable to it. On the contrary, it is difficult to handle $Λ_3(s)$, but, by virtue of the important relation

$$\frac{g(c)^3}{|c|^3} = \mu(c)\frac{c}{|c|} = \mu(c)η(c)$$

containing Möbius function μ, a direct computation using explicit form of $τ(m)$ in §1 gives rise to

$$Λ_3(s) = 16 \cdot 3^2ζ_F(3s - 3/2)L(s, η)^{-1},$$

as appeared in the former half part of introduction. This case is therefore remarkable.

An integral representation of routine form can be given to $Λ_n(s)$. Let $θ_n(u) = θ_n(z, v)$ be the $n$-ple convolution of $θ(u)$ as a function on $C/30$. The convolution of functions on $C/30$ is defined by

$$\int_{C/30} f(z + t)g(-t)|dt|,$$

where $|dt|$ is the Euclidean measure on $C$. Put $θ_n(v) = θ_n(0, v)$, let $Ξ_n(s)$ be

$$Ξ_n(s) = \int_{0}^{∞} (m_0^{-(n-1)}θ_n(v) - v^{2s})v^{2s}dv,$$

$m_0 = \frac{9√3}{2}$ being the surface area of $C/30$. Then,

$$Ξ_n(s) = \sum_{m≠0} \frac{τ(m)^n}{|m|^{2s+n}} \cdot (4π)^{-2s-n}M(2s + n, K_n^{1/3}),$$
and thus $\Lambda_n(s)$ takes place here. This integral representation itself is not very effective, but, to determine the region of holomorphy of $\Lambda_3(s)$, it is enough to reveal the behavior of the integrand

$$m_0^{-2}\theta_3(v) - v^2 = \sum_{m \neq 0} \tau(m)^3 v^3 K_{\frac{1}{3}}(4\pi|m|v)^3,$$

$(m \in \frac{1}{3^{\sqrt{-3}}0})$, as $v \to 0$, since this integrand is rapidly decreasing as $v \to \infty$. So we go back to the origin from where the cubic theta function comes from.

§3. Metaplectic Eisenstein series.

We define an Eisenstein series

$$E(u, \alpha) = E(z, v, \alpha) = \sum_{\Gamma_{\infty}\backslash \Gamma} \chi(\sigma) v(\sigma u)^{\alpha}, \quad (\text{Re} \alpha > 2),$$

with the metaplectic character $\chi$ in §1. Here, $v(u)$ stands for the $v$-coordinate of $u = (z, v) \in H$. If $u = (z, v), \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})$, then

$$v(\sigma u) = \frac{v}{|cz+d|^2 + c^2v^2}.$$

As a function of $z$ with period $3\alpha$, the Fourier expansion of $E(u, \alpha)$ is given by computing

$$\int_{\mathbb{C}/30} E(u, \alpha) e(-mz) |dz| = \int_{\mathbb{C}/30} \sum_{c,d} \left( \frac{c}{d} \right) \frac{v^{\alpha}}{|cz+d|^2 + |c|^2v^2}^{\alpha} e(-mz) |dz|,$$

$(c \equiv 0, d \equiv 1 (\text{mod} 3), (c, d) = 1)$. The result is

$$E(u, \alpha) = v^{\alpha} + \varphi(\alpha) v^{2-\alpha} + m_0^{-1} \sum_{m \neq 0} (2\pi)^\alpha \Gamma(\alpha)^{-1} |m|^{\alpha-1} \varphi_m(\alpha),$$

$$vK_{\alpha-1}(4\pi|m|v)e(mz), (m \in \frac{1}{3^{\sqrt{-3}}0}),$$

where $\varphi_m(\alpha)$ is a Dirichlet series. In particular,

$$\varphi(\alpha) = m_0^{-1} \frac{\pi}{\alpha-1} \varphi_0(\alpha),$$

and

$$\varphi_0(\alpha) = 4 \cdot 3^{-(3\alpha-2)} \left(1 - \frac{1}{3^{3\alpha-2}}\right)^{-1} \frac{\zeta_F(3\alpha-3)}{\zeta_F(3\alpha-2)}.$$
Since $\varphi_0(\alpha)$ has a pole of first order at $\frac{4}{3}$, the general theory of Eisenstein series shows that $E(u, \alpha)$ itself has a pole of first order at $\frac{4}{3}$, and produces there the residual form
\[
\rho(u) = \rho(z, v) = \lim_{\alpha \to \frac{4}{3}} (\alpha - \frac{4}{3}) E(u, \alpha).
\]
This is, together with $E(u, \alpha)$, a metaplectic automorphic function. The constant term of $\rho(u)$ is $v^{\frac{2}{3}}$ multiplied by a constant. Dividing $\rho(u)$ by the constant, there arises what we call cubic theta function.

Let $E_n(z, v, \alpha) = E_n(u, \alpha)$ be the $n$-ple convolution of $E(u, \alpha) = E(z, v, \alpha)$ as a function of $z$ with period $3\sigma$, and let
\[
\Xi_n(s, \alpha) = \int_0^\infty (m_0^{-(n-1)} E_n(0, v, \alpha) - (v^\alpha + \varphi(\alpha)v^{2-\alpha})^n)v^{2s} \frac{dv}{v}.
\]
Then,
\[
\Xi_n(s, \alpha) = m_0^{-n} 2^{-4s+n \alpha-2n} \pi^{-2s+n \alpha-2n} \Gamma(\alpha)^{-n} \cdot \sum_{m \neq 0} \frac{\varphi_m(\alpha)^n}{|m|^{2s-n \alpha+2n}} M(2s+n, K_\alpha).
\]

If enough information on the behavior of $E_n(0, v, \alpha)$ as $v \to 0$ is obtained, the region of holomorphy of the Dirichlet series contained in $\Xi_n(s, \alpha)$, and as a consequence, the region of holomorphy of $\Xi_3(s)$ or of $L(s, \eta)^{-1}$ will possibly be deduced. Toward this aim, we adopt termwise convolution of Eisenstein series.

§4. Termwise convolution.

To observe the behavior of $E_n(0, v, \alpha)$ as $v \to 0$, it suffices to observe the behavior of the value $E_3(v, \alpha)$ at $z = 0$ of the triple convolution of $E(u, \alpha) - v^\alpha = E(z, v, \alpha) - v^\alpha$, since the behavior of $v^\alpha$ is very clear. The expression $E(z, v, \alpha) - v^\alpha$ means the sum of all terms in the Eisenstein series such that $c \neq 0$.

Now,
\[
E_3(v, \alpha) = \int_{C/3} \int_{C/3} (E(z_1 + z_2, v, \alpha) - v^\alpha).
\]

\[
\cdot (E(-z_1, v, \alpha) - v^\alpha)(E(-z_2, v, \alpha) - v^\alpha)|dz_1||dz_2|
\]
\[
= \int_{C/3} \int_{C/3} \sum_{c, d} \frac{c_1}{d_1} \frac{v^\alpha}{(|c_1(z_1 + z_2) + d_1|^2 + |c_1|^2 v^2)^\alpha}.
\]

\[
\cdot \left( \frac{c_2}{d_2} \right) \frac{v^\alpha}{(|c_2 z_1| + d_2|^2 + |c_2|^2 v^2)^\alpha}.
\]

\[
\cdot \left( \frac{c_3}{d_3} \right) \frac{v^\alpha}{(|c_3 z_2| + d_3|^2 + |c_3|^2 v^2)^\alpha} dz_1||dz_2|.
\]
To proceed further, some notation should be prepared. We decompose $c_i$ as

$$
c_1 = c_0 n_2 n_3 c'_1 = (\lambda_0 c'_0)(\lambda'_2 n'_2)(\lambda'_3 n'_3)\lambda_1 \epsilon_1 m_1, $$
$$
c_2 = c_0 n_3 n_1 c'_2 = (\lambda_0 c'_0)(\lambda'_3 n'_3)(\lambda'_1 n'_1)\lambda_2 \epsilon_2 m_2, $$
$$
c_3 = c_0 n_1 n_2 c'_3 = (\lambda_0 c'_0)(\lambda'_1 n'_1)(\lambda'_2 n'_2)\lambda_3 \epsilon_3 m_3, $$

where $c'_i$ are mutually prime, so are $n_i$, too, and $(n_i, c'_i) = 1$. Among the symbols, $\lambda_i, \lambda'_i$ are powers of $-3$, $\epsilon_i^6 = 1$, and all others are $\equiv 1 \pmod{3}$. Thus,

$$[c_1, c_2, c_3] = c_0 n_1 n_2 n_3 c'_1 c'_2 c'_3 $$
$$= c_0(\lambda'_1 n'_1)(\lambda'_2 n'_2)(\lambda'_3 n'_3) \cdot (\lambda_1 \epsilon_1 m_1)(\lambda_2 \epsilon_2 m_2)(\lambda_3 \epsilon_3 m_3) $$

is a l.c.m. of $c_1, c_2, c_3$, and $\lambda_i, \lambda'_i$ are 1 except at most one, respectively. Furthermore we determine $k$ by

$$\frac{d_1}{c_1} + \frac{d_2}{c_2} + \frac{d_3}{c_3} = \frac{k}{[c_1, c_2, c_3]}.$$ 

On the other hand, putting

$$\gamma_1(z) = \gamma_1(z, \alpha) = \frac{1}{(|z|^2 + 1)^\alpha}, \quad (\Re \alpha > 1),$$

we denote by $\gamma_3(z)$ the triple convolution of $\gamma_1(z, \alpha)$ on $\mathbb{C}$, i.e.

$$\gamma_3(z) = \gamma_3(z, \alpha) $$
$$= \int_{\mathbb{C}} \int_{\mathbb{C}} \gamma_1(z + t_1 + t_2) \gamma_1(-t_1) \gamma_1(-t_2) |dt_1||dt_2|. $$

In addition, $S$ will stand for a complete system of representatives of $\alpha \pmod{c_0 n_1 n_2 n_3}$, and $\mu_i, (i = 1, 2, 3)$, will be parameters running through $S$.

A rather long computation in which cubic reciprocity law is repeatedly needed yields finally

$$E_3(v, \alpha) = \sum_{k \neq 0} \sum c_0, n \frac{1}{(\lambda_0 |n_1 n_2 n_3|^{4\alpha}} $$
$$\cdot \sum_{\lambda, \epsilon, \mu} (\mu_1 n_1')^{-1} (\mu_2 n_2')^{-1} (\mu_3 n_3')^{-1} j_k(c_0, n, \lambda, \epsilon, \mu) $$
$$\cdot \sum_{m \equiv \mu \pmod{c_0 n_1 n_2 n_3}} $$
$$\left( \frac{\lambda_1 \lambda_2 \lambda_3 \epsilon_1 \epsilon_2 \epsilon_3}{m_1} \right)^{-1} \left( \frac{\lambda_2 \lambda_3 \lambda_1 \epsilon_3 \epsilon_1}{m_2} \right)^{-1} \left( \frac{\lambda_3 \lambda_1 \lambda_2 \epsilon_1 \epsilon_2}{m_3} \right)^{-1}. $$
ON SEVERAL MULTIPLE ZETA FUNCTIONS

\[
\left( \frac{k}{m_1 m_2 m_3} \right) \left( \frac{m_1}{m_2} \right) \left( \frac{m_2}{m_3} \right) \left( \frac{m_3}{m_1} \right) \frac{1}{|m_1 m_2 m_3|^{2\alpha}} \cdot v^{4-3\alpha} \gamma_3 \left( \frac{k}{v c_0 n_1 n_2 n_3} \right). \\
+ \sum_{c_0, n} \sum_{\varepsilon} \frac{j_0(c_0, n, \varepsilon)}{|c_0|^{6\alpha} |n_1 n_2 n_3|^{4\alpha}} v^{4-3\alpha} \gamma_3(0).
\]

Here, \( j_k, (k \in \sigma) \), is a finite sum similar to the Jacobi sum, and symbols without index like \( n \) are abbreviations of triples like \( n_1, n_2, n_3 \).

Now, passing to the formal Mellin transform, the above formula turns finally into

\[
\int_0^\infty \left( E_3(v, \alpha) - \sum_{c_0, n} \sum_{\varepsilon} \frac{j_0(c_0, n, \varepsilon)}{|c_0|^{6\alpha} |n_1 n_2 n_3|^{4\alpha}} v^{4-3\alpha} \gamma_3(0) \right) \cdot v^{2s} \frac{dv}{v} = \sum_{k \neq 0} \sum_{c_0, n} \frac{1}{|c_0|^{2s+3\alpha+4} |n_1 n_2 n_3|^{2s+\alpha+4}} \cdot \\
\sum_{\lambda, \epsilon, \mu} \frac{(\mu_1)}{n_1'}^{-1} \left( \frac{\nu_2}{n_2} \right)^{-1} \left( \frac{\nu_3}{n_3} \right) \cdot \frac{1}{|\lambda_1 \lambda_2 \lambda_3|^{2s-\alpha+4}} \cdot \\
\sum_{m \equiv \mu(\text{mod} \, c_0 n_1 n_2 n_3)} \left( \frac{\lambda_1 \lambda_2 \lambda_3 \varepsilon_2 \varepsilon_3}{m_1} \right)^{-1} \left( \frac{\lambda'_2 \lambda_1 \varepsilon_1 \varepsilon_3}{m_2} \right)^{-1} \left( \frac{\lambda'_3 \lambda_1 \lambda_2 \varepsilon_1 \varepsilon_2}{m_3} \right)^{-1} \cdot \\
\left( \frac{k}{m_1 m_2 m_3} \right) \left( \frac{m_1}{m_2} \right) \left( \frac{m_2}{m_3} \right) \left( \frac{m_3}{m_1} \right) \frac{|k|^{2s-3\alpha+4}}{|m_1 m_2 m_3|^{2s-\alpha+4}} \cdot M(-2s + 3\alpha - 4, \gamma_3). 
\]

Actually this formula contains multiple zeta functions announced in the introduction, and so, we are lead to the hypotheses stated there.

It should be noted that all our arguments after §1 concern solely the former half part of the introduction. Corresponding arguments for the latter half are similar. The only difference is that we have to apply the differential operator \( \frac{\partial}{\partial z} \) on the triple convolution of \( E(z, v, \alpha) \).

Epilogue.

It seems to be fairly hard to investigate numbers \( \sigma_0 \) and \( \sigma'_0 \) in the introduction, but an observation of the double convolution of Eisenstein series compared with results
coming from Rankin's method, it is to a certain extent plausible that the double zeta function

$$\sum_{m_1, m_2} \frac{1}{m_1 m_2 |m_1 m_2|^{2\alpha}},$$

$(m_1 \equiv m_2 \equiv 1 \mod 3), \ (m_1, m_2) = 1$), is holomorphic in $\text{Re} \alpha > \frac{2}{3}$. So, for instance, the comparison of $\frac{2}{3}$ and $\sigma_0'$ concerning

$$\sum_{m_1, m_2, m_3} \frac{(m_1)}{m_2} \frac{(m_2)}{m_3} \frac{(m_3)}{m_1} \frac{\eta(m_1 m_2 m_3)}{|m_1 m_2 m_3|^{2\alpha}}$$

becomes a big question. Following hypotheses in the introduction, we can drive a vague imagination that, if $\sigma_0' = \frac{1}{3}$, then Riemann's hypothesis for $\zeta_F(s)$ can be valid, if $\sigma_0' = \frac{2}{3}$, then Riemann's hypothesis for $\zeta_F(s)$ is valid only up to $\text{Res} > \frac{5}{6}$, and if $\sigma_0' = \frac{5}{6}$, then Riemann's hypothesis is false, that is, there exists no region of the form $\text{Res} > 1 - \epsilon, (\epsilon > 0)$, in which $\zeta_F(s)$ has no zero.

By means of the Fourier coefficients $\{a_m\}$ of an automorphic form, a Dirichlet series with the same coefficients is defined whose properties can be investigated fairly precisely. As for the coefficients $\{a_m\}, \{b_m\}$ of two automorphic forms, the properties of the Dirichlet series with coefficients $\{a_m b_m\}$ may also be studied by Rankin method, for instance. Nevertheless, for the coefficients $\{a_m\}, \{b_m\}, \{c_m\}$ of three automorphic forms, it is very difficult to consider the properties of the Dirichlet series with coefficients $\{a_m b_m c_m\}$, and, at least, no generally applicable, functional analytic method is expected presently. But, if such method were invented, we would not need complicated discussions as in the present notes.

The author has not yet sufficiently determined zeros of $M(s, K_{\frac{3}{3}})$ and $M(s, \gamma_3)$.

Bibliography.

A suitable bibliography for the present notes is S.J.Patterson: The constant term of the cubic theta series, J. reine angew. Math., 336, (1982), 185-190, with its references.

In addition, more detailed versions of the present notes are available at

http://www.geocities.jp/okatayo119/

with file names Metap in Japanese and eMetap in English.