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On the applications of Shimura's mass formula

By

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Abstract

We explain how to compute the mass of the genus of maximal lattices for quadratic form of the sum of squares by applying Shimura's mass formula when the basic field is a real quadratic field (Section 1), and consider its applications in special cases (Section 2). This paper is also a survey on [Mu] to which several examples are added.

§ 1. Shimura's mass formula for computation

To apply Shimura's mass formula in [S99a, Theorem 5.8] to the case treated below, we first recall some basic facts following [S].

Let V be the row vector space F^n over a real quadratic field F of dimension n and φ the identity matrix 1_n of size n $(n > 1)$. For $x, y \in V$, we set $\varphi(x, y) = x\varphi \cdot {}^t y = x \cdot {}^t y$ and $\varphi[x] = \varphi(x, x) = x \cdot {}^t x$. We define

$$G = \{\gamma \in GL_n(F) \mid \gamma\varphi \cdot {}^t\gamma = \varphi\}, \quad G_+ = \{\gamma \in G \mid \det(\varphi) = 1\},$$

which are written as G^φ, G^φ_+ in [S99a] and [Mu], and also as $O^\varphi(V), SO^\varphi(V)$ in [S].

Let $G_{\mathbf{A}}$ be the adelization of G. For a $\mathfrak{g}$-lattice L in V, which is a finitely generated $\mathfrak{g}$-submodule in V containing a basis of V, and $\alpha \in G_{\mathbf{A}}$, we denote by $L\alpha$ the $\mathfrak{g}$-lattice in V such that $(L\alpha)_v = L_v\alpha_v$ for any finite prime v of F. Here $\mathfrak{g}$ is the ring of integers of F and L_v is the localization of L at v. We call $\{L\alpha \mid \alpha \in G_{\mathbf{A}}\}$ (resp. $\{L\alpha \mid \alpha \in G\}$) the *genus* (resp. *class*) of L with respect to G; we also call it the G-genus (resp. G-class) of L. It is known that the genus of L consists of finitely many classes (cf. [S, Lemma 9.21(iv) and (v)]) ·

Let $\{L_i\}_{i=1}^h$ be a complete set of representatives for G-classes in the G-genus of L. Then we set

$$\mathfrak{m}(L) = \sum_{i=1}^{h} [\Gamma_i : 1]^{-1},$$

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where $\Gamma_i = \{\gamma \in G \mid L_i\gamma = L_i\}$. This is independent of the choice of $\{L_i\}_{i=1}^h$. We call $\mathfrak{m}(L)$ the *mass* of the genus of L with respect to G. Similarly for G_+, we can define the *mass* of the genus of L with respect to G_+ and denote it by $\mathfrak{m}_+(L)$. It should be noted that $\mathfrak{m}_+(L) = 2\mathfrak{m}(L)$ (cf. [S99a, Lemma 5.6(1)]).

For each finite prime v of F, there exists $\alpha_v \in GL_n(F_v)$ such that

$$\alpha_v\varphi \cdot {}^t\alpha_v = \begin{bmatrix} 0 & 0 & 1_{r_v} \\ 0 & \theta_v & 0 \\ 1_{r_v} & 0 & 0 \end{bmatrix} \tag{1.1}$$

with an anisotropic symmetric matrix $\theta_v \in GL_{t_v}(F_v)$ of size t_v. Here F_v is the v-completion of F, and we say that θ_v is anisotropic if $\theta_v[x] = 0 \Longrightarrow x = 0$. In this paper, we call a matrix as in the right-hand side of (1.1) a *Witt decomposition* for φ over F_v (cf. [S, Lemma 1.3]). Then $n = 2r_v + t_v$ and t_v is determined only by φ and v. We call t_v the *core dimension* of φ at v. It is known that $t_v \leq 4$ for every finite prime v (cf. [S, Theorem 7.6(ii)]).

For a $\mathfrak{g}$-lattice L, we set

$$\tilde{L} = \{x \in V \mid 2\varphi(x, y) \in \mathfrak{g} \text{ for every } y \in L\}.$$

Then $\tilde{L}$ is a $\mathfrak{g}$-lattice in V, and $L \subset \tilde{L}$ if $\varphi[L] \subset \mathfrak{g}$. Let $\mathfrak{e}$ be the product of all finite primes v satisfying $\tilde{L}_v \neq L_v$.

Let L be a $\mathfrak{g}$-*maximal lattice with respect to* φ, that is, a $\mathfrak{g}$-lattice L in V which is maximal among $\mathfrak{g}$-lattices on which the values $\varphi[x]$ are contained in $\mathfrak{g}$. It is known that the genus of L consists of all $\mathfrak{g}$-maximal lattices (cf. [S, §9.7]) . Then, by applying an exact formula due to Shimura in [S99a, Theorem 5.8] to our case, $\mathfrak{m}_+(L)$ can be given as follows:

Theorem 1.1. *Let L be a $\mathfrak{g}$-maximal lattice with respect to φ. Then*

$$\mathfrak{m}_+(L) = 2D_F^{[\mu^2]}\prod_{k=1}^{[\mu]}\{D_F^{1/2}((2k-1)!(2\pi)^{-2k})^2\zeta_F(2k)\} \cdot [\tilde{L} : L]^{\mu}\prod_{v|\mathfrak{e}}\lambda_v$$
$$\cdot \begin{cases} 2^{-2\mu} & \text{if } n \text{ is odd,} \\ D_F^{1/2}((n/2-1)!(2\pi)^{-n/2})^2L(n/2,\,\psi_{K/F}) & \text{if } n \text{ is even.} \end{cases}$$

Here $\mu = (n-1)/2$ and D_F is the discriminant of F; if n is even, then $K = F(\sqrt{(-1)^{n/2}})$, $\psi_{K/F}$ is the Hecke character of F corresponding to K/F, and we set $\psi_{K/F} = 1$ when

$K = F$; λ_v is given as follows:

$$\lambda_v = \begin{cases} 1 & \text{if } t_v = 1, \\ 2^{-1} & \text{if } t_v = 2 \text{ and } \mathfrak{d}_v \neq \mathfrak{r}_v, \\ 2^{-1}(1+q_v)^{-1}(1-q_v^{1-n}) & \text{if } t_v = 3, \\ 2^{-1}(1+q_v)^{-1}(1-q_v^{1-n/2})(1-q_v^{-n/2}) & \text{if } t_v = 4, \end{cases}$$

where q_v is the norm of the prime ideal at v; $\mathfrak{r}_v$ is the maximal order of $K_v = F_v(\sqrt{-1})$ and $\mathfrak{d}_v$ is the different of K_v relative to F_v when $t_v = 2$.

We note that the case where $t_v = 2$, $\mathfrak{d}_v = \mathfrak{r}_v$, and $\tilde{M}_v \neq M_v$ in [S99a, Theorem 5.8] cannot be possible in our quadratic space $(F^n, 1_n)$, because of $\det(\theta_v) \equiv \pm\det(\varphi) = \pm 1$ modulo $\{a^2 \mid a \in F^\times\}$.

By virtue of Theorem 1.1, we can reduce the calculation of $\mathfrak{m}(L)$ to the following two arguments:

One is to compute the special values of the Dedekind zeta function of F and the L-function of F associated to the Hecke character of F corresponding to $F(\sqrt{-1})/F$. These values can be obtained by calculating values of the Riemann zeta function and Dirichlet L-functions, since $F(\sqrt{-1})/\mathbf{Q}$ is an abelian extension.

The other is to find all finite primes v satisfying $\tilde{L}_v \neq L_v$. The index $[\tilde{L}_v : L_v]$ can be computed by using [S99a, (3.2.1)], which needs a Witt decomposition for 1_n over F_v. To determine this, we first take an anisotropic matrix θ_p of a Witt decomposition for 1_n over $\mathbf{Q}_p$ for a rational prime p. Then the size of θ_p is ≤ 4. After that, we decompose θ_p on F_v for v lying above p. It should be noted that this method is useful only when the quadratic form in question is given by a matrix with entries in $\mathbf{Q}$.

To get a numerical example of the mass, let us consider the case where $F = \mathbf{Q}(\sqrt{5})$ and $\varphi = 1_4$. Then the quadratic form over $\mathbf{Q}$ given by $\varphi = 1_4$ is equivalent to the norm form β of the quaternion algebra B_0 over $\mathbf{Q}$ which is ramified only at 2 and the infinite prime. In other words, 1_4 is the matrix that represents β with respect to a suitable $\mathbf{Q}$-basis of B_0; see §2 below. Thus we first consider a Witt decomposition for β over $\mathbf{Q}_p$. It can be verified that the core dimension at v of the norm form of a quaternion algebra A is 4 if A is ramified at v, and it is 0 if A is unramified at v. From this fact, the core dimension of β at p is 4 if $p = 2$, otherwise 0. Next we consider β as the norm form of $B = B_0 \otimes_{\mathbf{Q}} F$ and ask whether β is decomposed over F_v for v lying above 2. Now, it is known that a quaternion algebra over a nonarchmedean local field splits over an arbitrary quadratic extension of the local field (cf. [D, VII, §2, Satz 4]). Since 2 remains prime in F, B splits at 2 (as an algebra), and consequently B is unramified at every finite prime. This implies that the core dimension t_v of the norm form β, or rather, of φ is 0 for every prime v. Then by virtue of [S99a, (3.2.1)], $[\tilde{L}_v : L_v] = 1$

holds for an arbitrary v. Hence we have $[\tilde{L} : L] = 1$ and $\mathfrak{e} = \mathfrak{g}$. Combining this with $\zeta_F(2) = \zeta(2)L(2, \chi)$ (χ is the Dirichlet character of F), by Theorem 1.1, we obtain the mass of the genus of maximal lattices as follows:

$$\mathfrak{m}(L) = 2^{-1}\mathfrak{m}_+(L) = 5^3(2\pi)^{-8}\zeta_F(2)^2 = 2^{-6} \cdot 3^{-2} \cdot 5^{-2}.$$

From the above argument, for a finite prime v dividing 2 of an arbitrary real quadratic field F, we can verify that

$$\begin{aligned} B \otimes_F F_v &\cong (B_0 \otimes_{\mathbf{Q}} \mathbf{Q}_2) \otimes_{\mathbf{Q}_2} F_v \\ &\cong \begin{cases} B_0 \otimes_{\mathbf{Q}} \mathbf{Q}_2 & \text{if 2 splits in } F, \\ M_2(F_v) & \text{otherwise.} \end{cases} \end{aligned}$$

Since $t_v = 0$ for the other primes v, we then find a Witt decomposition for 1_4 over F_v and the core dimension t_v for each prime v of F. A Witt decomposition for 1_n over F_v for an arbitrary n and v was given in [Mu, Lemma 3.3]. By combining that lemma with calculations of L-values, Theorem 1.1 can be stated in a simpler form as follows:

Theorem 1.2. *([Mu, Theorem 3.6]) Let $F = \mathbf{Q}(\sqrt{m})$ with a squarefree positive integer m, and let L be a $\mathfrak{g}$-maximal lattice with respect to φ. Let χ, χ', and χ'' be the Dirichlet characters corresponding to $F/\mathbf{Q}$, $\mathbf{Q}(\sqrt{-1})/\mathbf{Q}$, and $\mathbf{Q}(\sqrt{-m})/\mathbf{Q}$, respectively. Also let B_k and $B_{k,\psi}$ be the k-th Bernoulli number and k-th generalized Bernoulli number associated with a Dirichlet character ψ.*

(1) *If $n \equiv 0 \pmod 8$, then*

$$\mathfrak{m}(L) = n^{-2}B_{n/2}B_{n/2,\chi}\left(\prod_{k=1}^{[(n-1)/2]}(4k)^{-2}B_{2k}B_{2k,\chi}\right).$$

(2) *If $n \equiv \pm 1 \pmod 8$, then*

$$\mathfrak{m}(L) = \prod_{k=1}^{(n-1)/2}(4k)^{-2}B_{2k}B_{2k,\chi}.$$

(3) *If $n \equiv \pm 2 \pmod 8$, then*

$$\mathfrak{m}(L) = n^{-2}B_{n/2,\chi'}B_{n/2,\chi''}\left(\prod_{k=1}^{[(n-1)/2]}(4k)^{-2}B_{2k}B_{2k,\chi}\right) \cdot \begin{cases} 2^{-2} & \textit{if } m \equiv 1 \pmod 8, \\ 1 & \textit{if } m \equiv 3 \pmod 4, \\ 2^{-1} & \textit{otherwise.} \end{cases}$$

(4) *If* $n \equiv \pm 3 \pmod 8$, *then*

$$\mathfrak{m}(L) = \left(\prod_{k=1}^{(n-1)/2} (4k)^{-2} B_{2k} B_{2k,\chi} \right) \cdot \begin{cases} 2^{-2} \cdot 3^{-2} (2^{n-1} - 1)^2 & \text{if } m \equiv 1 \pmod 8, \\ 1 & \text{otherwise.} \end{cases}$$

(5) *If* $n \equiv 4 \pmod 8$, *then*

$$\mathfrak{m}(L) = n^{-2} B_{n/2} B_{n/2,\chi} \left(\prod_{k=1}^{[(n-1)/2]} (4k)^{-2} B_{2k} B_{2k,\chi} \right) \cdot \begin{cases} 2^{-2} \cdot 3^{-2} (2^{n/2-1} - 1)^2 (2^{n/2} - 1)^2 & \text{if } m \equiv 1 \pmod 8, \\ 1 & \text{otherwise.} \end{cases}$$

These are analogues of the formulas for $\mathfrak{m}_+(L)$ in the case where $F = \mathbf{Q}$ and $\varphi = 1_n$ in [S99a, Examples 5.16] to the case of real quadratic fields.

§ 2. Applications of the mass formula

We set again $F = \mathbf{Q}(\sqrt{5})$. As applications of the mass formula, we shall determine the number h of G-classes of the G-genus of maximal lattices in $V = F^n$ with respect to $\varphi = 1_n$ for $n = 2, 3, 4, 5, 6$. For a fixed $\mathfrak{g}$-lattice L in V and $q \in \mathfrak{g}$, we set

$$\Gamma(L) = \{\gamma \in G \mid L\gamma = L\},$$
$$n(L, q) = \{x \in L \mid \varphi[x] = q\}, \quad N(L, q) = \# n(L, q).$$

We first explain the case of $n = 4$, which was treated in [Mu, §4]. In this case, we consider a $\mathfrak{g}$-lattice L defined by

$$L = \sum_{i=1}^{4} \mathfrak{g}\alpha_i = \mathfrak{g}^4 \alpha, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & (1+\varepsilon)/2 & \varepsilon/2 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \tag{2.1}$$

with $\varepsilon = (1 + \sqrt{5})/2$. This is a $\mathfrak{g}$-maximal lattice with respect to φ, since $[\tilde{L} : L] = 1$ by using elementary divisors. Then the order of $\Gamma(L)$ becomes $2^6 \cdot 3^2 \cdot 5^2$ as shown below.

While, we have seen that $\mathfrak{m}(L) = 2^{-6} \cdot 3^{-2} \cdot 5^{-2}$. Hence, from the definition of the mass, the genus of L consists of one class.

We are going to see that $[\Gamma(L) : 1] = 2^6 \cdot 3^2 \cdot 5^2$ by a different way from that given in [Mu, §4]. Since there exists $\gamma \in \Gamma(L)$ such that $\det(\gamma) = -1$, it is sufficient to show that $[\Gamma_+(L) : 1] = 2^5 \cdot 3^2 \cdot 5^2$, where $\Gamma_+(L) = G_+ \cap \Gamma(L)$. First, as mentioned in §1, $\varphi = 1_4$ can be considered as the norm form β of the quaternion algebra $B = B_0 \otimes_{\mathbf{Q}} F$ over F. More precisely, the β is defined by $\beta(x, y) = 2^{-1}Tr_{B/F}(xy^\iota)$ and $\beta[x] = N_{B/F}(x) = xx^\iota$ for $x, y \in B$. Here ι is the main involution of B, $Tr_{B/F}(x)$ is the reduced trace, and $N_{B/F}(x)$ is the reduced norm of x. The quaternion algebra B_0 can be written in the form

$$B_0 = \mathbf{Q} + \mathbf{Q}a + \mathbf{Q}b + \mathbf{Q}ab$$

with $a, b \in B_0$ such that $a^2 = b^2 = -1$ and $ba = -ab$. By the isomorphism ξ : $x = (x_1, x_2, x_3, x_4) \longmapsto x_1 + x_2 a + x_3 b + x_4 ab$ of F^4 onto $B = B_0 \otimes_{\mathbf{Q}} F$, we have $1_4[x] = \beta[x\xi] = x_1^2 + x_2^2 + x_3^2 + x_4^2$. Then the mapping $\gamma \longmapsto \xi^{-1}\gamma\xi$ gives an isomorphism of G_+ onto the special orthogonal group $SO(\beta)$ of β. Hence if we set $\mathfrak{o} = L\xi$, then

$$\Gamma_+(L) \cong \Gamma_+(\mathfrak{o}) = \{\tau \in SO(\beta) \mid \mathfrak{o}\tau = \mathfrak{o}\},$$

and so we consider the order of $\Gamma_+(\mathfrak{o})$ instead of $\Gamma_+(L)$. It can be seen that $\mathfrak{o}$ is a maximal order in B.

The following Lemma 2.1 is fundamental to observe $\Gamma_+(\mathfrak{o})$ (cf. [S99a, Lemma 1.5]).

Lemma 2.1.

$$SO(\beta) = \{\tau_{x,y} \mid x, y \in B^\times \textit{ such that } N_{B/F}(x) = N_{B/F}(y)\},$$

where $\tau_{x,y}$ is defined by $z\tau_{x,y} = y^{-1}zx$ for $z \in B$.

In view of Lemma 2.1, we set $\tau_x = \tau_{x,x}$ and

$$\Gamma_0 = \{\tau_{x,y} \in SO(\beta) \mid x, y \in \mathfrak{o}^\times\}, \quad \Gamma_1 = \{\tau_x \mid x \in B^\times\},$$
$$\Gamma^*(\mathfrak{o}) = \{x \in B \mid x\mathfrak{o} = \mathfrak{o}x\}.$$

Also we set $n(\mathfrak{o}, q) = \{x \in \mathfrak{o} \mid N_{B/F}(x) = q\}$ and $N(\mathfrak{o}, q) = \#n(\mathfrak{o}, q)$ for $q \in \mathfrak{g}$.

Lemma 2.2. *Let the notation be as in above. Then the following three assertions hold:*

(1) $[\Gamma_0 : \Gamma_0 \cap \Gamma_1] = N(\mathfrak{o}, 1)$.

(2) $\Gamma_0 \cap \Gamma_1 \cong \mathfrak{o}^\times / \mathfrak{g}^\times$.

(3) $\Gamma_+(\mathfrak{o})/\Gamma_0 \cong (\Gamma_+(\mathfrak{o}) \cap \Gamma_1)/(\Gamma_0 \cap \Gamma_1) \cong \Gamma^*(\mathfrak{o})/F^\times \mathfrak{o}^\times$.

Proof. The mapping $\tau_{x,y} \longmapsto xy^{-1}$ leads the assertion (1). Noticing that $\tau_x = \tau_y$ if and only if $xy^{-1} \in \mathfrak{g}^\times$ for $x, y \in \mathfrak{o}^\times$, and considering the homomorphism $\tau_x \longmapsto x\mathfrak{g}^\times$, we have the isomorphism of (2). The last assertion (3) follows from the two mappings $\tau_x \longmapsto \tau_x\Gamma_0$ and $\tau_x \longmapsto xF^\times\mathfrak{o}^\times$. □

From Lemma 2.2, we have

$$[\Gamma_+(\mathfrak{o}) : 1] = N(\mathfrak{o}, 1)[\Gamma^*(\mathfrak{o}) : F^\times\mathfrak{o}^\times][\mathfrak{o}^\times : \mathfrak{g}^\times].$$

Furthermore, by virtue of the formula [E, (16)], we have

$$[\Gamma^*(\mathfrak{o}) : F^\times\mathfrak{o}^\times] = 2^r h_F H(\mathfrak{o})^{-1}.$$

Here r is the number of all finite primes which are ramified in B and h_F is the class number of F; $H(\mathfrak{o})$ is the class number of the two-sided $\mathfrak{o}$-ideals of B and satisfies $H(\mathfrak{o}) \leq h(B)$ for the class number $h(B)$ of B. (We do not explain it here; for a more detailed explanation, the reader is referred to Eichler's article [E, §4].) We know that $h_F = 1$ and $r = 0$ when $F = \mathbf{Q}(\sqrt{5})$, because $B = B_0 \otimes_{\mathbf{Q}} F$ is unramified at every finite prime as observed in §1. Moreover, it is known that $h(B) = 1$ (cf. [P, §9, TABELLE 2]), and so $H(\mathfrak{o}) = 1$. Consequently we have $[\Gamma^*(\mathfrak{o}) : F^\times\mathfrak{o}^\times] = 1$. Since

$$\mathfrak{o}^\times = \bigsqcup_{x \in n(\mathfrak{o}, 1)/\mathbf{Z}^\times} x\mathfrak{g}^\times$$

by $N_{F/\mathbf{Q}}(\varepsilon) = -1$, we have $[\mathfrak{o}^\times : \mathfrak{g}^\times] = 2^{-1}N(\mathfrak{o}, 1)$. Thus we obtain

$$[\Gamma_+(\mathfrak{o}) : 1] = 2^{-1}N(\mathfrak{o}, 1)^2.$$

Now, using the basis of L given before, we see that

$$N(\mathfrak{o}, 1) = N(L, 1) = 120 = 2^3 \cdot 3 \cdot 5$$

(cf. [Mu, §4]). Hence the order of $\Gamma_+(L)$ is

$$[\Gamma_+(\mathfrak{o}) : 1] = 2^{-1} \cdot 2^6 \cdot 3^2 \cdot 5^2 = 2^5 \cdot 3^2 \cdot 5^2.$$

Let us add further examples for $n = 2, 3, 5, 6$, which are not in [Mu].

If $n = 2$, then φ can be identified with the norm form of the quadratic extension $F(\sqrt{-1})/F$ and

$$\frac{h_+}{w} = \mathfrak{m}_+(L),$$

where h_+ is the number of G_+-classes in the G_+-genus of maximal lattices with respect to φ and w is the order of the group of all roots of unity in $F(\sqrt{-1})$. Moreover, a complete set of representatives for G_+-classes of the G_+-genus of maximal lattices can be described in terms of the ideal classes of $F(\sqrt{-1})$. These facts follow immediately from the results on the two-dimensional quadratic spaces in [S99b, §6.1]. By Theorem 1.2, we have $\mathfrak{m}_+(L) = 2\mathfrak{m}(L) = 2^{-2}$. This together with $w = 4$ shows that $h_+ = 1$. Since $h \leq h_+$ (cf. [S, Lemma 9.23(i)]), we have $h = 1$. We note that $L = \mathfrak{g}^2$ is a $\mathfrak{g}$-maximal lattice of F^2 with respect to 1_2 because of $[\tilde{L} : L] = 2^4$.

Let $n = 3$. We use the notation in the case $n = 4$ before. Then φ can be identified with the restriction β° of the norm form of the quaternion algebra $B = B_0 \otimes_{\mathbf{Q}} F$ to $T = \{x \in B \mid x^\iota = -x\}$ and $SO(\beta^\circ)$ is generated by τ_x for $x \in B^\times$ ([S99a, Lemma 1.4]). Furthermore, by the results on the three-dimensional quadratic spaces treated in [S, §12.2], $\mathfrak{o} \cap T$ is a $\mathfrak{g}$-maximal lattice in T with respect to β° and $\mathfrak{o}$ is the unique maximal order in B containing $\mathfrak{g}$ and $\mathfrak{o} \cap T$. It can be verified from these facts that $L = (\mathfrak{o} \cap T)\xi^{-1}$ is a $\mathfrak{g}$-maximal lattice in F^3 with respect to 1_3 and $\Gamma_+(L) \cong \Gamma_+(\mathfrak{o} \cap T) \cong \Gamma_+(\mathfrak{o}) \cap \Gamma_1$. Thus

$$[\Gamma_+(L) : 1] = [\mathfrak{o}^\times : \mathfrak{g}^\times] = 2^{-1}N(\mathfrak{o}, 1) = 60.$$

As clearly $-1_3 \notin \Gamma_+(L)$, we have $[\Gamma(L) : 1] = 120$. While Theorem 1.2 in this case shows that $\mathfrak{m}(L) = 2^{-3} \cdot 3^{-1} \cdot 5^{-1}$. Hence we have $h = 1$. We note that all $\mathfrak{O} \cap T$ for maximal orders $\mathfrak{O}$ in B that are not mutually same type form a complete set of representatives for the classes of the genus of maximal lattices with respect to β°, and thus h is the type number of B; see [S, §12.2]. We also note that L can be written in the form $\mathfrak{g}^3\alpha$ with

$$\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\varepsilon/2 & (1-\varepsilon)/2 & 1/2 \end{bmatrix}. \tag{2.2}$$

If $n = 5$, then $h = 1$. This example and the result on $(F^6, 1_6)$ presented below are due to T.Hiraoka and the author. To determine h, we follow the method explained in [Mu, §4]. Let L_4 be the maximal lattice with respect to 1_4 given in (2.1). Then $L = L_4 + \mathfrak{g}e_5$ is a $\mathfrak{g}$-maximal lattice in F^5 with respect to 1_5 because of $[\tilde{L} : L] = 2^2$. Here $\{e_i\}$ is the standard basis of F^5 and F^4 is embedding into F^5 in a natural way. It can be seen that for $\gamma = {}^t[{}^t\gamma_1 \cdots {}^t\gamma_5] \in M_5(F)$, γ belongs to $\Gamma(L)$ if and only if

$$\begin{cases} \gamma_i \in n(L, 1)\ (1 \leq i \leq 5),\ \gamma_i \cdot {}^t\gamma_j = 0\ (i \neq j), \\ 2^{-1}(\gamma_1 + (1+\varepsilon)\gamma_2 + \varepsilon\gamma_3) \in n(L, 1+\varepsilon), \\ 2^{-1}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \in n(L, 1). \end{cases} \tag{2.3}$$

We also see that $n(L, 1) = n(L_4, 1) \sqcup \{\pm e_5\}$; all elements of $n(L_4, 1)$ were given in [Mu, §4]. Then we can verify that

$$\Gamma(L) = \left\{ \begin{bmatrix} \gamma_0 & 0 \\ 0 & \delta \end{bmatrix} \in \Gamma(L) \mid \gamma_0 \in \Gamma(L_4),\ \delta = \pm 1 \right\}. \tag{2.4}$$

To show this, let $\gamma \in \Gamma(L)$ and γ_i be the ith row vecter of γ. Suppose $\gamma_i = e_5$ with some $1 \leq i \leq 4$. Then $2^{-1}(\gamma_1 + \cdots + \gamma_4)$ must be belonging to L, but it is impossible in our choice of L. Hence $\gamma_i \neq e_5$ and so $\gamma_i \in n(L_4, 1)$ for every $1 \leq i \leq 4$. Thus we have $\gamma_5 = \pm e_5$. At the same time, in view of [Mu, (4.3)], ${}^t[{}^t\gamma_1 \ \cdots {}^t\gamma_4] \in \Gamma(L_4)$, which proves (2.4). Here we should remark that in [Mu, page 142, line 9], "$\gamma \in \Gamma_1$" should be read "$\gamma \in \Gamma$". As a consequence, we have

$$[\Gamma(L) : 1] = 2[\Gamma(L_4) : 1] = 2^7 \cdot 3^2 \cdot 5^2.$$

By Theorem 1.2, $\mathfrak{m}(L) = 2^{-7} \cdot 3^{-2} \cdot 5^{-2}$, which implies $h = 1$.

Let $n = 6$. Then we find three maximal lattices $L = \mathfrak{g}^6\alpha$, $L' = \mathfrak{g}^6\alpha'$, $L'' = \mathfrak{g}^6\alpha''$ with respect to 1_6 given by

$$\alpha = \begin{bmatrix} \alpha_4 & 0 \\ 0 & 1_2 \end{bmatrix}, \quad \alpha' = \begin{bmatrix} \alpha_3 & 0 \\ 0 & \alpha_3 \end{bmatrix},$$

$$\alpha'' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix},$$

where α_4 is the matrix in (2.1) and α_3 is in (2.2). Let us compute the order of Γ for $\Gamma = \Gamma(L),\ \Gamma(L'),\ \Gamma(L'')$. In each cases, there is a suitable necessary and sufficient condition for $\gamma \in \Gamma$ such as (2.3). We will use the condition without a detailed explanation. First, the order of $\Gamma(L)$ becomes $2^3[\Gamma(L_4) : 1]$, which can be handled in the similar manner as in the case $n = 5$. Next we see that

$$\Gamma(L') = \left\{ \begin{bmatrix} \gamma & 0 \\ 0 & \gamma' \end{bmatrix}, \begin{bmatrix} 0 & \gamma \\ \gamma' & 0 \end{bmatrix} \mid \gamma,\ \gamma' \in \Gamma(L_3) \right\},$$

where L_3 is the lattice in F^3 given in (2.2). This follows from the fact that every element of $n(L', 1)$ can be written in the form $[x\ 0\ 0\ 0]$ or $[0\ 0\ 0\ x]$ with $x \in n(L_3, 1)$. Then $[\Gamma(L') : 1] = 2[\Gamma(L_3) : 1] = 2^7 \cdot 3^2 \cdot 5^2$. Finally, we look at $n(L'', 1) = n(2^{-1}\mathfrak{g}^6, 1) \cap L''$. Then it can be verified that $n(L'', 1) = n(L_0, 1)$, where $L_0 = \mathbf{Z}^6\alpha''$. From this, it follows that $\Gamma(L'') = \Gamma(L_0)$. Since the order of $\Gamma(L_0)$ is known, we have $[\Gamma(L'') : 1] = 2^6 \cdot 6!$. Consequently these maximal lattices are not mutually same class, and

$$[\Gamma(L) : 1]^{-1} + [\Gamma(L') : 1]^{-1} + [\Gamma(L'') : 1]^{-1} = 2^{-10} \cdot 3^{-1} \cdot 5^{-1}.$$

This coincides with the mass by Theorem 1.2. Therefore we conclude $h = 3$. We note that the order of $\Gamma(L')$ can be computed by applying [Ma, Theorem 1.4.6] to L'.

Summing up these results, we have

Theorem 2.3. *Let $F = \mathbf{Q}(\sqrt{5})$, $\varphi = 1_n$, and let $h(n)$ be the number of classes of the genus of $\mathfrak{g}$-maximal lattices with respect to φ. Then $h(n) = 1$ for $2 \leq n \leq 5$, and $h(6) = 3$.*

We shall end this paper with the following remark: The above applications of the mass formula provided the examples that we can determine the class number h of the genus. However we can not always determine h in this way. For example, if $F = \mathbf{Q}(\sqrt{229})$ and $\varphi = 1_6$, then for a maximal lattice L with respect to φ, Theorem 1.2 shows

$$\mathfrak{m}(L) = \frac{3^3 \cdot 101 \cdot 2203 \cdot 199403}{2^{10} \cdot 5}.$$

In view of the definition of the mass, we find that $h > 200000000$. It seems that it is almost impossible to determine h by the way explained above, though we are interested how these classes can be found.

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