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On New Expressions of the Painlevé Hierarchies

By

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§ 0. Introduction

A WKB theoretic study of the (P_J)-hierarchy was given in [KKoNT] for $J = \mathrm{I}$, II-1, II-2, and in [KoN] for $J = \mathrm{IV}$. The purpose of this article is to present a new expression of such Painlevé hierarchies. Here we discuss (P_{34})-hierarchy instead of $(P_{\text{II-1}})$-hierarchy; (P_{34})-hierarchy is another expression of $(P_{\text{II-1}})$-hierarchy ([CJP]), and it is more amenable to our scheme. In our expression, each member of the (P_J)-hierarchy, which will be denoted by $(P_J)_m$ $(m = 1, 2, 3, \cdots)$, is written down as a system of the first order nonlinear ordinary differential equations. One of the advantages of the new expression is that it is more suited for WKB analysis; for example the discussion given in [KoNT] on the description of the Stokes geometry of $(P_{\mathrm{IV}})_m$ is simpler and clearer than that in [KoN]. Our expression is also useful in studying relations between $(P_J)_m$ and degenerate Garnier systems; here we discuss this aspect of $(P_{\mathrm{I}})_m$ and $(P_{34})_m$ in parallel with the discussion of $(P_{\text{II-2}})_m$ and $(P_{\mathrm{IV}})_m$ given in [Ko]. By using these relations, we can obtain the Hamiltonian system of $(P_J)_m$, and hence we can construct the instanton-type solutions by using the method discussed in [T].

The plan of this article is as follows: In § 1, after recalling the definition of the (P_{I})-hierarchy, we will show that $(P_{\mathrm{I}})_m$ is equivalent to the restriction to an appropriate complex line of the degenerate Garnier system $G(m + 5/2; m)$. Although this was already shown by Shimomura in [S1], [S2] and [S3] (cf. also [KKoNT]), the emphasis of our analysis is on the study of the Hamiltonian of $G(m + 5/2; m)$. Note that the Hamiltonian of $(P_{\mathrm{I}})_m$ is also obtained by Takasaki in [Tks1], [Tks2] and [Tks3] by a different argument. In § 2, we study the (P_{34})-hierarchy; we first give a new expression of $(P_{34})_m$, which shall be denoted by $(\widetilde{P}_{34})_m$, and then give the Lax pair of $(\widetilde{P}_{34})_m$.

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We also show that $(\widetilde{P}_{34})_m$ is equivalent to the restriction to an appropriate complex line of the degenerate Garnier system $G(1, m + 3/2; m)$. In § 3, we will discuss the $(P_{\text{II-2}})$-hierarchy and the (P_{IV})-hierarchy, and give equivalent, but different, expressions $(\widetilde{P}_{\text{II-2}})_m$ and $(\widetilde{P}_{\text{IV}})_m$. We note that $(\widetilde{P}_{\text{II-2}})_m$ and $(\widetilde{P}_{\text{IV}})_m$ are denoted respectively by (P_{II}) and (P_{IV}) in [Ko].

In ending this introduction we note that all of equations in this article except those in § 1.3 and § 2.4 are with a large parameter η. The usual form can be obtained by simply setting η to one.

§ 1. (P_{I})-Hierarchy

§ 1.1. Equivalence of the (P_{I})-Hierarchy and the $(\widetilde{P}_{\text{I}})$-Hierarchy

We start our discussion by recalling the basic facts on the (P_{I})-hierarchy. One traditional approach to obtain the (P_{I})-hierarchy is to consider a certain reduction from the KdV hierarchy (cf. [Ku]). In this expression, we use F_n for $n = 0, 1, 2, \cdots$, a polynomial of $u = u(t)$ and its derivatives, defined by the following recursive relation:

$$F_0 = \frac{1}{2}, \tag{1.1}$$

$$\begin{aligned} F_{n+1} = &-\sum_{j=0}^{n-1} F_{n-j}F_{j+1} + 4u\sum_{j=0}^{n} F_{n-j}F_j \\ &+ 2\eta^{-2}\sum_{j=0}^{n} F_{n-j}\frac{d^2F_j}{dt^2} - \eta^{-2}\sum_{j=0}^{n}\frac{dF_{n-j}}{dt}\frac{dF_j}{dt} \quad (n \geq 0). \end{aligned} \tag{1.2}$$

Here η denotes a large parameter for the WKB analysis. For example, we have

$$F_1 = u, \qquad F_2 = 3u^2 + \eta^{-2}\frac{d^2u}{dt^2}, \tag{1.3}$$

$$F_3 = 10u^3 + \eta^{-2}\Big[10u\frac{d^2u}{dt^2} + 5\Big(\frac{du}{dt}\Big)^2\Big] + \eta^{-4}\frac{d^4u}{dt^4}. \tag{1.4}$$

We can confirm (cf. for example, [KKoNT, Appendix A]) that $\{F_n\}$ satisfies

$$\frac{dF_{n+1}}{dt} = \eta^{-2}\frac{d^3F_n}{dt^3} + 4u\frac{dF_n}{dt} + 2\frac{du}{dt}F_n. \tag{1.5}$$

Thus F_n is (a constant multiple of) the Gelfand-Dickey polynomial with a large parameter.

Definition 1.1 ((P_{I})-Hierarchy with a Large Parameter η, cf. [Ku] for $\eta = 1$)**.**
We set:

$$(P_{\text{I}})_m: \quad F_{m+1} + c_1F_m + \cdots + c_{m+1}F_0 + 2\gamma t = 0, \tag{1.6}$$

where γ $(\neq 0)$ and $\{c_n\}_{n=1}^{m+1}$ are constants.

Remark. Without loss of generality, we can choose $c_1 = 0$ by the translation of u, $c_{m+1} = 0$ by the translation of t, and fix γ to an arbitrary nonzero constant by the scalings of u and t.

By using (1.3) and (1.4) one can readily obtain the concrete forms of $(P_{\mathrm{I}})_1$ and $(P_{\mathrm{I}})_2$ (see Appendix A). In particular, $(P_{\mathrm{I}})_1$ is nothing but the first Painlevé equation with a large parameter.

Another approach to the (P_{I})-hierarchy is given in [S1], [S2] and [S3], where the monodromy preserving deformation of a certain system of linear ordinary differential equations is considered. A form somewhat different from his expression is used in [KKoNT]:

Definition 1.2 (($\widetilde{P}_{\mathrm{I}}$)-Hierarchy with a Large Parameter η)**.** We set:

$$
(1.7) \qquad (\widetilde{P}_{\mathrm{I}})_m : \begin{cases} \eta^{-1}\dfrac{du_j}{dt} = 2v_j & (j = 1, 2, \cdots, m), \quad (1.7.\mathrm{a}) \\ \eta^{-1}\dfrac{dv_j}{dt} = 2(u_{j+1} + u_1 u_j + w_j) & (j = 1, 2, \cdots, m), \quad (1.7.\mathrm{b}) \\ u_{m+1} = \widetilde{\gamma} t, & (1.7.\mathrm{c}) \end{cases}
$$

where w_n is a polynomial of $\{u_l, v_l\}$ that is determined by the following recursive relation:

$$
(1.8) \quad w_n = \frac{1}{2}\sum_{k=1}^{n} u_k u_{n-k+1} + \sum_{k=1}^{n-1} u_k w_{n-k} - \frac{1}{2}\sum_{k=1}^{n-1} v_k v_{n-k} + \widetilde{c}_0\bigl(2u_n - \sum_{k=1}^{n-1} u_k u_{n-k}\bigr) + \widetilde{c}_n.
$$

Here $\widetilde{\gamma}$ ($\neq 0$) and $\{\widetilde{c}_j\}_{j=0}^{m}$ are constants.

Remark. Without loss of generality, we can choose $\widetilde{c}_0 = \widetilde{c}_m = 0$ and fix $\widetilde{\gamma}$ to an arbitrary nonzero constant (cf. Remark after Definition 1.1 and (1.10) below).

See [KKoNT, § 1.1] for the concrete forms of $(\widetilde{P}_{\mathrm{I}})_1$, $(\widetilde{P}_{\mathrm{I}})_2$ and $(\widetilde{P}_{\mathrm{I}})_3$. It was shown that $(P_{\mathrm{I}})_m$ is the same equation with $(\widetilde{P}_{\mathrm{I}})_m$ as follows:

Theorem 1.3 ([KKoNT, Appendix B], [S3])**.** *If u is a solution of $(P_{\mathrm{I}})_m$, then $\{u_j, v_j\}$ defined by*

$$
(1.9) \qquad u_j = -2^{-2j+1}\mathcal{F}_j, \quad v_j = -2^{-2j}\eta^{-1}\frac{d\mathcal{F}_j}{dt} \qquad (1 \leq j \leq m)
$$

satisfies $(\widetilde{P}_{\mathrm{I}})_m$ whose constants are chosen so that

$$
(1.10) \qquad \widetilde{\gamma} = 4^{-m}\gamma, \quad \widetilde{c}_n = 2^{-2n-3}\sum_{k=0}^{n+1} c_{n-k+1}c_k \quad (0 \leq n \leq m) \quad \textit{with} \quad c_0 = 1.
$$

Here $\mathcal{F}_n$ is defined by

$$\mathcal{F}_n = c_0 F_n + c_1 F_{n-1} + \cdots + c_n F_0 \quad (1 \le n \le m). \tag{1.11}$$

Conversely if $\{u_j, v_j\}$ satisfies $(\widetilde{P}_{\mathrm{I}})_m$ with (1.10), $u = -2(u_1 + \widetilde{c}_0)$, which is obtained from (1.9) with $j = 1$ and $c_1 = 4\widetilde{c}_0$, is a solution of $(P_{\mathrm{I}})_m$.

It is known that $(\widetilde{P}_{\mathrm{I}})_m$ (and also $(P_{\mathrm{I}})_m$) can be obtained as a compatibility condition of a system of linear equations. Following the traditional terminology, we call them as the Lax pair. The Lax pair of $(\widetilde{P}_{\mathrm{I}})_m$ is given in [S3] and [KKoNT] as follows:

Definition 1.4 (Lax Pair of $(\widetilde{P}_{\mathrm{I}})_m$)**.** We set:

$$(\widetilde{L}_{\mathrm{I}})_m : \quad \eta^{-1}\widetilde{\gamma}\frac{\partial \vec{\psi}}{\partial x} = A\vec{\psi}, \quad \eta^{-1}\frac{\partial \vec{\psi}}{\partial t} = B\vec{\psi} \quad \text{with} \quad \overrightarrow{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{1.12}$$

where

$$A = \begin{pmatrix} \frac{1}{2}V(x) & U(x) \\ \frac{1}{4}\left\{2x^{m+1} - (x - 2\widetilde{c}_0)U(x) + 2W(x) + 2\widetilde{\gamma}t\right\} & -\frac{1}{2}V(x) \end{pmatrix}, \tag{1.13}$$

$$B = \begin{pmatrix} 0 & 2 \\ \frac{1}{2}x + u_1 + \widetilde{c}_0 & 0 \end{pmatrix}. \tag{1.14}$$

Here U, V and W are given by

$$U(x) = x^m - \sum_{j=1}^{m} u_j x^{m-j}, \quad V(x) = \sum_{j=1}^{m} v_j x^{m-j}, \quad W(x) = \sum_{j=1}^{m} w_j x^{m-j}, \tag{1.15}$$

where $\{w_j\}$ is defined by (1.8).

In the next subsection, we will determine the Hamiltonian of $(P_{\mathrm{I}})_m$. As a preparation for it, we determine here the equation which ψ_1 satisfies, where ψ_1 is the first component of the solution $\overrightarrow{\psi}$ of (1.12). First, we can confirm that ψ_1 solves

$$\frac{\partial^2 \psi_1}{\partial x^2} + \eta q_1 \frac{\partial \psi_1}{\partial x} + \eta^2 q_2 \psi_1 = 0, \tag{1.16}$$

$$\frac{\partial \psi_1}{\partial t} = \mathcal{C}\frac{\partial \psi_1}{\partial x} + \eta \mathcal{D}\psi_1, \tag{1.17}$$

where

$$q_1 = -\frac{1}{U(x)}\frac{\partial U}{\partial x}, \quad q_2 = \frac{1}{\widetilde{\gamma}^2}\det A - \frac{\eta^{-1}}{2\widetilde{\gamma}}\frac{\partial V}{\partial x} + \eta^{-1}\frac{V(x)}{2\widetilde{\gamma}U(x)}\frac{\partial U}{\partial x}, \tag{1.18}$$

$$\mathcal{C} = \frac{2\widetilde{\gamma}}{U(x)}, \qquad \mathcal{D} = -\frac{V(x)}{U(x)}. \tag{1.19}$$

Next we factorize $U(x)$ as

$$U(x) = \prod_{j=1}^{m}(x - b_j(t)). \tag{1.20}$$

We then obtain

$$q_1(x) = -\sum_{j=1}^{m}\frac{1}{x-b_j}, \tag{1.21}$$

$$\begin{aligned} q_2(x) &= \frac{1}{\widetilde{\gamma}^2}\det A - \frac{\eta^{-1}}{2\widetilde{\gamma}}\frac{\partial V}{\partial x} + \frac{\eta^{-1}}{2\widetilde{\gamma}}\sum_{j=1}^{m}\frac{V(x)}{x-b_j} \\ &= \frac{1}{\widetilde{\gamma}^2}\det A - \frac{\eta^{-1}}{2\widetilde{\gamma}}\frac{\partial V}{\partial x} + \frac{\eta^{-1}}{2\widetilde{\gamma}}\sum_{j=1}^{m}\frac{V(x)-V(b_j)}{x-b_j} + \frac{\eta^{-1}}{2\widetilde{\gamma}}\sum_{j=1}^{m}\frac{V(b_j)}{x-b_j}. \end{aligned} \tag{1.22}$$

We note that $x = b_j$ $(1 \le j \le m)$, which is a singular point of q_1 and q_2, is an apparent singular point of (1.16) because (1.12) is not singular there. In order to transform q_2 into a more appropriate form, we use the following:

Proposition 1.5. *Let $\{w_j\}$ be a polynomial of $\{u_j, v_j\}$ defined by* (1.8) (*it is not assumed that $\{u_j, v_j\}$ is a solution of $(\widetilde{P}_{\mathrm{I}})_m$*). *Then*

$$x^{m+1}U - \frac{1}{2}(x - 2\widetilde{c}_0)U^2 + UW + \frac{1}{2}V^2 = \frac{1}{2}x^{2m+1} + x^m C(x) - R(x), \tag{1.23}$$

where U, V and W are defined by (1.15), *and*

$$C(x) = \sum_{j=0}^{m}\widetilde{c}_j x^{m-j}, \tag{1.24}$$

$$R(x) = \sum_{n=1}^{m}x^{m-n}\Big[\frac{1}{2}\sum_{j+k=m+n+1}u_j u_k + \sum_{j+k=m+n}(u_j w_k - \frac{1}{2}v_j v_k - \widetilde{c}_0 u_j u_k)\Big]. \tag{1.25}$$

Since this proposition follows from (1.8) by a straightforward computation, we omit the proof here. By using Proposition 1.5, we obtain

$$\det A = -\frac{1}{4}x^{2m+1} - \frac{1}{2}x^m C(x) - \frac{1}{2}\widetilde{\gamma}tU + \frac{1}{2}R(x). \tag{1.26}$$

Thus q_2 becomes the following form:

$$q_2(x) = -\frac{1}{4\widetilde{\gamma}^2}x^{2m+1} - \frac{1}{2\widetilde{\gamma}^2}x^m C(x) - \frac{t}{2\widetilde{\gamma}}x^m + L(x) + \frac{\eta^{-1}}{2\widetilde{\gamma}}\sum_{j=1}^{m}\frac{V(b_j)}{x-b_j}, \tag{1.27}$$

$$L(x) = -\frac{t}{2\widetilde{\gamma}}U + \frac{1}{2\widetilde{\gamma}^2}R(x) - \eta^{-1}\frac{1}{2\widetilde{\gamma}}\frac{\partial V}{\partial x} + \eta^{-1}\sum_{j=1}^{m}\frac{V(x)-V(b_j)}{2\widetilde{\gamma}(x-b_j(t))}. \tag{1.28}$$

Summing up, we obtain the following:

Proposition 1.6. *The first component* ψ_1 *of the solution* $\overrightarrow{\psi}$ *of* (1.12) *satisfies* (1.16) *and* (1.17) *with* (1.21), (1.27) *and* (1.19).

By the same argument in [Ko], we can show the following:

Theorem 1.7. *The compatibility condition of* (1.16) *and* (1.17) *with* (1.21), (1.27) *and* (1.19) *is equivalent to* $(\widetilde{P}_{\mathrm{I}})_m$.

§ 1.2. Hamiltonians of $(P_{\mathrm{I}})_m$ and the Degenerate Garnier Systems

In this subsection we will find the degenerate Garnier system whose restriction to an appropriate complex line is equivalent to $(\widetilde{P}_{\mathrm{I}})_m$. This result enables us to find the Hamiltonian for $(\widetilde{P}_{\mathrm{I}})_m$ (and hence for $(P_{\mathrm{I}})_m$ in view of Theorem 1.3). Our method here is same with that used in [Ko]; it is shown in [Ko] that $(P_{\text{II-2}})_m$ (resp. $(P_{\mathrm{IV}})_m$) is exactly the same equation with the restriction to an appropriate complex line of some degenerate Garnier system studied by Liu and Okamoto (cf. [L]) (resp. that studied by Kawamuko (cf. [Kwm])).

As is well-known, the degenerate Garnier systems are obtained through the monodromy preserving deformation of some second order linear ordinary differential equation (cf. e.g. [O], [Ki], [IKSY], [L] and [Kwm]). As is mentioned in the previous section, Shimomura already consider the monodromy preserving deformation of an 2×2 system of linear ordinary differential equations to obtain $(\widetilde{P}_{\mathrm{I}})_m$ (cf. [S1], [S2] and [S3]). Although we consider the monodromy preserving deformation of some single equation in order to obtain the Hamiltonian of $(\widetilde{P}_{\mathrm{I}})_m$, our discussion here can be considered as the reformulation of Shimomura's one.

Throughout this subsection we set $\widetilde{c}_0 = 0$ in $(\widetilde{P}_{\mathrm{I}})_m$. To find the appropriate degenerate Garnier system for $(\widetilde{P}_{\mathrm{I}})_m$, we transform the variables and constants in (1.16) with (1.21) and (1.27) by

$$x = \Theta z, \quad b_j = \Theta\lambda_j, \quad \mu_j = \frac{\Theta}{2\widetilde{\gamma}}V(b_j) \quad (1 \le j \le m), \tag{1.29}$$

$$t_1 = \frac{\Theta^{m+2}}{2\widetilde{\gamma}}t + \frac{\Theta^{m+2}}{2\widetilde{\gamma}^2}\widetilde{c}_m, \quad t_k = \frac{\Theta^{m+k+1}}{2\widetilde{\gamma}^2}\widetilde{c}_{m-k+1} \; (2 \le k \le m), \tag{1.30}$$

where Θ is a non-zero constant determined by $\Theta^{2m+3} = 4\widetilde{\gamma}^2$. We then obtain the equation of the following form with $g = m$:

$$\frac{d^2\psi}{dz^2} + \eta p_1 \frac{d\psi}{dz} + \eta^2 p_2 \psi = 0, \tag{1.31}$$

where

$$p_1 = -\eta^{-1}\sum_{j=1}^{g}\frac{1}{z-\lambda_j}, \tag{1.32}$$

$$p_2 = -\big[z^{2g+1}+\sum_{j=1}^{g}t_j z^{g+j-1}+\sum_{j=1}^{g}h_j z^{g-j}\big]+\eta^{-1}\sum_{j=1}^{g}\frac{\mu_j}{z-\lambda_j}. \tag{1.33}$$

We consider the monodromy preserving deformation of (1.31).

First, we find that (1.31) has an irregular singular point at $z=\infty$ whose Poincaré rank is $g+3/2$. It also has a regular singular point at $z=\lambda_j$ $(1\le j\le g)$. We assume that these singular points $\lambda_1,\lambda_2,\cdots,\lambda_g$ are apparent ones (recall that $z=b_j$ is an apparent singular point in (1.16)). The Riemann scheme of (1.31) becomes the following (see [O, p. 609] for the definition of the following Riemann scheme):

$$\left\{\begin{array}{ccccccccccc} z=\lambda_1 & \cdots & z=\lambda_g & \multicolumn{8}{c}{\overbrace{\hphantom{\frac{\eta T_0}{g+3/2}\quad 0\quad \frac{\eta T_1}{g+1/2}\quad 0\quad \cdots\quad 0\quad \frac{\eta T_{g+1}}{1/2}\quad \frac{1}{4}}}^{z=\infty\,(1/2)}} \\ 0 & \cdots & 0 & \dfrac{\eta T_0}{g+3/2} & 0 & \dfrac{\eta T_1}{g+1/2} & 0 & \cdots & 0 & \dfrac{\eta T_{g+1}}{1/2} & \dfrac{1}{4} \\ 2 & \cdots & 2 & -\dfrac{\eta T_0}{g+3/2} & 0 & -\dfrac{\eta T_1}{g+1/2} & 0 & \cdots & 0 & -\dfrac{\eta T_{g+1}}{1/2} & \dfrac{1}{4} \end{array}\right\}.$$

Here $\{T_j\}_{j=0}^{g+1}$ is recursively defined by

$$T_0=1,\quad T_1=0,\quad 2T_0T_{n+1}+\sum_{j=1}^{n}T_jT_{n-j+1}=t_{g-n+1}\quad (1\le n\le g). \tag{1.34}$$

For example we have

$$T_0=1,\quad T_1=0,\quad T_2=\frac{1}{2}t_g,\quad T_3=\frac{1}{2}t_{g-1},\quad T_4=t_{g-2}-\frac{1}{8}{t_g}^2. \tag{1.35}$$

Concerning the assumption of the apparent singular points, we can show the following by the same argument as in [L, Proposition 2.1] (cf. also [O]):

Lemma 1.8. *The singular points* $\lambda_1,\cdots,\lambda_g$ *in* (1.31) *are apparent ones if and only if* $\{h_i\}$ *in* p_2 *is given by* $h_i=\overline{h}_i$ *for* $1\le i\le g$, *where* $\overline{h}_i$ *is a rational function in* $\{\lambda_j,\mu_k,t_l\}$ *defined by the following*:

$$\overline{h}_i=\sum_{j=1}^{g}N_jN^{i,j}\Big[{\mu_j}^2-\big\{{\lambda_j}^{2g+1}+\sum_{k=1}^{g}t_k{\lambda_j}^{g+k-1}\big\}\Big] - \eta^{-1}\sum_{\substack{j,l=1\\ j\ne l}}^{g}\frac{N_jN^{i,j}+N_lN^{i,l}}{\lambda_j-\lambda_l}\mu_j, \tag{1.36}$$

where

$$\Lambda(x) = (x-\lambda_1)\cdots(x-\lambda_g), \quad N_j = \frac{1}{\Lambda'(\lambda_j)}, \quad N^{i,j} = (-1)^{i-1} e^{(j)}_{i-1}(\lambda) \tag{1.37}$$

and $e^{(j)}_l$ *is the* l*-th symmetric polynomial in* $\{\lambda_k;\ k \neq j\}$, *i.e.*

$$\prod_{\substack{n=1\\ n\neq j}}^{g} (X+\lambda_n) = \sum_{l=0}^{g-1} e^{(j)}_l X^{g-l-1}. \tag{1.38}$$

We consider $t = (t_1, \cdots, t_g)$ as a deformation parameter, and we would like to determine the condition on $\{\lambda_j, \mu_k\}$ so that the monodromy data (in our case Stokes multipliers of (1.31) at $z=\infty$) is preserved. It is known (cf. [O], [U], [JMU], [IKSY]) that the monodromy data is preserved if and only if there exist rational functions $\mathcal{A}_j$, $\mathcal{B}_j$ $(1 \leq j \leq g)$ in z so that (1.31) and

$$\frac{\partial \psi}{\partial t_j} = \mathcal{A}_j \frac{\partial \psi}{\partial z} + \eta \mathcal{B}_j \psi \quad (1 \leq j \leq g) \tag{1.39}$$

are completely integrable. We obtain the following:

Theorem 1.9. *The monodromy data of* (1.31) *is preserved if and only if* $\{\lambda_j, \mu_k\}$ *satisfies the following completely integrable Hamiltonian system with time variables* $t = (t_1, \cdots, t_g)$:

$$\frac{\partial \lambda_j}{\partial t_k} = \eta \frac{\partial H_k}{\partial \mu_j}, \quad \frac{\partial \mu_j}{\partial t_k} = -\eta \frac{\partial H_k}{\partial \lambda_j} \quad (1 \leq j, k \leq g), \tag{1.40}$$

where

$$H_k = \sum_{p=1}^{g} a_{k,g-p}(t) \overline{h}_p. \tag{1.41}$$

Here $\{a_{j,k}(t)\}$ *is determined by the following recursion relation for each* j:

$$(2g-2k-1) a_{j,k}(t) + \sum_{l=2}^{k} (2g-2k+l-1) t_{g-l+2} a_{j,k-l}(t) = \delta_{k,g-j}. \tag{1.42}$$

Following the commonly used terminology (cf. [Ki]), we refer to (1.40) as

$$G(g+5/2; g).$$

As is well-known, (1.40) becomes P_{I} when $g=1$. Equation (1.40) with $g=2$ is obtained in [Ki].

This theorem will be proved in the next subsection. In the course of the proof of Theorem 1.9, we will find that $\mathcal{A}_j$ in (1.39) has the following form:

$$\mathcal{A}_j = \sum_{k=1}^{g} \frac{1}{z-\lambda_k} \sum_{l=0}^{g-1} N_k N^{g-l,k} a_{j,l}(t). \tag{1.43}$$

Remark. We will show in the next subsection that $\{a_{j,k}(t)\}$ satisfies

$$\sum_{\substack{p,k,l\geq 0\\ p+k+l=m}} (2g-2m+2p-1) a_{j,k} T_p T_l = \delta_{j,g-m} \quad (0 \leq m \leq g-1), \tag{1.44}$$

where $\{T_l\}$ is defined by (1.34), or in a matrix form,

$$\begin{pmatrix} T_0 & 0 & \cdots & 0 \\ T_1 & T_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ T_{g-1} & \cdots & T_1 & T_0 \end{pmatrix} \begin{pmatrix} a_{1,g-1} & a_{1,g-2} & \cdots & a_{1,0} \\ a_{2,g-1} & a_{2,g-2} & \cdots & a_{2,0} \\ \vdots & \vdots & & \vdots \\ a_{g,g-1} & a_{g,g-2} & \cdots & a_{g,0} \end{pmatrix} \begin{pmatrix} T_0 & 0 & \cdots & 0 \\ T_1 & T_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ T_{g-1} & \cdots & T_1 & T_0 \end{pmatrix} = \mathrm{diag}(1, \frac{1}{3}, \dots, \frac{1}{2g-1}). \tag{1.45}$$

Once this Theorem 1.9 is established, we can determine the Hamiltonian of $(\widetilde{P}_{\mathrm{I}})_m$ as follows:

Theorem 1.10. *Let K be a rational function of $\{\lambda_j, \mu_k, t\}$ defined by*

$$K := \Theta^{1/2} H_1 \Big|_{t_1=\Theta^{1/2}t+2\Theta^{-m-1}\tilde{c}_m,\ t_j=2\Theta^{-(m-j+2)}\tilde{c}_{m-j+1}\ (2\leq j\leq m)}, \tag{1.46}$$

where H_1 is a Hamiltonian of $G(m+5/2;m)$ defined in (1.41) *and*

$$\Theta^{2m+3} = 4\tilde{\gamma}^2. \tag{1.47}$$

Then $\{u_j, v_j\}$ is a solution of $(\widetilde{P}_{\mathrm{I}})_m$ if and only if $\{\lambda_j, \mu_k\}$ defined by

$$U = \prod_{j=1}^{m} (x - \Theta\lambda_j), \quad \mu_j = \Theta^{-m-1/2} V(\Theta\lambda_j) \quad (1 \leq j \leq m) \tag{1.48}$$

is a solution of the following Hamiltonian system:

$$\frac{d\lambda_j}{dt} = \eta \frac{\partial K}{\partial \mu_j}, \quad \frac{d\mu_j}{dt} = -\eta \frac{\partial K}{\partial \lambda_j} \quad (1 \leq j \leq m). \tag{1.49}$$

Proof. By the same argument given in [Ko, § 2.3], it is enough to verify the following equations:

$$p_1 = \Theta q_1(\Theta z), \qquad p_2\Big|_{h=\overline{h}} = \Theta^2 q_2(\Theta z), \qquad \Theta^{-1}\mathcal{C} = \mathcal{A}_1 \tag{1.50}$$

with (1.29) and (1.30), where Θ is determined by (1.47). In fact these relations (1.50) imply that the compatibility condition of (1.16) and (1.17) is equivalent to that of (1.31) and (1.39) with $j = 1$. The first and third equation of (1.50) follows by a straightforward computation. To show the second relation, we need to prove the following:

$$\Theta^2 L(\Theta z) = -\sum_{j=1}^{m} h_j z^{m-j}. \tag{1.51}$$

To prove (1.51), we note that $\{h_j\}$ is uniquely determined by the condition that $\lambda_1, \cdots, \lambda_m$ are apparent singular points of (1.31) (cf. Lemma 1.8). Since $x = b_1, \cdots, b_m$ are also apparent singular points, the second equation of (1.50) follows from the uniqueness of $\{h_j\}$. □

Remark. Takasaki obtained the Hamiltonian of $(P_{\mathrm{I}})_m$ in [Tks1], [Tks2] and [Tks3] by considering the Hamiltonians of the so-called string equations of type $(2, 2g+1)$ and the associated commuting flows. The Hamiltonians that Takasaki obtained is related to ours by the the following canonical transformation: Let us transform $\{t_j\}$ by

$$s_{2g-2n+3} = \frac{2}{2g-2n+3} T_n \quad (2 \leq n \leq g+1). \tag{1.52}$$

We also define $\{L_{2n-1}\}_{n=1}^{g}$ by

$$\begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_g \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 3 & & \\ & & \ddots & \\ & & & 2g-1 \end{pmatrix} \begin{pmatrix} T_0 & & & \\ T_1 & T_0 & & \\ \vdots & \ddots & \ddots & \\ T_{g-1} & \cdots & T_1 & T_0 \end{pmatrix} \begin{pmatrix} L_1 \\ L_3 \\ \vdots \\ L_{2g-1} \end{pmatrix} \tag{1.53}$$

Then by using (1.45), we can verify that

$$(\lambda, \mu, H, t) \mapsto (\lambda, \mu, L, s) \tag{1.54}$$

is a canonical transformation. The Hamiltonians which Takasaki obtained in [Tks3] are $L_1, L_3, \cdots, L_{2g-1}$.

§ 1.3. Proof of Theorem 1.9

In this section we give a proof of Theorem 1.9. This theorem can be shown by following the argument given in [L] and [Kwm]. In the following we set the large parameter η to one to make equations simple.

We first transform (1.31) into the so-called SL-type, i.e. the equation without the first derivative term, by the following change of the unknown function:

$$\psi = \prod_{j=1}^{g} (z-\lambda_j)^{-1/2}\varphi. \tag{1.55}$$

Then (1.31) becomes

$$\frac{d^2\varphi}{dz^2} - Q\varphi = 0, \tag{1.56}$$

where

$$Q = z^{2g+1} + \sum_{j=1}^{g} t_j z^{g+j-1} + \sum_{j=1}^{g} \overline{h}_j z^{g-j} - \sum_{j=1}^{g} \frac{\nu_j}{z-\lambda_j} + \sum_{j=1}^{g} \frac{3}{4(z-\lambda_j)^2} \tag{1.57}$$

(we set here $h_i = \overline{h}_i$ to guarantee that λ_j is a non-logarithmic singular point) with

$$\nu_j = \mu_j - \frac{1}{2} \sum_{\substack{k=1\\k\neq j}}^{g} \frac{1}{\lambda_j - \lambda_k} \quad (1 \le j \le g). \tag{1.58}$$

Remark. If we expand Q near $z = \lambda_j$ as

$$Q = \frac{1}{(z-\lambda_j)^2} \sum_{n=0}^{\infty} Q_n^{(j)} (z-\lambda_j)^k, \tag{1.59}$$

then it is easy to see $Q_0^{(j)} = 3/4$ and $Q_1^{(j)} = -\nu_j$. Our assumption that $z = \lambda_j$ is non-logarithmic gives (cf. [O, (2.4)], [Kwm, Lemma 3.3], [L, p.583])

$$Q_2^{(j)} = (\nu_j)^2. \tag{1.60}$$

It was shown in [O, Proposition 1.2 (p.584), Proposition 1.3 (p.585)] that the monodromy data of (1.31) is preserved if and only if the monodromy data of (1.56) is preserved, and that the monodromy data of (1.56) is preserved if and only if there exists rational functions $\mathcal{A}_1, \cdots, \mathcal{A}_g$ such that (1.56) and the following equations are completely integrable:

$$\frac{\partial \varphi}{\partial t_j} = \mathcal{A}_j \frac{\partial \varphi}{\partial z} - \frac{1}{2} \frac{\partial \mathcal{A}_j}{\partial z} \varphi. \tag{1.61}$$

Note that the compatibility condition of (1.56) and (1.61) is given by

$$\Theta_j := \frac{1}{2} \frac{\partial^3 \mathcal{A}_j}{\partial z^3} - 2Q \frac{\partial \mathcal{A}_j}{\partial z} - \mathcal{A}_j \frac{\partial Q}{\partial z} + \frac{\partial Q}{\partial t_j} = 0 \quad (1 \le j \le g), \tag{1.62}$$

$$\Xi_{i,j} := \frac{\partial \mathcal{A}_j}{\partial t_i} - \frac{\partial \mathcal{A}_j}{\partial t_j} + \mathcal{A}_j \frac{\partial \mathcal{A}_i}{\partial z} - \mathcal{A}_i \frac{\partial \mathcal{A}_j}{\partial z} = 0 \quad (1 \le i, j \le g). \tag{1.63}$$

We also note that the transformation

$$(\lambda, \mu, H, t) \mapsto (\lambda, \nu, K, t) \tag{1.64}$$

is canonical, where K_j for $1 \le j \le g$ is obtained by substituting (1.58) into H_j. Thus if we establish the following theorem for (1.56) we obtain Theorem 1.9:

Theorem 1.11. *The monodromy data of* (1.56) *is preserved if and only if* $\{\lambda_j, \mu_k\}$ *satisfies the following completely integrable Hamiltonian system with time variables* $t = (t_1, \cdots, t_g)$:

$$\frac{\partial \lambda_j}{\partial t_k} = \frac{\partial K_k}{\partial \mu_j} = \sum_{p=1}^{g} a_{k,g-p}(t) \frac{\partial \overline{h}_p}{\partial \nu_j}, \tag{1.65}$$

$$\frac{\partial \mu_j}{\partial t_k} = -\frac{\partial K_k}{\partial \lambda_j} = -\sum_{p=1}^{g} a_{k,g-p}(t) \frac{\partial \overline{h}_p}{\partial \lambda_j} \tag{1.66}$$

for $1 \le j \le g$.

Now we prove Theorem 1.11. We divide our proof into several steps.

1st Step: Analytic Properties of $\{\mathcal{A}_j\}$.

Lemma 1.12. *If* $\mathcal{A}_j$ *is rational in* z *and satisfies* (1.62), *then* $\Lambda(x)\mathcal{A}_j$ *is a polynomial in* z *whose degree is at most* $j-1$.

This lemma can be proved by the same argument as in [L] and [Kwm, Lemma 3.1]. See these references for its details.

2nd Step: Local Expansion near $z = \infty$. To determine $\beta_j^{(k)}$ in (1.97), we compute the expansion of $\mathcal{A}_j$ near ∞ as follows:

Lemma 1.13. *If* $\mathcal{A}_j$ *is rational in* z *and satisfies* (1.62), *then we obtain*

$$\mathcal{A}_j = \sum_{k=0}^{g-1} a_{j,k}(t) z^{-k-1} + O(z^{-g-1}) \quad (|z| \to \infty), \tag{1.67}$$

where $\{a_{j,k}(t)\}$ *is defined by* (1.42).

Proof. By substituting the expansion

$$\mathcal{A}_j = \sum_{k=0}^{\infty} \mathcal{A}_{j,k}^{(\infty)} z^{-k-1}, \quad Q = x^{2g+1} \sum_{l=0}^{\infty} Q_l^{(\infty)} z^{-l} \tag{1.68}$$

into (1.62), we obtain

$$\Theta_j = -z^{2g-1}\sum_{n=0}^{g-1} z^{-n} \sum_{k+l=n} (2g-2k-l-1)Q_l^{(\infty)}\mathcal{A}_{j,k}^{(\infty)} + x^{g+j-1} + O(z^{g-1}). \tag{1.69}$$

Hence $\Theta_j = 0$ gives

$$\sum_{k+l=n} (2g-2k-l-1)Q_l^{(\infty)}\mathcal{A}_{j,k}^{(\infty)} = \delta_{n,g-j} \quad (0 \leq n \leq g-1). \tag{1.70}$$

Since

$$Q_0^{(\infty)} = 1, \quad Q_1^{(\infty)} = 0, \quad Q_l^{(\infty)} = t_{g-l+2} \quad (2 \leq l \leq g+1), \tag{1.71}$$

we can verify that $\mathcal{A}_{j,k}^{(\infty)} = a_{j,k}(t)$. □

Remark. In this proof, we also show that if $\mathcal{A}_j$ satisfies (1.67), then

$$\Theta_j = O(z^{g-1}) \quad (|z| \to \infty) \tag{1.72}$$

Lemma 1.14. *If $\Theta_j = 0$ holds for $1 \leq j \leq g$, then*

$$\mathcal{A}_j = T(z)^{-1}\int_0^z \frac{\partial T}{\partial t_j}dz + O(z^{-g-1}) \quad (|z| \to \infty), \tag{1.73}$$

where

$$T(z) = z^{g+1/2}\sum_{l=0}^{g+1} T_l z^{-l} \tag{1.74}$$

and $\{T_l\}_{l=0}^{g+1}$ is defined by (1.34).

Proof. First we note that the following holds:

$$Q(z) = T(z)^2 + O(z^{-g-1}) \quad (|z| \to \infty). \tag{1.75}$$

From $\Theta_j = 0$, we obtain

$$2Q^{1/2}\frac{\partial}{\partial z}(Q^{1/2}\mathcal{A}_j) = \frac{\partial Q}{\partial t_j} + \frac{1}{2}\frac{\partial^3 \mathcal{A}_j}{\partial z^3} = \frac{\partial Q}{\partial t_j} + O(z^{-4}). \tag{1.76}$$

Hence

$$\frac{\partial}{\partial z}(Q^{1/2}\mathcal{A}_j) = \frac{\partial Q^{1/2}}{\partial t_j} + O(x^{-z-9/2}). \tag{1.77}$$

Then, by substituting (1.73) into (1.77), we can confirm (1.73) holds. □

We note that (1.45) can be obtained by considering the expansion of the both side of (1.76) near $z = \infty$.

Proposition 1.15. *If $\mathcal{A}_j$ satisfies* (1.62) *for $1 \le j \le g$, then*

$$\Xi_{i,j} = O(z^{-g-1}) \quad (|z| \to \infty). \tag{1.78}$$

Proof. From (1.73), we fist obtain

$$\frac{\partial A_j}{\partial t_i} - \frac{\partial A_i}{\partial t_j} = T(z)^{-2}\Big(-\frac{\partial T}{\partial t_i}\int_0^z \frac{\partial T}{\partial t_j}dz + \frac{\partial T}{\partial t_j}\int_0^z \frac{\partial T}{\partial t_i}dz\Big) + O(z^{-g-1}). \tag{1.79}$$

On the other hand, since

$$\begin{aligned}\mathcal{A}_i\frac{\partial \mathcal{A}_j}{\partial z} &= \Big(T^{-1}\int_0^z \frac{\partial T}{\partial t_i}dz + O(z^{-g-1})\Big)\Big(-T^{-2}\frac{\partial T}{\partial z}\int_0^z \frac{\partial T}{\partial t_j}dz + T^{-1}\frac{\partial T}{\partial t_j} + O(z^{-g-2})\Big)\\ &= -T^{-3}\frac{\partial T}{\partial z}\int_0^z \frac{\partial T}{\partial t_i}dz\int_0^z \frac{\partial T}{\partial t_j}dz + T^{-2}\frac{\partial T}{\partial t_j}\int_0^z \frac{\partial T}{\partial t_i}dz + O(z^{-g-2}).\end{aligned} \tag{1.80}$$

Hence we obtain

$$\mathcal{A}_i\frac{\partial \mathcal{A}_j}{\partial z} - \mathcal{A}_j\frac{\partial \mathcal{A}_i}{\partial z} = T^{-2}\Big(\frac{\partial T}{\partial t_j}\int_0^z \frac{\partial T}{\partial t_i}dz - \frac{\partial T}{\partial t_i}\int_0^z \frac{\partial T}{\partial t_j}dz\Big) + O(z^{-g-2}). \tag{1.81}$$

Thus (1.78) follows. □

3rd Step: Local expansion near $z = \lambda_j$. We prove the following general property:

Proposition 1.16 (cf. [Kwm, Proposition 3.3] and [L, Theorem 5.1])**.** *Assume that Q and $\mathcal{A}_j$ have the following local expansion near $z = \lambda_j$:*

$$Q = \frac{1}{(z-\lambda_l)^2}\sum_{n=0}^{\infty} Q_n^{(l)}(z-\lambda_l)^n \quad \Big(Q_0^{(l)} = \frac{3}{4}, \quad Q_1^{(l)} = -\nu_l, \quad Q_2^{(l)} = {\nu_l}^2\Big), \tag{1.82}$$

$$\mathcal{A}_j = \frac{1}{z-\lambda_l}\sum_{n=0}^{\infty} \mathcal{A}_{j,n}^{(l)}(z-\lambda_l)^n. \tag{1.83}$$

By substituting these expansion into (1.62) *and* (1.63)*, we obtain the following local expansion of Θ_j and $\Xi_{i,j}$:*

$$\Theta_j = \frac{1}{(z-\lambda_l)^4}\sum_{n=0}^{\infty} \Theta_{j,n}^{(l)}(z-\lambda_l)^n, \quad \Xi_{i,j} = \frac{1}{(z-\lambda_l)^3}\sum_{n=0}^{\infty} \Xi_{i,j,n}^{(l)}(z-\lambda_l)^n. \tag{1.84}$$

Note that $\Theta_{j,0}^{(l)}$ and $\Xi_{i,j,0}^{(l)}$ are trivially vanish. If $\Theta_{j,1}^{(l)} = \Theta_{j,3}^{(l)} = 0$ hold for $1 \le j \le g$, then λ_l and ν_l satisfy

$$\frac{\partial \lambda_l}{\partial t_j} = 2\mathcal{A}_{j,0}^{(l)}\nu_l - \mathcal{A}_{j,1}^{(l)}, \tag{1.85}$$

$$\frac{\partial \nu_l}{\partial t_j} = \mathcal{A}_{j,0}^{(l)}Q_3^{(l)} + \mathcal{A}_{j,2}^{(l)}\nu_l - \frac{3}{2}\mathcal{A}_{j,3}^{(l)}, \tag{1.86}$$

and we obtain $\Theta^{(l)}_{j,2} = \Theta^{(l)}_{j,4} = 0$ *and* $\Xi^{(l)}_{i,j,1} = 0$ *for* $1 \le i, j \le g$. *If we further assume*

$$\Theta^{(l)}_{j,5} = 0 \quad \textit{and} \quad \Xi^{(l)}_{i,j,2} = \Xi^{(l)}_{i,j,3} = \Xi^{(l)}_{i,j,4} = \Xi^{(l)}_{i,j,5} = 0 \tag{1.87}$$

for $1 \le i, j \le g$, *then* (1.85) *and* (1.86) *are completely integrable in the sense of Frobenius.*

Proof. To prove this proposition, we explicitly compute $\Theta^{(l)}_{j,k}$ and $\Xi^{(l)}_{i,j,k}$ for $0 \le k \le 5$. The higher order coefficients of Θ_j are given as follows:

$$\Theta^{(l)}_{j,1} = \frac{3}{2}\Big[\frac{\partial \lambda_l}{\partial t_j} + \mathcal{A}^{(l)}_{j,1} - 2\mathcal{A}^{(l)}_{j,0}\nu_l\Big], \qquad \Theta^{(l)}_{j,2} = -\frac{2}{3}\nu_l \mathcal{A}^{(l)}_{j,1}, \tag{1.88}$$

$$\Theta^{(l)}_{j,3} = -\Big[\frac{\partial \nu_l}{\partial t_j} + \frac{3}{2}\mathcal{A}^{(l)}_{j,3} - \mathcal{A}^{(l)}_{j,2}\nu_l - \mathcal{A}^{(l)}_{j,0}Q^{(l)}_3\Big], \quad \Theta^{(l)}_{j,4} = -\frac{2}{3}\mathcal{A}^{(l)}_{j,1}Q^{(l)}_3 - 2\nu\mathcal{A}^{(l)}_{j,3}, \tag{1.89}$$

$$\begin{aligned}\Theta^{(l)}_{j,5} = & -\frac{4}{3}Q^{(l)}_2 F^{(l)}_{j,1} - (Q^{(l)}_5 + 4\nu_l Q^{(l)}_4)\mathcal{A}^{(l)}_{j,0} - 3Q^{(l)}_3\mathcal{A}^{(l)}_{j,2} - 4{\nu_l}^2\mathcal{A}^{(l)}_{j,3} + 5\nu_l\mathcal{A}^{(l)}_{j,4} \\ & + \frac{15}{2}\mathcal{A}^{(l)}_{j,5} + \frac{\partial Q^{(l)}_3}{\partial t_j}.\end{aligned} \tag{1.90}$$

We also obtain the following:

$$\Xi^{(l)}_{i,j,1} = \frac{2}{3}(\mathcal{A}^{(l)}_{j,0}\Theta^{(l)}_{i,1} - \mathcal{A}^{(l)}_{i,0}\Theta^{(l)}_{j,1}), \qquad \Xi^{(l)}_{i,j,2} = \frac{\partial \mathcal{A}^{(l)}_{j,0}}{\partial t_i} - \frac{\partial \mathcal{A}^{(l)}_{i,0}}{\partial t_j} - 2Z_{0,2},$$

$$\Xi^{(l)}_{i,j,3} = \frac{\partial \mathcal{A}^{(l)}_{j,1}}{\partial t_i} - \frac{\partial \mathcal{A}^{(l)}_{i,1}}{\partial t_j} - 2\nu_l Z_{0,2} - 3Z_{0,3}, \quad \Xi^{(l)}_{i,j,4} = \frac{\partial \mathcal{A}^{(l)}_{j,2}}{\partial t_i} - \frac{\partial \mathcal{A}^{(l)}_{i,2}}{\partial t_j} - 4\nu_l Z_{0,3} - 4Z_{0,4},$$

$$\Xi^{(l)}_{i,j,5} = \frac{\partial \mathcal{A}^{(l)}_{j,3}}{\partial t_i} - \frac{\partial \mathcal{A}^{(l)}_{i,3}}{\partial t_j} - 6\nu_l Z_{0,4} - 5Z_{0,5} - Z_{2,3}.$$

Here and in what follows, we fix i, j, l to simplify the notation, and we put

$$Z_{p,q} = \mathcal{A}^{(l)}_{i,p}\mathcal{A}^{(l)}_{j,q} - \mathcal{A}^{(l)}_{j,p}\mathcal{A}^{(l)}_{i,q} \tag{1.91}$$

Now the first assertion follows from these concrete expressions. We next show the completely integrability of (1.85) and (1.86). To this purpose it suffices to show

$$\frac{\partial}{\partial t_i}\big(2\mathcal{A}^{(l)}_{j,0}\nu_l - \mathcal{A}^{(l)}_{j,1}\big) - \frac{\partial}{\partial t_j}\big(2\mathcal{A}^{(l)}_{i,0}\nu_l - \mathcal{A}^{(l)}_{i,1}\big) = 0. \tag{1.92}$$

$$\frac{\partial}{\partial t_i}\big(\frac{3}{2}\mathcal{A}^{(l)}_{j,3} - \mathcal{A}^{(l)}_{j,2}\nu_l - \mathcal{A}^{(l)}_{j,0}Q^{(l)}_3\big) - \frac{\partial}{\partial t_j}\big(\frac{3}{2}\mathcal{A}^{(l)}_{i,3} - \mathcal{A}^{(l)}_{i,2}\nu_l - \mathcal{A}^{(l)}_{i,0}Q^{(l)}_3\big) = 0. \tag{1.93}$$

First we verify (1.92). The left-hand side of (1.92) becomes

$$2\nu_l\Big(\frac{\partial \mathcal{A}^{(l)}_{j,0}}{\partial t_i} - \frac{\partial \mathcal{A}^{(l)}_{i,0}}{\partial t_j}\Big) + 2\Big[\mathcal{A}^{(l)}_{j,0}\frac{\partial \nu_l}{\partial t_i} - \mathcal{A}^{(l)}_{i,0}\frac{\partial \nu_l}{\partial t_j}\Big] - \Big[\frac{\partial \mathcal{A}^{(l)}_{j,1}}{\partial t_i} - \frac{\partial \mathcal{A}^{(l)}_{i,1}}{\partial t_j}\Big]. \tag{1.94}$$

To eliminate terms in (1.94) which include t-derivatives, we use (1.86) for the second term, and $\Xi^{(l)}_{i,j,2} = \Xi^{(l)}_{i,j,3} = 0$ for the first and the third terms. Then we obtain

$$(1.94) = 4\nu_l Z_{0,2} + [3Z_{0,3} - 2\nu_l Z_{0,2} + 2Q^{(l)}_3 Z_{0,0}] - [2\nu_l Z_{0,2} + 3Z_{0,3}] = 0. \tag{1.95}$$

Thus (1.92) holds. In a similar manner we can show (1.93); the left-hand side of (1.93) becomes

$$\begin{aligned}\frac{3}{2}\Big(\frac{\partial \mathcal{A}^{(l)}_{j,3}}{\partial t_i} - \frac{\partial \mathcal{A}^{(l)}_{i,3}}{\partial t_j}\Big) - \nu_l\Big(\frac{\partial \mathcal{A}^{(l)}_{j,2}}{\partial t_i} - \frac{\partial \mathcal{A}^{(l)}_{i,2}}{\partial t_j}\Big) - \Big(\mathcal{A}^{(l)}_{j,2}\frac{\partial \nu_l}{\partial t_i} - \mathcal{A}_{i,2}\frac{\partial \nu_l}{\partial t_j}\Big) \\ - Q^{(l)}_3\Big(\frac{\partial \mathcal{A}^{(l)}_{j,0}}{\partial t_i} - \frac{\partial \mathcal{A}^{(l)}_{i,0}}{\partial t_j}\Big) - \Big(\mathcal{A}^{(l)}_{j,0}\frac{\partial Q^{(l)}_3}{\partial t_i} - \mathcal{A}^{(l)}_{i,0}\frac{\partial Q^{(l)}_3}{\partial t_j}\Big).\end{aligned} \tag{1.96}$$

We can then show that (1.96) vanishes by using (1.86) for the third term, $\Theta^{(l)}_{j,5} = 0$ for the fifth term, and $\Xi^{(l)}_{i,j,2} = \Xi^{(l)}_{i,j,4} = \Xi^{(l)}_{i,j,5} = 0$ for the first, the second and fourth terms. □

4th Step: Derivation of the Nonlinear Equations. At this step, we combine local analytic properties near the singular points obtained at the previous steps to determine the nonlinear equations which govern the monodromy preserving deformation. First, from Lemma 1.12 we can assume that $\mathcal{A}_j$ has the following form:

$$\mathcal{A}_j = \sum_{k=1}^{g} \frac{\beta^{(k)}_j}{z-\lambda_k} = \frac{\beta^{(1)}_j}{z-\lambda_1} + \cdots + \frac{\beta^{(g)}_j}{z-\lambda_g} \tag{1.97}$$

Hence we obtain

$$\mathcal{A}^{(\infty)}_{j,k} = \beta^{(1)}_j {\lambda_1}^k + \cdots + \beta^{(g)}_j {\lambda_g}^k, \tag{1.98}$$

where $\mathcal{A}^{(\infty)}_{j,k}$ is a coefficients of (1.67). Hence from Lemma 1.13 we obtain

$$\beta^{(1)}_j {\lambda_1}^k + \cdots + \beta^{(g)}_j {\lambda_g}^k = a_{j,k}(t) \quad (1 \le j \le g). \tag{1.99}$$

Because it is linear in $\beta^{(k)}_j$, we can solve (1.99) as follows:

$$\beta^{(k)}_j = \sum_{p=0}^{g-1} N_k N^{g-p,k} a_{j,p}(t). \tag{1.100}$$

See, e.g. [L, Proposition 2.1 (p.570)] for the inverse of the Vandermonde matrix appearing in the left-hand side of (1.99). We also obtain from (1.97) that

$$\mathcal{A}^{(l)}_{j,0} = \beta^{(l)}_j, \quad \mathcal{A}^{(l)}_{j,n} = (-1)^{n-1} \sum_{k\neq l} \frac{\beta^{(k)}_j}{(\lambda_l - \lambda_k)^n} \quad (n \ge 1). \tag{1.101}$$

Hence it follows from Proposition 1.16 that $\Theta_j = 0$ gives the following differential equations of $\{\lambda_j, \mu_k\}$:

$$\frac{\partial \lambda_l}{\partial t_j} = 2\beta_j^{(l)}\nu_l - \sum_{\substack{k=1\\k\neq l}}^{g} \frac{\beta_j^{(k)}}{\lambda_l - \lambda_k}, \tag{1.102}$$

$$\frac{\partial \nu_l}{\partial t_j} = -\sum_{\substack{k=1\\k\neq l}}^{g} \Big[\frac{\beta_j^{(k)}\nu_l}{(\lambda_l - \lambda_k)^2} - \frac{3}{2}\frac{\beta_j^{(k)}}{(\lambda_l - \lambda_k)^3}\Big] + \beta_j^{(l)} Q_3^{(l)}. \tag{1.103}$$

Proposition 1.17. *Assume $\mathcal{A}_j$ is given by* (1.97) *satisfying* (1.99)*. Then the condition $\Theta_j = \Xi_{i,j} = 0$ for $1 \le i, j \le g$ is equivalent to* (1.102) *and* (1.103)*, where Θ_j and $\Xi_{i,j}$ are defined by* (1.62) *and* (1.63) *respectively. Moreover if one of these conditions is satisfied, then* (1.102) *and* (1.103) *are completely integrable.*

Proof. Since the necessity is obvious from the discussion given above, we show the sufficiency. By its definition Θ_j is a rational function in z which is holomorphic except $z = \lambda_1, \cdots, \lambda_g$. If we assume that (1.102) and (1.103) hold, then Proposition 1.16 tells us that Θ_j has zeros at $\{\lambda_1, \cdots, \lambda_g\}$. Thus Θ_j is a polynomial which has, at least, g zeros. On the other hand, Θ_j is of $O(z^{g-1})$ near $z = \infty$ (cf. (1.72)). Thus Θ_j should vanish identically.

It also follows from Proposition 1.16 that $\Lambda(z)\Xi_{i,j}$ is polynomial in z. From (1.78), we obtain $\Lambda(z)\Xi_{i,j} \to 0$ as $|z| \to \infty$. Hence $\Xi_{i,j}$ identically vanishes.

The completely integrability follows from Proposition 1.16. $\square$

5th Step: Derivation of the Hamiltonians. To finish the proof of Theorem 1.11, we will show that the nonlinear equations (1.102) and (1.103) can be expressed as the Hamiltonian system (1.40). To this purpose we put

$$q_2^{(k)} = (\lambda_k)^{2g+1} + \sum_{i=1}^{g} t_i (\lambda_k)^{g+i-1} + \sum_{i=1}^{g} h_i (\lambda_k)^{g-i} - \sum_{\substack{i=1\\i\neq k}}^{g} \frac{\nu_i}{\lambda_k - \lambda_i} + \sum_{\substack{i=1\\i\neq k}}^{g} \frac{3}{4(\lambda_k - \lambda_i)^2}. \tag{1.104}$$

Then we can easily confirm that

$$Q_2^{(k)} = q_2^{(k)}|_{h=\overline{h}}. \tag{1.105}$$

We can also confirm from its definition that

$$\frac{\partial q_2^{(l)}}{\partial \nu_l} = 0, \qquad \frac{\partial q_2^{(k)}}{\partial \nu_l} = -\frac{1}{\lambda_k - \lambda_l} \quad (k \neq l). \tag{1.106}$$

Hence (1.102) becomes

$$\frac{\partial \lambda_l}{\partial t_j} = 2\beta_j^{(l)}\nu_l - \sum_{\substack{k=1\\k\neq l}}^{g} \beta_j^{(k)} \frac{\partial q_2^{(k)}}{\partial \nu_l} = \sum_{k=1}^{g} \beta_j^{(k)} \Big(2\nu_l \delta_{l,k} - \frac{\partial q_2^{(k)}}{\partial \nu_l} \Big). \tag{1.107}$$

In a similar manner, since we obtain

$$\frac{\partial q_2^{(l)}}{\partial \lambda_l}\Big|_{h=\bar{h}} = Q_3^{(l)}, \qquad \frac{\partial q_2^{(k)}}{\partial \lambda_l} = -\frac{\nu_l}{(\lambda_k-\lambda_l)^2} + \frac{3}{2(\lambda_k-\lambda_l)^3} \quad (k \neq l) \tag{1.108}$$

from the definition of $q_2^{(l)}$, we find that (1.103) becomes

$$\frac{\partial \nu_l}{\partial t_j} = \sum_{\substack{k=1\\k\neq l}}^{g} \beta_j^{(k)} \frac{\partial q_2^{(k)}}{\partial \lambda_l} + \beta_j^{(l)} \frac{\partial q_2^{(l)}}{\partial \lambda_l} = \sum_{k=1}^{g} \beta_j^{(k)} \frac{\partial q_2^{(k)}}{\partial \lambda_l}. \tag{1.109}$$

It then follows from

$$q_2^{(k)}|_{h=\bar{h}} = (\nu_k)^2 \tag{1.110}$$

(cf. (1.60) and (1.105)) and the chain-rule of differentiations that

$$\frac{\partial q_2^{(k)}}{\partial \lambda_l}\Big|_{h=\bar{h}} = -\sum_{p=1}^{g} \frac{\partial q_2^{(k)}}{\partial h_p}\Big|_{h=\bar{h}} \frac{\partial \bar{h}_p}{\partial \lambda_l}, \tag{1.111}$$

$$2\nu_l\delta_{l,k} - \frac{\partial q_2^{(k)}}{\partial \nu_l}\Big|_{h=\bar{h}} = \sum_{p=1}^{g} \frac{\partial q_2^{(k)}}{\partial h_p}\Big|_{h=\bar{h}} \frac{\partial \bar{h}_p}{\partial \nu_l}. \tag{1.112}$$

Hence the relation $\dfrac{\partial q_2^{(k)}}{\partial h_p} = (\lambda_k)^{g-p}$ together with (1.99) entails the following:

$$\frac{\partial \lambda_l}{\partial t_j} = \sum_{p,k=1}^{g} \beta_j^{(k)} \frac{\partial q_2^{(k)}}{\partial h_p}\Big|_{h=\bar{h}} \frac{\partial \bar{h}_p}{\partial \nu_l} = \sum_{p,k=1}^{g} (\lambda_k)^{g-p} \beta_j^{(k)} \frac{\partial \bar{h}_p}{\partial \nu_l} = \sum_{p=1}^{g} a_{j,g-p} \frac{\partial \bar{h}_p}{\partial \nu_l}, \tag{1.113}$$

$$\frac{\partial \nu_l}{\partial t_j} = -\sum_{p,k=1}^{g} \beta_j^{(k)} \frac{\partial q_2^{(k)}}{\partial h_p}\Big|_{h=\bar{h}} \frac{\partial \bar{h}_p}{\partial \lambda_l} = -\sum_{p,k=1}^{g} (\lambda_k)^{g-p} \beta_j^{(k)} \frac{\partial \bar{h}_p}{\partial \lambda_l} = -\sum_{p=1}^{g} a_{j,i} \frac{\partial \bar{h}_p}{\partial \lambda_l}. \tag{1.114}$$

Thus the proof of Theorem 1.11 completes. □

§ 2. (P_{34})-Hierarchy

§ 2.1. (P_{34})-Hierarchy and Its Equivalent Form

It is known that the second Painlevé equation has an equivalent form, which is called P_{34} (cf. [I]). In [CJP], Clarkson, Joshi and Pickering discussed its higher order version, which is referred to as (P_{34})-hierarchy, and construct the Bäcklund transformations between each member of (P_{34})-hierarchy and that of $(P_{\text{II-1}})$-hierarchy. Here we follow [CJP] for the formulation of this hierarchy:

Definition 2.1 ((P_{34})-Hierarchy with a Large Parameter η)**.** We set:

$$(2.1)\quad (P_{34})_m: \quad 2\eta^{-2}(\mathcal{F}_m + 2\gamma t)\frac{d^2\mathcal{F}_m}{dt^2} - \eta^{-2}\Big(\frac{d\mathcal{F}_m}{dt} + 2\gamma\Big)^2 + 4u(\mathcal{F}_m + 2\gamma t)^2 = -\kappa^2,$$

where $\mathcal{F}_m$ is given in (1.11), and γ $(\neq 0)$, κ and $\{c_j\}_{j=0}^m$ are constants with $c_0 = 1$.

See Appendix A for the concrete forms of $(P_{34})_1$ and $(P_{34})_2$ with a large parameter.

Remark. Without loss of generality, we can choose $c_m = 0$ by the translation of t, and fix γ to an arbitrary nonzero constant by the scalings of u and t. Thus $(P_{34})_m$ essentially contains m constants.

Next we introduce the $(\widetilde{P}_{34})$-hierarchy by the following:

Definition 2.2 ($(\widetilde{P}_{34})$-Hierarchy with a Large Parameter η)**.** We set:

$$(2.2)\qquad (\widetilde{P}_{34})_m: \quad \begin{cases} \eta^{-1}\dfrac{du_j}{dt} = 2v_j \quad (j = 1, 2, \cdots, m), & \text{(2.2.a)} \\ \eta^{-1}\dfrac{dv_j}{dt} = 2(u_1u_j + u_{j+1} + w_j) \quad (j = 1, 2, \cdots, m), & \text{(2.2.b)} \\ \quad u_{m+1} = -w_m + \widetilde{c}_0 u_m - \widetilde{\gamma} t(u_1 + \widetilde{c}_0) & \\ \qquad\qquad + \dfrac{(v_m - \eta^{-1}\widetilde{\gamma}/2)^2 - \widetilde{\kappa}^2}{2(u_m - \widetilde{\gamma} t)}, & \text{(2.2.c)} \end{cases}$$

where $\{w_j\}$ is a polynomial of $\{u_l, v_l\}$ defined by (1.8), and $\widetilde{\gamma}$ $(\neq 0)$, $\widetilde{\kappa}$ and $\{\widetilde{c}_j\}_{j=0}^m$ are constants.

Remark. Without loss of generality, we can choose $\widetilde{c}_{m-1} = \widetilde{c}_m = 0$ and fix $\widetilde{\gamma}$ to an arbitrary nonzero constant (cf. Remark after Definition 2.1 and (2.3) below).

We note a similarity of the form of $(\widetilde{P}_{\text{I}})_m$ and $(\widetilde{P}_{34})_m$. In fact the difference appears only at (1.7.c) and (2.2.c) ((1.7.a) and (1.7.b) are exactly the same with (2.2.a) and (2.2.b)).

The equivalence of $(P_{34})_m$ and $(\widetilde{P}_{34})_m$ is given through the following correspondence:

Theorem 2.3. *If u is a solution of $(P_{34})_m$, then $\{u_j, v_j\}$ defined by* (1.9) *satisfies $(\widetilde{P}_{34})_m$ whose constants are chosen as follows*:

$$(2.3)\qquad \widetilde{\gamma} = 4^{m-1}\gamma, \quad \widetilde{\kappa} = 2^{-m}\kappa, \quad \widetilde{c}_n = 2^{-2n-3}\sum_{k=0}^{n+1} c_{n-k+1}c_k.$$

Conversely if $\{u_j, v_j\}$ *satisfies* $(\widetilde{P}_{34})_m$ *with* (2.3), $u = -2(u_1 + \widetilde{c}_0)$ (*which follows from* (1.9) *with* $j = 1$, *i.e.* $u_1 = -\mathcal{F}_1/2$ *and* $c_1 = 4\widetilde{c}_0$) *is a solution of* (P_{34}).

The purpose of this subsection is to prove Theorem 2.3. After preparing several lemmas (Lemmas 2.4 through 2.6) we give its proof at the end of this subsection. First of all we determine the recursive relation by which $\mathcal{F}_n$ is determined:

Lemma 2.4. $\{\mathcal{F}_n\}$ *defined by* (1.11) *satisfies the following*:

$$\tag{2.4} \mathcal{F}_{n+1} = -\sum_{j=0}^{n-1} \mathcal{F}_{n-j}\mathcal{F}_{j+1} + 4u \sum_{j=0}^{n} \mathcal{F}_{n-j}\mathcal{F}_j + 2\eta^{-2} \sum_{j=0}^{n} \mathcal{F}_{n-j} \frac{d^2\mathcal{F}_j}{dt^2} - \eta^{-2} \sum_{j=0}^{n} \frac{d\mathcal{F}_{n-j}}{dt}\frac{d\mathcal{F}_j}{dt} + \frac{1}{4}\sum_{j=0}^{n+1} c_{n-j+1}c_j.$$

(2.4) with $\mathcal{F}_0 = 1/2$ enables us to determine $\mathcal{F}_n$ for $n \geq 1$ recursively. This Lemma can be proved by (1.11) and (1.2). Since the proof is straightforward, we omit its details here.

We then transform (2.4) to an equivalent form by using

$$\tag{2.5} \mathcal{G}_n = \eta^{-1}\frac{d\mathcal{F}_n}{dt},$$

$$\tag{2.6} \mathcal{W}_n = \eta^{-1}\frac{d\mathcal{G}_n}{dt} + 2\mathcal{F}_1\mathcal{F}_n - \mathcal{F}_{n+1}.$$

Lemma 2.5. *If* $\{\mathcal{F}_n\}$ *satisfies* (2.4), *then a sequence* $\{\mathcal{F}_n, \mathcal{G}_n, \mathcal{W}_n\}$ *satisfies the following recursion relation for* $n \geq 1$:

$$\tag{2.7} \mathcal{W}_n = -2\sum_{j=1}^{n-1} \mathcal{F}_{n-j}\mathcal{W}_j - \sum_{j=0}^{n-1} \mathcal{F}_{n-j}\mathcal{F}_{j+1} + \sum_{j=1}^{n-1} \mathcal{G}_{n-j}\mathcal{G}_j + 2c_1\Big(\mathcal{F}_n + \sum_{j=1}^{n-1} \mathcal{F}_{n-j}\mathcal{F}_j\Big) - \frac{1}{4}\sum_{j=0}^{n+1} c_{n-j+1}c_j.$$

Conversely if $\mathcal{F}_1$ *is given, then* (2.5), (2.6) *and* (2.7) *determine* $\{\mathcal{F}_n, \mathcal{G}_n, \mathcal{W}_n\}_{n\geq 1}$ *uniquely and recursively, and* $\{\mathcal{F}_n\}$ *thus obtained satisfies* (2.4) *with* $u = \mathcal{F}_1 - c_0/2$.

Proof. We first assume $\{\mathcal{F}_n\}$ satisfies (2.4). It then follows from (2.5) and (2.6) that

$$\tag{2.8} \eta^{-1}\frac{d\mathcal{F}_n}{dt} = \mathcal{G}_n,$$

$$\tag{2.9} \eta^{-2}\frac{d^2\mathcal{F}_n}{dt^2} = -2\mathcal{F}_1\mathcal{F}_n + \mathcal{W}_n + \mathcal{F}_{n+1}.$$

By (2.8) and (2.9) we can eliminate $\dfrac{d\mathcal{F}_n}{dt}$ and $\dfrac{d^2\mathcal{F}_n}{dt^2}$ in (2.4) to obtain

$$
\begin{aligned}
(2.10)\quad \mathcal{F}_{n+1} = & -\sum_{j=0}^{n-1}\mathcal{F}_{n-j}\mathcal{F}_{j+1} + 4u\sum_{j=0}^{n}\mathcal{F}_{n-j}\mathcal{F}_j \\
& + 2\sum_{j=0}^{n}\mathcal{F}_{n-j}(-2\mathcal{F}_1\mathcal{F}_n + \mathcal{W}_n + \mathcal{F}_{n+1}) - \sum_{j=0}^{n}\mathcal{G}_{n-j}\mathcal{G}_j + \frac{1}{4}\sum_{j=0}^{n+1}c_{n-j+1}c_j \\
= & \sum_{j=0}^{n-1}\mathcal{F}_{n-j}\mathcal{F}_{j+1} + 2\mathcal{F}_0\mathcal{F}_{n+1} + 2\sum_{j=0}^{n}\mathcal{F}_{n-j}\mathcal{W}_j - \sum_{j=0}^{n}\mathcal{G}_{n-j}\mathcal{G}_j \\
& + 2c_1\sum_{j=0}^{n}\mathcal{F}_{n-j}\mathcal{F}_j + \frac{1}{4}\sum_{j=0}^{n+1}c_{n-j+1}c_j.
\end{aligned}
$$

Since $\mathcal{F}_0 = 1/2$ and $\mathcal{G}_0 = \mathcal{W}_0 = 0$, we obtain (2.7).

To show the converse, we first observe that $\{\mathcal{F}_n, \mathcal{G}_n, \mathcal{W}_n\}$ is uniquely determined. If $\mathcal{F}_1$ is given, then $\mathcal{G}_1$ and $\mathcal{W}_1$ are determined from (2.5) and (2.7). Next we assume $\{\mathcal{F}_j, \mathcal{G}_j, \mathcal{W}_j\}_{j=1}^{n}$ is given. Since (2.6) implies

$$
(2.11)\qquad \mathcal{F}_{n+1} = \eta^{-1}\frac{d\mathcal{G}_n}{dt} + 2\mathcal{F}_1\mathcal{F}_n - \mathcal{W}_n,
$$

$\{\mathcal{F}_j, \mathcal{G}_j, \mathcal{W}_j\}_{j=1}^{n}$ uniquely determines $\mathcal{F}_{n+1}$. Then $\mathcal{G}_{n+1}$ is determined by (2.5), and $\mathcal{W}_{n+1}$ is determined by (2.7). Thus $\{\mathcal{F}_n, \mathcal{G}_n, \mathcal{W}_n\}$ is uniquely and recursively determined. Finally, by substituting (2.5) and (2.6) into (2.7), we can confirm that $\{\mathcal{F}_n\}$ satisfies (2.4). □

In order to rewrite (2.1) in terms of $\{\mathcal{F}_n, \mathcal{G}_n, \mathcal{W}_n\}$, we introduce the following quantity I_m:

$$
(2.12)\qquad I_m := 2\eta^{-2}(\mathcal{F}_m + 2\gamma t)\frac{d^2\mathcal{F}_m}{dt^2} - \eta^{-2}\Big(\frac{d\mathcal{F}_m}{dt} + 2\gamma\Big)^2 + 4u(\mathcal{F}_m + 2\gamma t)^2 + \kappa^2.
$$

Lemma 2.6. *We obtain*

$$
\begin{aligned}
(2.13)\quad I_m = 2(\mathcal{F}_m + 2\gamma t)\Big[\mathcal{W}_m - c_1\mathcal{F}_m + \mathcal{F}_{m+1} + 4\gamma t\big(\mathcal{F}_1 - \frac{1}{2}c_1\big)\Big] \\
- (\mathcal{G}_m + 2\gamma)^2 + \kappa^2
\end{aligned}
$$

to find that $(P_{34})_m$ is equivalent to

$$
(2.14)\qquad \mathcal{F}_{m+1} = -\mathcal{W}_m + c_1\mathcal{F}_m - 4\gamma t\big(\mathcal{F}_1 - \frac{1}{2}c_1\big) + \frac{(\mathcal{G}_m + 2\gamma)^2 - \kappa^2}{2(\mathcal{F}_m + 2\gamma t)}.
$$

Proof. Since

$$(2.15)\quad I_m = 2(\mathcal{F}_m + 2\gamma t)\Big[\eta^{-2}\frac{d^2\mathcal{F}_m}{dt^2} + 2u(\mathcal{F}_m + 2\gamma t)\Big] - \eta^{-2}\Big(\frac{d\mathcal{F}_m}{dt} + 2\gamma\Big)^2 + \kappa^2,$$

we obtain (2.13) by eliminating $\dfrac{d\mathcal{F}_m}{dt}$ and $\dfrac{d^2\mathcal{F}_m}{dt^2}$ in (2.15) by (2.8) and (2.9). □

We now arrive at the following proposition that relates a solution of $(P_{34})_m$ with the structure of $\{\mathcal{F}_n, \mathcal{G}_n, \mathcal{W}_n\}$:

Proposition 2.7. *If u is a solution of $(P_{34})_m$, then $\mathcal{F}_n$ defined by* (1.11)*, $\mathcal{G}_n$ defined by* (2.5) *and $\mathcal{W}_n$ defined by* (2.6) *satisfy* (2.7) *and* (2.14)*.*

Conversely if $\{\mathcal{F}_n\}_{n=1}^{m+1}$, $\{\mathcal{G}_n\}_{n=1}^{m}$ and $\{\mathcal{W}_n\}_{n=1}^{m}$ are given so that (2.5)*,* (2.6)*,* (2.7) *and* (2.14) *are satisfied, then $u = \mathcal{F}_1 - c_0/2$ satisfies $(P_{34})_m$.*

Proof. We first assume that u is a solution of $(P_{34})_m$. Since $\mathcal{F}_n$ satisfies (2.4), it then follows from Lemma 2.5 that $\{\mathcal{F}_n\}$, $\{\mathcal{G}_n\}$ and $\{\mathcal{W}_n\}$ satisfy (2.7). They also satisfy (2.14) by Lemma 2.6.

Conversely, if we assume (2.5) through (2.7) and (2.14), then Lemma 2.5 implies that $\{\mathcal{F}_n\}$ satisfies (2.4) with $u = \mathcal{F}_1 - c_0/2$. Since it follows from Lemma 2.6 that (2.14) implies $I_m = 0$, u is a solution of $(P_{34})_m$. □

Proof of Theorem 2.3. Let us introduce $\{u_j, v_j, w_j\}$ by the following:

$$(2.16)\qquad u_j = -2^{-2j+1}\mathcal{F}_j, \quad v_j = -2^{-2j}\mathcal{G}_j \quad w_j = -2^{-2j-1}\mathcal{W}_j.$$

We then find that (2.5) and (2.6) are coincident with (2.2.a) and (2.2.b) respectively, (2.7) reads as (1.8), and (2.14) becomes (2.2.c). Thus Theorem 2.3 follows from Proposition 2.7. □

§ 2.2. Lax Pairs of (P_{34})-Hierarchy and $(\widetilde{P}_{34})$-Hierarchy

In this subsection we derive the Lax pair of $(\widetilde{P}_{34})_m$ from that of $(P_{34})_m$. The Lax pair of the (P_{34})-hierarchy is given in [GP]. Let us first recall its explicit form with the addition of the large parameter.

Definition 2.8 (Lax Pair of $(P_{34})_m$)**.** We set:

$$(2.17)\qquad (L_{34})_m: \quad 4\eta^{-1}\gamma x\frac{\partial\vec{\psi}}{\partial x} = A\vec{\psi}, \quad \eta^{-1}\frac{\partial\vec{\psi}}{\partial t} = B\vec{\psi}$$

with

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & -A_{11} \end{pmatrix} = A_0 + \frac{1}{4(\mathcal{F}_m + 2\gamma t)} \begin{pmatrix} 0 & 0 \\ I_m & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ x-u & 0 \end{pmatrix}, \tag{2.18}$$

$$A_0 = \frac{1}{2} \begin{pmatrix} -\eta^{-1}(\frac{\partial \mathcal{F}}{\partial t} + 2\gamma) & 2(\mathcal{F} + 2\gamma t) \\ -\eta^{-2}\frac{\partial^2 \mathcal{F}}{\partial t^2} + 2(x-u)(\mathcal{F} + 2\gamma t) & \eta^{-1}(\frac{\partial \mathcal{F}}{\partial t} + 2\gamma) \end{pmatrix}, \tag{2.19}$$

where

$$\mathcal{F} = \sum_{j=0}^{m} (4x)^{m-j} \mathcal{F}_j \tag{2.20}$$

and I_m is given in (2.12).

We note that the Lax pair (2.17) is slightly different from that given in [GP]; the second term in A is missing in [GP]. Without this term, the compatibility condition gives only $\dfrac{dI_m}{dt} = 0$, not $I_m = 0$, where I_m is defined by (2.12). This is the reason why we add the second term in A. The addition of this term is obtained by using the idea given in [GJP2]. A straightforward computation gives

$$\eta^{-1}\frac{\partial \mathcal{A}}{\partial t} - 4\eta^{-1}\gamma x \frac{\partial B}{\partial x} + [A, B] = \frac{1}{4(\mathcal{F}_m + 2\gamma t)} \begin{pmatrix} -I_m & 0 \\ -\eta^{-1}\dfrac{\partial_t \mathcal{F}_m + 2\gamma}{\mathcal{F}_m + 2\gamma t} I_m & I_m \end{pmatrix}. \tag{2.21}$$

Hence the compatibility condition of (2.17) coincides with $(P_{34})_m$.

We derive a Lax pair of $(\widetilde{P}_{34})_m$ from that given in (2.17) by rewriting each component of A and B using(1.9) and (2.16). First we note that

$$\mathcal{F} = \sum_{j=0}^{m} (4x)^{m-j} \mathcal{F}_j = (4x)^m \mathcal{F}_0 + \sum_{j=1}^{m} (4x)^{m-j} \mathcal{F}_j = 2^{2m-1} U(x), \tag{2.22}$$

$$\eta^{-1}\frac{\partial \mathcal{F}}{\partial t} = \sum_{j=1}^{m} (4x)^{m-j} \cdot \eta^{-1}\frac{d\mathcal{F}_j}{dt} = \sum_{j=1}^{m} (4x)^{m-j} \mathcal{G}_j = -2^{2m} V(x), \tag{2.23}$$

where

$$U(x) = x^m - \sum_{j=1}^{m} u_j x^{m-j}, \quad V(x) = \sum_{j=1}^{m} v_j x^{m-j}. \tag{2.24}$$

Hence we obtain

$$\begin{aligned} A_{11} &= -\frac{1}{2}\eta^{-1}(\frac{\partial \mathcal{F}}{\partial t} + 2\gamma) = 2^{2m-1}[V(x) - \frac{1}{2}\eta^{-1}\widetilde{\gamma}], \\ A_{12} &= \mathcal{F} + 2\gamma t = 2^{2m-1}[U(x) + \widetilde{\gamma} t]. \end{aligned}$$

It also follows from (2.13) that

$$A_{21} = -\frac{1}{2}\eta^{-2}\frac{\partial^2\mathcal{F}}{\partial t^2} + (x-u)(\mathcal{F}+2\gamma t) + \frac{1}{2}[\mathcal{W}_m - c_1\mathcal{F}_m + \mathcal{F}_{m+1} + 4\gamma tu] - \frac{(\mathcal{G}_m+2\gamma)^2-\kappa^2}{4(\mathcal{F}_m+2\gamma t)}. \tag{2.25}$$

To rewrite A_{21} further we use the following:

Lemma 2.9. *If u satisfies $(P_{34})_m$, then*

$$\mathcal{W} = \frac{1}{2}(4x)^{m+1} - (4x-2\mathcal{F}_1)\mathcal{F} + \eta^{-1}\frac{\partial\mathcal{G}}{\partial t} - \mathcal{F}_{m+1}, \tag{2.26}$$

where $\mathcal{F}$ is given in (2.20) *and*

$$\mathcal{G} = \sum_{j=1}^{m}(4x)^{m-j}\mathcal{G}_j, \quad \mathcal{W} = \sum_{j=1}^{m}(4x)^{m-j}\mathcal{W}_j. \tag{2.27}$$

Proof. Multiplying $(4x)^{m-n}$ with (2.6), and adding it for $n = 1, 2, \dots, m$, we obtain

$$\begin{aligned} \sum_{n=1}^{m}(4x)^{m-n}\mathcal{W}_n &= \eta^{-1}\frac{\partial\mathcal{G}}{\partial t} + 2\mathcal{F}_1\sum_{n=1}^{m}(4x)^{m-n}\mathcal{F}_n - \sum_{n=1}^{m}(4x)^{m-n}\mathcal{F}_{n+1} \\ &= \eta^{-1}\frac{\partial\mathcal{G}}{\partial t} + 2\mathcal{F}_1(\mathcal{F}-\mathcal{F}_0(4x)^m) \\ &\qquad - 4x(\mathcal{F}-\mathcal{F}_0(4x)^m - \mathcal{F}_1(4x)^{m-1}) - \mathcal{F}_{m+1} \\ &= \eta^{-1}\frac{\partial\mathcal{G}}{\partial t} + \frac{1}{2}(4x)^{m+1} - (4x-2\mathcal{F}_1)\mathcal{F} - \mathcal{F}_{m+1}. \end{aligned} \tag{2.28}$$

This completes the proof. □

As $\dfrac{\partial^2\mathcal{F}}{\partial t^2} = \eta\dfrac{\partial\mathcal{G}}{\partial t}$, it follows from (2.26) that the right-hand side of (2.25) becomes

$$\begin{aligned} &-\frac{1}{2}\Big[\mathcal{W}(x) - \frac{1}{2}(4x)^{m+1} + (4x-\mathcal{F}_1)\mathcal{F} + \mathcal{F}_{m+1}\Big] + (x-u)(\mathcal{F}+2\gamma t) \\ &\qquad + \frac{1}{2}\Big[\mathcal{W}_m - c_1\mathcal{F}_m + \mathcal{F}_{m+1} + 4\gamma tu\Big] - \frac{(\mathcal{G}_m+2\gamma)^2-\kappa^2}{4(\mathcal{F}_m+2\gamma t)} \\ &= \frac{1}{2}\Big[\frac{1}{2}(4x)^{m+1} - \mathcal{W}(x) - (2x-c_1)\mathcal{F} + 4\gamma t(x-u)\Big] \\ &\qquad + \frac{1}{2}\mathcal{W}_m + 2\gamma tu - \frac{(\mathcal{G}_m+2\gamma)^2-\kappa^2}{4(\mathcal{F}_m+2\gamma t)} - \frac{1}{2}c_1\mathcal{F}_m. \end{aligned} \tag{2.29}$$

We then use (2.14) to obtain

$$A_{21} = 4^m x^{m+1} - (x - \frac{1}{2}c_1)\mathcal{F} - \frac{1}{2}\mathcal{W} + 2\gamma t(x-u) - \frac{1}{2}F_{m+1}. \tag{2.30}$$

Hence it follows from (2.22), (2.23) and

$$\mathcal{W} = -2^{2m+1}W(x) \quad \text{with} \quad W(x) = \sum_{j=1}^{m} w_j x^{m-j} \tag{2.31}$$

that

$$\begin{aligned} A_{21} &= 2^{2m-1}\big[2x^{m+1} - (x - \frac{1}{2}c_1)U + 2W + \widetilde{\gamma}t(x + 2u_1 + \frac{1}{2}c_1) + 2u_{m+1}\big] \\ &= 2^{2m-1}\big[2x^{m+1} - (x - 2\widetilde{c}_0)U + 2W + \widetilde{\gamma}t(x + 2u_1 + 2\widetilde{c}_0) + 2u_{m+1}\big]. \end{aligned}$$

Here we used the relation $c_1 = 4\widetilde{c}_0$, which follows from (1.10). Summing up the results obtained so far, we find that A can be written as follows:

$$A = 2^{2m-1}\begin{pmatrix} V(x) - \frac{1}{2}\eta^{-1}\widetilde{\gamma} & U(x) + 4\widetilde{\gamma}t \\ \begin{array}{l} 2x^{m+1} - (x - 2\widetilde{c}_0)U + 2W \\ \quad + \widetilde{\gamma}t(x + 2u_1 + 2\widetilde{c}_0) + 2u_{m+1} \end{array} & -(V(x) - \frac{1}{2}\eta^{-1}\widetilde{\gamma}) \end{pmatrix}. \tag{2.32}$$

After the transformation of the unknown vector of (2.17)

$$\vec{\psi} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \overrightarrow{\psi}^{\dagger}, \tag{2.33}$$

we obtain a Lax pair of $(\widetilde{P}_{34})_m$.

Definition 2.10 (Lax Pair of $(\widetilde{P}_{34})_m$)**.** We set

$$(L_{34})_m : \quad \eta^{-1}\widetilde{\gamma}x\frac{\partial\vec{\psi}}{\partial x} = A\vec{\psi}, \quad \eta^{-1}\frac{\partial\vec{\psi}}{\partial t} = B\vec{\psi} \quad \text{with} \quad \overrightarrow{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.34}$$

where

$$A = \begin{pmatrix} \frac{1}{2}V(x) - \frac{1}{4}\eta^{-1}\widetilde{\gamma} & U + 4\widetilde{\gamma}t \\ \begin{array}{l} \frac{1}{4}\{2x^{m+1} - (x - 2\widetilde{c}_0)U + 2W \\ \quad +\widetilde{\gamma}t\,(x + 2u_1 + 2\widetilde{c}_0) + 2u_{m+1}\} \end{array} & -\{\frac{1}{2}V(x) - \frac{1}{4}\eta^{-1}\widetilde{\gamma}\} \end{pmatrix}, \tag{2.35}$$

$$B = \begin{pmatrix} 0 & 2 \\ \frac{1}{2}x + u_1 + \widetilde{c}_0 & 0 \end{pmatrix}. \tag{2.36}$$

Then we obtain the following theorem:

Theorem 2.11. *The compatibility condition of* (2.34) *is expressed by* $(\widetilde{P}_{34})_m$.

Although Theorem 2.11 is evident from our discussions given above, we can also confirm it by direct computations. To show it, let us define Δ_j $(j = 1, 2, 3)$ by

$$\eta^{-1}\frac{\partial A}{\partial t} - \eta^{-1}\widetilde{\gamma}x\frac{\partial B}{\partial x} + AB - BA = \begin{pmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & -\Delta_1 \end{pmatrix}.$$

Then we obtain

Proposition 2.12. *We have*

$$\Delta_1 = \frac{1}{2}\sum_{j=1}^{m}\{\eta^{-1}\frac{dv_j}{dt} - 2u_1u_j - 2u_{j+1} - 2w_j\}x^{m-j}, \tag{2.37}$$

$$\Delta_2 = \sum_{j=1}^{m}\{-\eta^{-1}\frac{du_j}{dt} + 2v_j\}x^{m-j}, \tag{2.38}$$

$$\begin{aligned}\Delta_3 = &-\frac{1}{4}(x + 2u_1 + 2\widetilde{c}_0)\Delta_2 + \frac{1}{2}\eta^{-1}\Big[(u_1 + 2\widetilde{c}_0)\frac{\partial U}{\partial t} + \frac{\partial W}{\partial t}\Big] \\ &+ \frac{1}{2}\eta^{-1}\frac{d}{dt}[u_{m+1} + \widetilde{\gamma}t(u_1 + \widetilde{c}_0)] + \frac{1}{2}\widetilde{\gamma}(u_1 + \widetilde{c}_0).\end{aligned} \tag{2.39}$$

Proposition 2.13. *If $\{u_j, v_j\}$ satisfies $(\widetilde{P}_{34})_m$, we obtain*

$$\text{(i)}\ \frac{dw_j}{dt} = (u_1 + 2\widetilde{c}_0)\frac{du_j}{dt}, \quad \text{(ii)}\ \eta^{-1}\frac{d}{dt}[u_{m+1} + \widetilde{\gamma}t(u_1 + \widetilde{c}_0)] + \widetilde{\gamma}(u_1 + \widetilde{c}_0) = 0.$$

Theorem 2.11 readily follows from Proposition 2.12 and Proposition 2.13: in fact, if (2.34) is compatible, we obtain $\Delta_1 = \Delta_2 = \Delta_3 = 0$. As is easily confirmed from Proposition 2.12, $\Delta_1 = \Delta_2 = 0$ implies that $\{u_j, v_j\}$ satisfies $(\widetilde{P}_{34})_m$. Then, from Proposition 2.13, we obtain $\Delta_3 = 0$. Thus the compatibility condition is equivalent to $(\widetilde{P}_{34})_m$.

Since Proposition 2.12 can be verified by straightforward computations, we omit its proof. The proof of Proposition 2.13 is also computational, but it contains some delicate points. For the sake of the convenience of the reader, we briefly describe how we proceed. The key lemma is the following

Lemma 2.14. *If $\{w_j\}$ is a polynomial of $\{u_j, v_j\}$ defined by* (1.8), *we obtain*

$$\begin{aligned}\frac{dw_n}{dt} - (u_1 + 2\widetilde{c}_0)\frac{du_n}{dt} = &\frac{1}{2}\sum_{k=1}^{n}\frac{du_{n-k}}{dt}\{-\frac{dv_k}{dt} + 2(u_1u_k + u_{k+1} + w_k)\} \\ &+ \frac{1}{2}\sum_{k=1}^{n-1}\{\frac{du_{n-k}}{dt} - 2v_{n-k}\}\frac{dv_k}{dt} \\ &+ \sum_{k=1}^{n-1}\{\frac{dw_{n-k}}{dt} - (u_1 + 2\widetilde{c}_0)\frac{du_{n-k}}{dt}\}u_k.\end{aligned} \tag{2.40}$$

This lemma can be verified by differentiating (1.8) with respect to t. Using this lemma, we obtain Proposition 2.13 (i) by the induction on j. The proof of Proposition 2.13 (ii) proceeds as follows. Since u_{m+1} is given by (2.2.c), we obtain

$$
\begin{aligned}
\frac{d}{dt}[u_{m+1}+\tilde{\gamma}t(u_1+\tilde{c}_0)] &= \frac{d}{dt}\Big[-w_m+\tilde{c}_0u_m+\frac{(v_m-\frac{1}{2}\eta^{-1}\tilde{\gamma})^2-\tilde{\kappa}^2}{2(u_m-\tilde{\gamma}t)}\Big] \\
&= -\frac{dw_m}{dt}+\tilde{c}_0\frac{du_m}{dt}+\frac{v_m-\frac{1}{2}\eta^{-1}\tilde{\gamma}}{u_m-\tilde{\gamma}t}\frac{dv_m}{dt} \\
&\quad -\frac{(v_m-\frac{1}{2}\eta^{-1}\tilde{\gamma})^2-\tilde{\kappa}^2}{2(u_m-\tilde{\gamma}t)^2}\Big\{\frac{du_m}{dt}-\frac{1}{2}\eta^{-1}\tilde{\gamma}\Big\}.
\end{aligned}
\tag{2.41}
$$

We then use (2.2.a) and (2.2.b) with $j=m$ and Proposition 2.13 (i) with $j=m$ on the left-hand side of (2.41):

$$
\begin{aligned}
&\eta^{-1}\frac{d}{dt}[u_{m+1}+\tilde{\gamma}t(u_1+\tilde{c}_0)] \\
&\quad= -2(u_1+2\tilde{c}_0)v_m+2\tilde{c}_0v_m+2\frac{v_m-\frac{1}{2}\eta^{-1}\tilde{\gamma}}{u_m-\tilde{\gamma}t}(u_1u_m+u_{m+1}+w_m) \\
&\quad\quad -\frac{(v_m-\frac{1}{2}\eta^{-1}\tilde{\gamma})^2-\tilde{\kappa}^2}{(u_m-2\tilde{\gamma}t)^2}(v_m-\frac{1}{2}\eta^{-1}\tilde{\gamma}).
\end{aligned}
\tag{2.42}
$$

Since it follows from (2.2.c) that

$$
\begin{aligned}
u_1u_m+u_{m+1}+w_m &= u_1u_m+\tilde{c}_0u_m-\tilde{\gamma}t(u_1+\tilde{c}_0)+\frac{(v_m-\frac{1}{2}\eta^{-1}\tilde{\gamma})^2-\tilde{\kappa}^2}{2(u_m-\tilde{\gamma}t)} \\
&= (u_1+\tilde{c}_0)(u_m-\tilde{\gamma}t)+\frac{(v_m-\frac{1}{2}\eta^{-1}\tilde{\gamma})^2-\tilde{\kappa}^2}{2(u_m-\tilde{\gamma}t)}
\end{aligned}
\tag{2.43}
$$

we then conclude that

$$
\begin{aligned}
&\eta^{-1}\frac{d}{dt}[u_{m+1}+\tilde{\gamma}t(u_1+\tilde{c}_0)] \\
&\quad= -2(u_1+2\tilde{c}_0)v_m+2\tilde{c}_0v_m \\
&\quad\quad +2\frac{v_m-\frac{1}{2}\eta^{-1}\tilde{\gamma}}{u_m-\tilde{\gamma}t}\Big[(u_1+\tilde{c}_0)(u_m-\tilde{\gamma}t)+\frac{(v_m-\frac{1}{2}\eta^{-1}\tilde{\gamma})^2-\tilde{\kappa}^2}{2(u_m-\tilde{\gamma}t)}\Big] \\
&\quad\quad -\frac{(v_m-\frac{1}{2}\eta^{-1}\tilde{\gamma})^2-\tilde{\kappa}^2}{(u_m-\tilde{\gamma}t)^2}(v_m-\frac{1}{2}\eta^{-1}\tilde{\gamma})
\end{aligned}
\tag{2.44}
$$

$$= -\widetilde{\gamma}(u_1 + \widetilde{c}_0).$$

Thus we have confirmed Proposition 2.13 (ii).

In the next subsection we will consider the relation between $(P_{34})_m$ and the degenerate Garnier systems, as we did in § 1.2 for $(P_{\rm I})_m$. To study the relation we need to derive a system of equations for one unknown functions from the Lax pair (2.34) whose compatibility condition coincides with $(P_{34})_m$. To this purpose, we first transform the unknown function of (2.34) by

$$\overrightarrow{\psi} = x^{\widetilde{\kappa}/(2\widetilde{\gamma})}\overrightarrow{\varphi} \quad \text{with} \quad \overrightarrow{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{2.45}$$

The background of this transformation will be explained in Remark after Lemma 2.15. Then the first component φ_1 solves

$$\frac{\partial^2 \varphi_1}{\partial x^2} + \eta q_1 \frac{\partial \varphi_1}{\partial x} + \eta^2 q_2 \varphi_1 = 0, \tag{2.46}$$

$$\frac{\partial \varphi_1}{\partial t} = \mathcal{C}\frac{\partial \varphi_1}{\partial x} + \eta \mathcal{D}\varphi_1, \tag{2.47}$$

where

$$\begin{aligned} q_1 &= -\frac{1}{\widetilde{\gamma}x}\operatorname{tr}(A - \frac{1}{2}\widetilde{\kappa}) - \eta^{-1}\frac{\widetilde{\gamma}x}{U + \widetilde{\gamma}t}\frac{\partial}{\partial x}\Big(\frac{U + \widetilde{\gamma}t}{\widetilde{\gamma}x}\Big) \\ &= -\frac{1}{\widetilde{\gamma}x}\operatorname{tr}(A - \frac{1}{2}\widetilde{\kappa}) + \eta^{-1}\frac{1}{x} - \eta^{-1}\frac{1}{U + \widetilde{\gamma}t}\frac{\partial U}{\partial x}, \end{aligned} \tag{2.48}$$

$$\begin{aligned} q_2 &= \frac{1}{\widetilde{\gamma}^2 x^2}\det(A - \frac{1}{2}\widetilde{\kappa}) - \eta^{-1}\frac{\partial}{\partial x}\Big(\frac{\frac{1}{2}V - \frac{1}{4}\eta^{-1}\widetilde{\gamma}}{\widetilde{\gamma}x}\Big) \\ &\quad + \eta^{-1}\frac{\frac{1}{2}V - \frac{1}{4}\eta^{-1}\widetilde{\gamma}}{\widetilde{\gamma}x} \cdot \frac{\widetilde{\gamma}x}{U + \widetilde{\gamma}t} \cdot \frac{\partial}{\partial x}\Big(\frac{U + \widetilde{\gamma}t}{\widetilde{\gamma}x}\Big) \\ &= \frac{1}{\gamma^2 x^2}\det(A - \frac{1}{2}\widetilde{\kappa}) - \eta^{-1}\frac{1}{2\widetilde{\gamma}x}\frac{\partial V}{\partial x} + \eta^{-1}\frac{V - \frac{1}{2}\eta^{-1}\widetilde{\gamma}}{2\widetilde{\gamma}x(U + \widetilde{\gamma}t)}\frac{\partial U}{\partial x}, \end{aligned} \tag{2.49}$$

$$\mathcal{C} = \frac{2\widetilde{\gamma}x}{U + \widetilde{\gamma}t}, \quad \mathcal{D} = -\frac{V - \frac{1}{2}\eta^{-1}\widetilde{\gamma}}{U + \widetilde{\gamma}t}. \tag{2.50}$$

Lemma 2.15.

$$\operatorname{tr}(A - \frac{1}{2}\widetilde{\kappa}) = -\widetilde{\kappa}, \tag{2.51}$$

$$\det(A - \frac{1}{2}\widetilde{\kappa}) = -\frac{1}{4}x^{2m+1} - \frac{1}{2}x^m C(x) - \frac{1}{2}\widetilde{\gamma}tx^{m+1} + x\widetilde{R}(x). \tag{2.52}$$

Here $\widetilde{R}$ is a polynomial in x of degree $m-1$.

Remark. Note that $\det(A - \widetilde{\kappa}/2)$ vanishes at $x = 0$. Hence the origin is not a double pole but a simple pole of q_2 defined by (2.49); This is the reason why we first transform the unknown function by (2.45).

Proof of Lemma 2.15. The first assertion (2.51) is self-evident. To prove (2.52) we fist note

$$\begin{aligned}(2.53)\quad & 4\det(A - \frac{1}{2}\widetilde{\kappa}) \\ &= \widetilde{\kappa}^2 - (V - \frac{1}{2}\eta^{-1}\widetilde{\gamma})^2 \\ &\quad - (U + \widetilde{\gamma}t)[2x^{m+1} - (x - 2\widetilde{c}_0)U + 2W + \widetilde{\gamma}t(x + 2u_1 + 2\widetilde{c}_0) + 2u_{m+1}].\end{aligned}$$

Hence we find

$$\begin{aligned}(2.54)\quad 4\det(A - \frac{1}{2}\widetilde{\kappa})\Big|_{x=0} &= \widetilde{\kappa}^2 - (v_m - \frac{1}{2}\eta^{-1}\widetilde{\gamma})^2 \\ &\quad + (u_m - 2\widetilde{\gamma}t)[-2\widetilde{c}_0 u_m + 2w_m + 2\widetilde{\gamma}t(u_1 + \widetilde{c}_0) + 2u_{m+1}].\end{aligned}$$

Then the definition of u_{m+1} given in (2.2.c) entails that the righthand side of (2.54) vanishes. Hence $\det(A - \widetilde{\kappa}/2)$ vanishes at $x = 0$. We next rewrite the righthand side of (2.53) as follows:

$$\begin{aligned}(2.55)\quad & 4\det(A - \frac{1}{2}\widetilde{\kappa}) \\ &= -(V^2 + 2x^{m+1}U - (x - 2\widetilde{c}_0)U^2 + 2UW) \\ &\quad + \widetilde{\kappa}^2 + \eta^{-1}\widetilde{\gamma}V - \frac{1}{4}\eta^{-2}\widetilde{\gamma}^2 - U[\widetilde{\gamma}t(x + 2u_1 + 2\widetilde{c}_0) + 2u_{m+1}] \\ &\quad - \widetilde{\gamma}t[2x^{m+1} - (x - 2\widetilde{c}_0)U + 2W + \widetilde{\gamma}t(x + 2u_1 + 2\widetilde{c}_0) + 2u_{m+1}].\end{aligned}$$

Applying (1.23) to the first line of the right-hand side of (2.55), we obtain

$$(2.56)\qquad 4\det(A - \frac{1}{2}\widetilde{\kappa})\Big|_{x=0} = -(x^{2m+1} + 2x^m C(x) + 2\widetilde{\gamma}tx^{m+1}) + 4x\widetilde{R}(x),$$

where

$$\begin{aligned}(2.57)\qquad 4x\widetilde{R}(x) &= 2R(x) - 2\widetilde{\gamma}t[(u_1 + 2\widetilde{c}_0)U + W] - 2u_{m+1}(U + \widetilde{\gamma}t) \\ &\quad - \widetilde{\gamma}^2t^2(x + 2u_1 + 2\widetilde{c}_0) + \widetilde{\kappa}^2 + \eta^{-1}\widetilde{\kappa}V - \frac{1}{4}\eta^{-2}\widetilde{\gamma}^2.\end{aligned}$$

Since $\det(A - \widetilde{\kappa}/2)$ vanishes at $x = 0$, $\widetilde{R}(x)$ is a polynomial in x, as is required (2.52). □

Further we factorize $U + \widetilde{\gamma}t$ as

$$(2.58)\qquad U + \widetilde{\gamma}t = \prod_{j=1}^{m}(x - b_j(t)),$$

and rewrite q_1 and q_2 using $b_j(t)$. Clearly we find

$$q_1 = -\frac{\widetilde{\kappa}/\widetilde{\gamma} - \eta^{-1}}{x} - \eta^{-1}\sum_{j=1}^{m}\frac{1}{x-b_j}. \tag{2.59}$$

From (2.52) and (2.58), we also obtain

$$\begin{aligned} q_2 = &-\left[\frac{1}{4\widetilde{\gamma}^2}x^{2m-1} + \frac{1}{2\widetilde{\gamma}^2}x^{m-2}C(x) + \frac{1}{2\widetilde{\gamma}}tx^{m-1}\right] \\ &+ \frac{L(x)}{x} + \eta^{-1}\sum_{j=1}^{m}\frac{V(b_j) - \frac{1}{2}\eta^{-1}\widetilde{\gamma}}{2\widetilde{\gamma}x(x-b_j)}, \end{aligned} \tag{2.60}$$

where

$$L(x) = \frac{1}{\widetilde{\gamma}^2}\widetilde{R}(x) - \eta^{-1}\frac{1}{2\widetilde{\gamma}}\frac{\partial V}{\partial x} + \eta^{-1}\sum_{j=1}^{m}\frac{V(x)-V(b_j)}{2\widetilde{\gamma}(x-b_j)} \tag{2.61}$$

is a polynomial in x of order $m-1$. Thus we show the following:

Proposition 2.16. *If we define $\overrightarrow{\varphi}$ by* (2.45) *for the solution $\overrightarrow{\psi}$ of* (2.34)*, the first component φ_1 of $\overrightarrow{\varphi}$ solves* (2.46) *and* (2.47) *with* (2.59), (2.60) *and* (2.50).

By the same argument in [Ko], we also obtain

Theorem 2.17. $(\widetilde{P}_{34})_m$ *is the compatibility condition of* (2.46) *and* (2.47) *with* (2.59), (2.60) *and* (2.50).

§ 2.3. Hamiltonians of $(\widetilde{P}_{34})_m$ and the Degenerate Garnier Systems

In this subsection we will determine the Hamiltonian for $(\widetilde{P}_{34})_m$. Throughout this subsection, we set $\widetilde{c}_m = 0$. To begin with, let us consider the following change of variables and constants:

$$x = \Theta z, \quad b_j = \Theta\lambda_j, \quad \lambda_j\mu_j = \frac{1}{2\widetilde{\gamma}}(V(b_j) - \frac{1}{2}\eta^{-1}\widetilde{\gamma}) \quad (1 \le j \le m), \tag{2.62}$$

$$t_1 = \frac{2}{\widetilde{\gamma}}\Theta^{m+1}t + \frac{1}{2\widetilde{\gamma}^2}\Theta^{m+1}\widetilde{c}_{m-1}, \quad t_k = \frac{1}{2\widetilde{\gamma}^2}\Theta^{m+k-1}\widetilde{c}_{m-k} \quad (2 \le k \le m), \tag{2.63}$$

$$\kappa_0 = \frac{\widetilde{\kappa}}{\widetilde{\gamma}}, \tag{2.64}$$

where Θ is a nonzero constants satisfying $\Theta^{2m+1} = 4\widetilde{\gamma}^2$. Then Eq. (2.46) with (2.59) and (2.60) assume the following form (2.65) with $g = m$:

$$\frac{d^2\psi}{dz^2} + \eta p_1\frac{d\psi}{dz} + \eta^2 p_2\psi = 0, \tag{2.65}$$

where

$$p_1 = -\frac{\kappa_0 - \eta^{-1}}{z} - \eta^{-1}\sum_{j=1}^{g}\frac{1}{z-\lambda_j}, \tag{2.66}$$

$$p_2 = -[z^{2g-1} + \sum_{j=1}^{g} t_j z^{g+j-2} + \sum_{j=1}^{g} h_j z^{g-j-1}] + \eta^{-1}\sum_{j=1}^{g}\frac{\lambda_j\mu_j}{z(z-\lambda_j)}. \tag{2.67}$$

We then consider the monodromy preserving deformation of (2.65) to obtain the degenerate Garnier system.

Equation (2.65) has an irregular singular point at $z=\infty$ whose Poincaré rank is $g+1/2$. The origin is a regular singular point. It also has a regular singular point at $z=\lambda_j$ $(1\le j\le g)$. We assume that these singular points $\{\lambda_j\}_{j=1}^{g}$ are apparent ones. The Riemann scheme of (1.31) is of the following form:

$$\left\{\begin{matrix} z=0 & z=\lambda_1 \cdots z=\lambda_g & \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{z=\infty\,(1/2)} \\ 0 & 0 \quad \cdots \quad 0 & \frac{\eta T_0}{g+3/2} \quad 0 \quad \frac{\eta T_1}{g+1/2} \quad 0 \quad \cdots \quad 0 \quad \frac{\eta T_g}{1/2} \quad \frac{1}{4} \\ \eta\kappa_0 & 2 \quad \cdots \quad 2 & -\frac{\eta T_0}{g+3/2} \quad 0 \quad -\frac{\eta T_1}{g+1/2} \quad 0 \quad \cdots \quad 0 \quad -\frac{\eta T_g}{1/2} \quad \frac{1}{4} \end{matrix}\right\}.$$

Here $\{T_j\}_{j=0}^{g}$ are functions of $\{t_l\}_{l=1}^{g}$ which are recursively defined by

$$T_0 = 1, \quad 2T_{n+1} + \sum_{j=1}^{n} T_j T_{n-j+1} = t_{g-n} \quad (1\le n\le g-1). \tag{2.68}$$

In parallel with Lemma 1.8 we can show that the condition that $\lambda_1,\cdots,\lambda_g$ are apparent singular points is equivalent to $h_i = \overline{h}_i$ for $1\le i\le g$, where $\overline{h}_i$ is a rational functions in $\{\lambda_j,\mu_k,t_l\}$ defined by the following:

$$\begin{aligned} \overline{h}_i = &\sum_{j=1}^{g} N_j N^{i,j}[\lambda_j{\mu_j}^2 - \kappa_0\mu_j - \{{\lambda_j}^{2g} + t_g{\lambda_j}^{2g-1} + \cdots + t_1{\lambda_j}^{g}\}] \\ &- \eta^{-1}\sum_{\substack{j,l=1\\ j\ne l}}^{g}\frac{N_j N^{i,j} + N_l N^{i,l}}{\lambda_j - \lambda_l}\lambda_j\mu_j. \end{aligned} \tag{2.69}$$

The monodromy data of (2.65) is preserved, where $t=(t_1,\cdots,t_g)$ is considered as a deformation parameter, if and only if there exist rational functions $\mathcal{A}_j$, $\mathcal{B}_j$ $(1\le j\le g)$ in z so that (2.65) and

$$\frac{\partial\psi}{\partial t_j} = \mathcal{A}_j\frac{\partial\psi}{\partial z} + \eta\mathcal{B}_j\psi \quad (1\le j\le g) \tag{2.70}$$

are completely integrable. We obtain the following:

Theorem 2.18. *The monodromy data of* (2.65) *with* (2.66) *and* (2.67) *is preserved if and only if* $\{\lambda_j, \mu_k\}$ *satisfies the following completely integrable Hamiltonian system with time variables* $t = (t_1, \cdots, t_g)$:

$$\frac{\partial \lambda_j}{\partial t_k} = \eta \frac{\partial H_k}{\partial \mu_j}, \quad \frac{\partial \mu_j}{\partial t_k} = -\eta \frac{\partial H_k}{\partial \lambda_j} \quad (1 \leq j, k \leq g) \tag{2.71}$$

with

$$H_k = \sum_{p=1}^{g} a_{k,g-p}(t)\overline{h}_p. \tag{2.72}$$

Here $\{a_{j,k}(t)\}$ *is determined by the following recursion relation for each* j:

$$(2g-2k-1)a_{j,k}(t) + \sum_{l=1}^{k}(2g-2k+l-1)t_{g-l+1}a_{j,k-l}(t) = \delta_{k,g-j}. \tag{2.73}$$

We refer to (2.71) as $G(1, g+3/2; g)$.

In the course of the proof of Theorem 2.18, we will find that $\mathcal{A}_j$ in (2.70) has the following form:

$$\mathcal{A}_j = \sum_{k=1}^{g} \frac{z}{z-\lambda_k} \sum_{p=0}^{g-1} N_k N^{g-p,k} a_{j,p}(t). \tag{2.74}$$

By the same argument as in the proof of Theorem 1.10, we obtain the following:

Theorem 2.19. *Let* K *be a rational function of* $\{\lambda_j, \mu_k, t\}$ *defined by*

$$K := 4\Theta^{1/2} H_1 \Big|_{t_1 = 4\Theta^{1/2}t + 2\Theta^{-m}\widetilde{c}_{m-1},\ t_j = 2\Theta^{-(m-j+2)}\widetilde{c}_{m-j}\ (2 \leq j \leq m)}, \tag{2.75}$$

where H_1 *is the Hamiltonian of* $G(1, m+3/2; m)$ *defined by* (2.72) *with* $\kappa_0 = 2\Theta^{-m-1/2}\widetilde{\kappa}$ *and* $\Theta^{2m+1} = 4\widetilde{\gamma}^2$. *Then* $\{u_j, v_j\}$ *is a solution of* $(P_{34})_m$ *if and only if* $\{\lambda_j, \mu_k\}$ *defined by*

$$U + \widetilde{\gamma}t = \prod_{j=1}^{m}(x - \Theta\lambda_j), \quad \lambda_j \mu_j = \Theta^{-m-1/2}V(\Theta\lambda_j) - \eta^{-1} \quad (1 \leq j \leq m) \tag{2.76}$$

is a solution of the following Hamiltonian system:

$$\frac{d\lambda_j}{dt} = \eta \frac{\partial K}{\partial \mu_j}, \quad \frac{d\mu_j}{dt} = -\eta \frac{\partial K}{\partial \lambda_j} \quad (1 \leq j \leq m). \tag{2.77}$$

Remark. As is mentioned in the beginning of this section, $(P_{34})_m$ is equivalent to $(P_{\text{II-1}})_m$. The Hamiltonian structure of $(P_{\text{II-1}})_m$ is given by Mazzocco and Mo in [MM].

§ 2.4. Proof of Theorem 2.18

As in the proof of Theorem 1.9 in § 1.3, we prove Theorem 2.18 after eliminating the first order term of (2.65) by the following change of the unknown functions:

$$\psi = z^{(\kappa_0 - 1)/2} \prod_{j=1}^{m} (z - \lambda_j)^{-1/2} \varphi. \tag{2.78}$$

Then φ solves the following equation:

$$\frac{d^2\varphi}{dz^2} - Q\varphi = 0, \tag{2.79}$$

where

$$Q = z^{2g-1} + \sum_{j=1}^{g} t_j z^{g+j-2} + \sum_{j=1}^{g} \overline{h}_j' z^{g-j-1} + \frac{{\kappa_0}^2 - 1}{4x^2} - \sum_{j=1}^{g} \frac{\nu_j}{z - \lambda_j} + \sum_{j=1}^{g} \frac{3}{4(z - \lambda_j)^2} \tag{2.80}$$

Here we put

$$\overline{h}_i' = \overline{h}_i \quad (1 \le i \le g-1), \quad \overline{h}_g' = \overline{h}_g + \sum_{j=1}^{g} \mu_j - \frac{\kappa_0 - 1}{2} \sum_{j=1}^{g} \frac{1}{\lambda_j}, \tag{2.81}$$

$$\nu_j = \mu_j - \frac{\kappa_0 - 1}{2\lambda_j} - \frac{1}{2} \sum_{\substack{k=1 \\ k \ne j}}^{g} \frac{1}{\lambda_j - \lambda_k} \quad (1 \le j \le g). \tag{2.82}$$

Each $\overline{h}_i'$ should be considered as a function of $\{\lambda_j, \nu_k, t_l\}$ through the relation (2.82). Then the monodromy data of (2.65) is preserved if and only if (2.79) is preserved. This condition is equivalent to the condition that there exist rational functions $\mathcal{A}_1, \cdots, \mathcal{A}_g$ so that (2.79) and the following equations is completely integrable:

$$\frac{\partial \varphi}{\partial t_j} = \mathcal{A}_j \frac{\partial \varphi}{\partial z} - \frac{1}{2} \frac{\partial \mathcal{A}_j}{\partial z} \varphi. \tag{2.83}$$

Proposition 2.20. *The following transformation is canonical*:

$$(\lambda, \mu, H, t) \to (\lambda, \nu, K, t). \tag{2.84}$$

Proof. It follows from (2.82) that

$$d\nu_j = d\mu_j + \frac{\kappa_0 - 1}{2(\lambda_j)^2} d\lambda_j + \frac{1}{2} \sum_{\substack{k=1 \\ k \ne j}}^{g} \frac{d\lambda_j - d\lambda_k}{(\lambda_j - \lambda_k)^2}. \tag{2.85}$$

Hence we obtain

$$\sum_{j=1}^{g} d\nu_j \wedge d\lambda_j = \sum_{j=1}^{g} d\mu_j \wedge d\lambda_j - \frac{1}{2} \sum_{j=1}^{g} \sum_{\substack{k=1 \\ k \ne j}}^{g} \frac{d\lambda_k \wedge d\lambda_j}{(\lambda_j - \lambda_k)^2} = \sum_{j=1}^{g} d\mu_j \wedge d\lambda_j. \tag{2.86}$$

Thus the proof completes if we show $K_j = H_j$ for $1 \le j \le g$ where (2.82) is assumed. Since $a_{j,0} = 0$ holds for $1 \le j \le g-1$, we obtain

$$K_j = \sum_{p=1}^{g} a_{j,g-p}\overline{h}_p' = \sum_{p=1}^{g-1} a_{j,g-p}\overline{h}_p' = \sum_{p=1}^{g-1} a_{j,g-p}\overline{h}_p = H_j. \tag{2.87}$$

For $j = g$, we first obtain

$$\begin{aligned} \overline{h}_g' - \sum_{p=1}^{g} \nu_p & \\ &= \Big\{\overline{h}_g + \sum_{j=1}^{g}\mu_j - \frac{\kappa_0 - 1}{2}\sum_{j=1}^{g}\frac{1}{\lambda_j}\Big\} - \sum_{p=1}^{g}\Big\{\mu_p - \frac{\kappa_0-1}{2\lambda_p} - \frac{1}{2}\sum_{k\neq p}\frac{1}{\lambda_p - \lambda_k}\Big\} \\ &= \overline{h}_g + \frac{1}{2}\sum_{p=1}^{g}\sum_{k\neq p}\frac{1}{\lambda_p - \lambda_k} = \overline{h}_g. \end{aligned} \tag{2.88}$$

Hence, by using $a_{g,0} = 1/(2g-1)$, we obtain

$$K_g = \sum_{p=1}^{g-1} a_{g,g-p}\overline{h}_p' + a_{g,0}\overline{h}_g' - \frac{1}{2g-1}\sum_{p=1}^{g}\nu_p = \sum_{p=1}^{g-1} a_{g,g-p}\overline{h}_p' + a_{g,0}\Big(\overline{h}_g' - \sum_{p=1}^{g}\nu_p\Big) = H_g.$$

This completes the proof. □

Thus Theorem 2.18 follows from the following:

Theorem 2.21. *The monodromy data of* (2.79) *with* (2.80) *is preserved if and only if* $\{\lambda_j, \mu_k\}$ *satisfies the following completely integrable Hamiltonian system with time variables* $t = (t_1, \cdots, t_g)$:

$$\frac{\partial \lambda_j}{\partial t_k} = \eta\frac{\partial K_k}{\partial \mu_j}, \quad \frac{\partial \mu_j}{\partial t_k} = -\eta\frac{\partial K_k}{\partial \lambda_j} \quad (1 \le j, k \le g) \tag{2.89}$$

with

$$K_k = \sum_{p=1}^{g} a_{k,g-p}(t)\overline{h}_p' - \frac{\delta_{k,g}}{2g-1}\sum_{p=1}^{g}\nu_p. \tag{2.90}$$

The proof of Theorem 2.21 proceeds in the same manner as that of Theorem 1.11 given in § 1.3. Here we content ourselves with outlining the derivation of the Hamiltonian system (2.89).

First we can confirm that $\Lambda(z)\mathcal{A}_j$, where $\Lambda(z)$ is defined by (1.37), is a polynomial of degree g, and it vanishes at the origin. Hence we can assume that $\mathcal{A}_j$ has the following form:

$$\mathcal{A}_j = \gamma_j + \sum_{k=1}^{g}\frac{\beta_j^{(k)}}{z - \lambda_k} \quad \text{with} \quad \gamma_j = \sum_{k=1}^{g}\frac{\beta_j^{(k)}}{\lambda_k}. \tag{2.91}$$

This $\{\mathcal{A}_j\}$ has the following expansion near $z = \infty$:

$$\mathcal{A}_j = \sum_{k=0}^{g-1} a_{j,k}(t) z^{-k} + O(z^{-g}) \quad (|z| \to \infty). \tag{2.92}$$

Hence we can determine $\{\beta_j^{(k)}\}$ by comparing (2.91) and (2.92). The equations which correspond to (1.102) and (1.103) are

$$\frac{\partial \lambda_l}{\partial t_j} = -\gamma_j + 2\beta_j^{(l)} \nu_l - \sum_{\substack{k=1 \\ k \neq l}}^{g} \frac{\beta_j^{(k)}}{\lambda_l - \lambda_k}, \tag{2.93}$$

$$\frac{\partial \nu_l}{\partial t_j} = -\sum_{\substack{k=1 \\ k \neq l}}^{g} \Big[\frac{\beta_j^{(k)} \nu_l}{(\lambda_l - \lambda_k)^2} - \frac{3}{2} \frac{\beta_j^{(k)}}{(\lambda_l - \lambda_k)^3} \Big] + \beta_j^{(l)} Q_3^{(l)}. \tag{2.94}$$

Then a similar argument of the 5th Step in § 1.3 gives

$$\frac{\partial \lambda_l}{\partial t_j} = -\gamma_j + \frac{\partial}{\partial \nu_l} \sum_{p=1}^{g} a_{j,g-p} \overline{h}'_p, \qquad \frac{\partial \nu_l}{\partial t_j} = -\frac{\partial}{\partial \lambda_l} \sum_{p=1}^{g} a_{j,g-p} \overline{h}'_p. \tag{2.95}$$

Since $\gamma_j = a_{j,0} = \delta_{j,g}/(2g-1)$, we obtain the Hamiltonian system (2.89).

§ 3. $(P_{\text{II-2}})$-Hierarchy and (P_{IV})-Hierarchy

The purpose of this section is to show the equivalence of (P_J)-hierarchy ($J =$ II-2, IV) used in [KKoNT] and [KoN] and (P_J)-hierarchy ($J =$ II, IV) used in [Ko]. We note that (P_{II})-hierarchy (resp. (P_{IV})-hierarchy) in [Ko] is the same as $(\widetilde{P}_{\text{II-2}})$-hierarchy (resp. $(\widetilde{P}_{\text{IV}})$-hierarchy) in this article. We also show in § 3.2 that the equivalence extends to the underlying Lax pairs.

§ 3.1. (P_{II})-Hierarchy, (P_{IV})-Hierarchy and Their Equivalent Forms

The $(P_{\text{II-2}})$-hierarchy and the (P_{IV})-hierarchy are obtained by Gordoa, Joshi and Pickering in [GJP1] through a certain reduction of the dispersive water wave hierarchy. To write down the $(P_{\text{II-2}})$-hierarchy and the (P_{IV})-hierarchy, we first define $\{K_n, L_n\}$, polynomials of $u = u(t)$, $v = v(t)$ and their derivatives, by the following recursion relation:

$$\begin{cases} K_{n+1} = \dfrac{1}{2}(uK_n + 2L_n - \eta^{-1}\dfrac{dK_n}{dt}), \\ L_{n+1} = \dfrac{1}{4}\displaystyle\sum_{j=0}^{n}(vK_{n-j}K_j - L_{n-j}L_j + \eta^{-1}K_{n-j}\dfrac{dL_j}{dt}), \end{cases} \tag{3.1}$$

for $n \geq 0$ with $K_0 = 2$ and $L_0 = 0$ (this recursion formula (3.1) is obtained in [N1] (cf. [N2])). First few members of $\{K_n, L_n\}$ are given by

$$\begin{pmatrix} K_1 \\ L_1 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \begin{pmatrix} K_2 \\ L_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} u^2 + 2v \\ 2uv \end{pmatrix} + \eta^{-1}\frac{d}{dt}\begin{pmatrix} -u \\ v \end{pmatrix}, \tag{3.2}$$

$$\begin{pmatrix} K_3 \\ L_3 \end{pmatrix} = \left(\frac{1}{2}\right)^2 \begin{pmatrix} u^3 + 6uv \\ 3u^2 v + 3v^2 \end{pmatrix} + 3u\eta^{-1}\frac{d}{dt}\begin{pmatrix} -u \\ v \end{pmatrix} + \eta^{-2}\frac{d^2}{dt^2}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.3}$$

Then the $(P_{\text{II-2}})$-hierarchy can be formulated as follows (see [N1] and [N2] for the fact that $(P_{\text{II-2}})_m$ defined here is the same with that of [GJP1]):

Definition 3.1 ($P_{\text{II-2}}$-Hierarchy with a Large Parameter)**.** We set:

$$(P_{\text{II-2}})_m : \begin{cases} K_{m+1} + \displaystyle\sum_{j=1}^{m} c_j K_{m-j+1} + 2\gamma t = 0, \\ L_{m+1} + \displaystyle\sum_{j=1}^{m} c_j L_{m-j+1} = 2\kappa, \end{cases} \tag{3.4}$$

where γ ($\neq 0$), κ and $\{c_j\}_{j=1}^{m}$ are constants.

Remark. Without loss of generality, we can choose $c_1 = 0$ by the translation of u, fix γ to an arbitrary nonzero number by the scalings of u, v and t. We also note that $(P_{\text{II-2}})_m$ defined here can be obtained from that in [KKoNT] by the following replacement of constants:

$$\gamma \to \frac{1}{2}g, \quad \kappa \to \frac{1}{2}\delta, \quad c_j \to c_{m-j+1} \ (j = 1, 2, \cdots, m), \tag{3.5}$$

where $\{\gamma, \kappa, c_j\}$ are constants used here, and $\{g, \delta, c_{m-j+1}\}$ are those in [KKoNT].

To write down the (P_{IV})-hierarchy obtained in [GJP1], it is convenient to introduce the following symbols $\{\mathcal{K}_n, \mathcal{L}_n\}$ for given constants $\{c_j\}$:

$$\begin{pmatrix} \mathcal{K}_n \\ \mathcal{L}_n \end{pmatrix} = c_0 \begin{pmatrix} K_n \\ L_n \end{pmatrix} + c_1 \begin{pmatrix} K_{n-1} \\ L_{n-1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} K_0 \\ L_0 \end{pmatrix} \tag{3.6}$$

for $n = 1, 2, \cdots$. By using these symbols $\{\mathcal{K}_n, \mathcal{L}_n\}$, $(P_{\text{II-2}})_m$ is expressed as

$$\mathcal{K}_{m+1} + 2\gamma t = 0, \quad \mathcal{L}_{m+1} = 2\kappa \tag{3.7}$$

with $c_0 = 1$ and $c_{m+1} = 0$. Using these symbols, we can present the (P_{IV})-hierarchy with a large parameter as follows:

Definition 3.2 ((P_{IV})-Hierarchy with a Large Parameter)**.** We set:

$$(3.8)\qquad (P_{\mathrm{IV}})_m : \begin{cases} \eta^{-1}\dfrac{d\mathcal{K}_m}{dt} = 2\mathcal{L}_m + u\mathcal{K}_m + 2\gamma t u - 4\theta_1 - 2\eta^{-1}\gamma, \\ \eta^{-1}(\mathcal{K}_m + 2\gamma t)\dfrac{d\mathcal{L}_m}{dt} = -v(\mathcal{K}_m + 2\gamma t)^2 + (\mathcal{L}_m - 2\theta_1)^2 - 4{\theta_2}^2, \end{cases}$$

where γ, θ_1, θ_2 and $\{c_j\}_{j=1}^m$ are constants (we set $c_0 = 1$ for convenience).

Remark. This time, we can choose $c_m = 0$ by the translation of t, and fix γ to an arbitrary nonzero number by the scalings of u, v and t without loss of generality.

We now introduce the ($\widetilde{P}_{\mathrm{II\text{-}2}}$)-hierarchy and the ($\widetilde{P}_{\mathrm{IV}}$)-hierarchy, which seem to be amenable to explicit computations. (see [KT] for example).

Definition 3.3 (($\widetilde{P}_{\mathrm{II\text{-}2}}$)-Hierarchy with a Large Parameter)**.** We set:

$$(3.9)\qquad (\widetilde{P}_{\mathrm{II\text{-}2}}) : \begin{cases} \eta^{-1}\dfrac{du_j}{dt} = -2[u_1u_j + v_j + u_{j+1}] + 2c_ju_1 \quad (j = 1, 2, \cdots, m), \\ \eta^{-1}\dfrac{dv_j}{dt} = 2[v_1u_j + v_{j+1} + w_j] - 2c_jv_1 \qquad (j = 1, 2, \cdots, m), \\ u_{m+1} = \gamma t, \quad v_{m+1} = \kappa. \end{cases}$$

Here $\{u_j, v_j\}_{j=1}^m$ are unknown functions, γ ($\neq 0$), κ and $\{c_j\}_{j=1}^m$ are constants, and $\{w_n\}$ are polynomials of $\{u_j, v_j\}$ recursively defined by

$$(3.10)\qquad w_n = \sum_{j=1}^{n-1} u_{n-j}w_j + \sum_{j=1}^{n} u_{n-j+1}v_j + \frac{1}{2}\sum_{j=1}^{n-1} v_{n-j}v_j - \sum_{j=1}^{n-1} c_{n-j}w_j.$$

Definition 3.4 (($\widetilde{P}_{\mathrm{IV}}$)-Hierarchy with a Large Parameter)**.** We set:

$$(3.11)\qquad (\widetilde{P}_{\mathrm{IV}}) : \begin{cases} \eta^{-1}\dfrac{du_j}{dt} = -2[u_1u_j + v_j + u_{j+1}] + 2c_ju_1 \quad (j = 1, 2, \cdots, m), \\ \eta^{-1}\dfrac{dv_j}{dt} = 2[v_1u_j + v_{j+1} + w_j] - 2c_jv_1 \qquad (j = 1, 2, \cdots, m) \end{cases}$$

with

$$(3.12)\qquad u_{m+1} = -(\gamma t u_1 + \theta_1 + \frac{1}{2}\eta^{-1}\gamma),$$

$$(3.13)\qquad v_{m+1} = -w_m - \gamma t v_1 - \frac{(v_m - \theta_1)^2 - {\theta_2}^2}{2(u_m - \gamma t - c_m)}.$$

Here $\{u_j, v_j\}_{j=1}^m$ are unknown functions, γ ($\neq 0$), θ_1, θ_2 and $\{c_j\}_{j=1}^m$ are constants, and $\{w_n\}$ are polynomials of $\{u_j, v_j\}$ recursively defined by (3.10).

The principal aim of this subsection is to show that $(\widetilde{P}_{\text{II-2}})_m$ and $(\widetilde{P}_{\text{IV}})_m$ are respectively equivalent to $(P_{\text{II-2}})_m$ and $(P_{\text{IV}})_m$; to be more explicit we prove the following:

Theorem 3.5. *If $\{u, v\}$ is a solution of $(P_{\text{II-2}})_m$, then $\{u_j, v_j\}$ defined by*

$$u_j = -\frac{1}{2}\mathcal{K}_j + c_j, \quad v_j = \frac{1}{2}\mathcal{L}_j \quad (1 \le j \le m) \tag{3.14}$$

is a solution of $(P_{\text{II-2}})_m$. Conversely, if $\{u_j, v_j\}$ is a solution of $(\widetilde{P}_{\text{II-2}})_m$, then

$$u = -2u_1, \quad v = 2v_1 \tag{3.15}$$

gives a solution of $(P_{\text{II-2}})_m$.

Theorem 3.6. *If $\{u, v\}$ is a solution of $(P_{\text{IV}})_m$, then $\{u_j, v_j\}$ defined by* (3.14) *is a solution of $(\widetilde{P}_{\text{IV}})_m$. Conversely, if $\{u_j, v_j\}$ is a solution of $(\widetilde{P}_{\text{IV}})_m$, then $\{u, v\}$ defined by* (3.15) *gives a solution of $(P_{\text{IV}})_m$.*

Note that (3.15) is coincident with (3.14) with $j = 1$. As the logical structure of the proof of Theorem 3.5 and that of Theorem 3.6 are the same, here we give a proof of Theorem 3.6. To prove Theorem 3.6 we need the following:

Proposition 3.7. *Let $\{\mathcal{K}_n, \mathcal{L}_n\}$ be polynomials of u, v and their derivatives defined in* (3.6) *with $c_0 = 1$. Then we have*

$$\begin{cases} \mathcal{K}_{n+1} = \dfrac{1}{2}(u\mathcal{K}_n - \eta^{-1}\dfrac{d\mathcal{K}_n}{dt} + 2\mathcal{L}_n) + 2c_{n+1}, \\ \mathcal{L}_{n+1} = \dfrac{1}{4}\displaystyle\sum_{j=0}^{n}(v\mathcal{K}_{n-j}\mathcal{K}_j - \mathcal{L}_{n-j}\mathcal{L}_j + \eta^{-1}\mathcal{K}_{n-j}\dfrac{d\mathcal{L}_j}{dt} - 4c_{n-j+1}\mathcal{L}_j) \end{cases} \tag{3.16}$$

Proof. By the definition of $\mathcal{K}_{n+1}$, we find

$$\mathcal{K}_{n+1} = \sum_{j=0}^{n+1} c_j K_{n-j+1} = \frac{1}{2}\sum_{j=0}^{n} c_j(uK_{n-j} + 2L_{n-j} - \eta^{-1}\frac{dK_{n-j}}{dt}) + c_{n+1}K_0. \tag{3.17}$$

Then it follows from (3.1) and $K_0 = 2$ that

$$\mathcal{K}_{n+1} = \frac{1}{2}(u\mathcal{K}_n + 2\mathcal{L}_n - \eta^{-1}\frac{d\mathcal{K}_n}{dt}) + 2c_{n+1}. \tag{3.18}$$

This shows the first relation of (3.16). To prove the second relation, we first use (3.6) to obtain

$$\sum_{j=0}^{n}\{v\mathcal{K}_{n-j}\mathcal{K}_j - \mathcal{L}_{n-j}\mathcal{L}_j + \eta^{-1}\mathcal{K}_{n-j}\frac{d\mathcal{L}_j}{dt}\} \tag{3.19}$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{n-j} \sum_{l=0}^{j} c_k c_l \{ v K_{n-j-k} K_{j-l} - L_{n-j-k} K_{j-l} + \eta^{-1} K_{n-j-k} \frac{dL_{j-l}}{dt} \}$$

Then by changing the order of summations and using the second equation of (3.1), we obtain

$$\begin{aligned} (3.20) \quad & \sum_{j=0}^{n} \{ v \mathcal{K}_{n-j} \mathcal{K}_j - \mathcal{L}_{n-j} \mathcal{L}_j + \eta^{-1} \mathcal{K}_{n-j} \frac{d\mathcal{L}_j}{dt} \} \\ & = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \sum_{j=0}^{n-k-l} c_k c_l \{ v K_{n-j-k-l} K_j - L_{n-j-k-l} K_j + \eta^{-1} K_{n-j-k-l} \frac{dL_j}{dt} \} \\ & = 4 \sum_{k=0}^{n} \sum_{l=0}^{n-k} c_k c_l L_{n-k-l+1}. \end{aligned}$$

Hence the right-hand side of the second relation of (3.16) becomes

$$\sum_{k=0}^{n} \sum_{l=0}^{n-k} c_k c_l L_{n-k-l+1} - \sum_{j=0}^{n} c_{n-j+1} \mathcal{L}_j = \sum_{k=0}^{n} c_k \mathcal{L}_{n-k+1} - \sum_{j=0}^{n} c_{n-j+1} \mathcal{L}_j = c_0 \mathcal{L}_{n+1}.$$

This completes the proof. □

Proof of Theorem 3.6. By the first equation of (3.16), we find

$$\eta^{-1} \frac{d\mathcal{K}_j}{dt} = u \mathcal{K}_j + 2\mathcal{L}_j - 2\mathcal{K}_{j+1} + 4c_{j+1}. \tag{3.21}$$

We next introduce $\{\mathcal{W}_j\}$ by

$$\eta^{-1} \frac{d\mathcal{L}_j}{dt} = -v \mathcal{K}_j + 2\mathcal{L}_{j+1} + 4\mathcal{W}_j. \tag{3.22}$$

Clearly these $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_m$ are polynomials of u, v and their derivatives. Then by substituting (3.22) into the second equation of (3.16), we find that $\{\mathcal{W}_j\}$ satisfy

$$2\mathcal{W}_n = -\sum_{j=1}^{n-1} \mathcal{K}_{n-j} \mathcal{W}_j - \frac{1}{2} \sum_{j=1}^{n} (\mathcal{K}_{n-j+1} - 2c_{n-j+1}) \mathcal{L}_j + \frac{1}{4} \sum_{j=1}^{n-1} \mathcal{L}_{n-j} \mathcal{L}_j \tag{3.23}$$

for $n = 1, 2, \ldots$. Hence (3.16) is rewritten as the combination of (3.21), (3.22) and (3.23). We also note that we can show from (3.23) and (3.14) that $w_j = \mathcal{W}_j$ holds for $j = 1, 2, \cdots, m$ by induction.

Next by using (3.21) we can eliminate $d\mathcal{K}_m/dt$ in the left-hand side of the first equation of (3.8). Hence the first equation of (3.8) is rewritten as

$$\mathcal{K}_{m+1} = -\gamma t u + 2\kappa_1 + \eta^{-1} \gamma. \tag{3.24}$$

Furthermore (3.22) entails

$$\begin{aligned}\eta^{-1}(\mathcal{K}_m+2\gamma t)\frac{d\mathcal{L}_m}{dt}&+v(\mathcal{K}_m+2\gamma t)^2-(\mathcal{L}_m-2\theta_1)^2+4{\theta_2}^2\\ &=(\mathcal{K}_m+2\gamma t)\{\eta^{-1}\frac{d\mathcal{L}_m}{dt}+v(\mathcal{K}_m+2\gamma t)\}-(\mathcal{L}_m-2\theta_1)^2+4{\theta_2}^2\\ &=(\mathcal{K}_m+2\gamma t)(2\mathcal{L}_{m+1}+4\mathcal{W}_m+2\gamma tv)-(\mathcal{L}_m-2\theta_1)^2+4{\theta_2}^2.\end{aligned}\tag{3.25}$$

Hence the second equation of (3.16) is rewritten as

$$\mathcal{L}_{m+1}=-2\mathcal{W}_m-\gamma tv+\frac{(\mathcal{L}_m-2\theta_1)^2-4{\theta_2}^2}{2(\mathcal{K}_m+2\gamma t)}.\tag{3.26}$$

Thus $(P_{\mathrm{IV}})_m$ is equivalent to (3.24) and (3.26).

It is now clear that $(P_{\mathrm{IV}})_m$ is equivalent to $(\widetilde{P}_{\mathrm{IV}})_m$ by (3.14) and $w_j=\mathcal{W}_j$. Here (3.21) and (3.22) correspond to (3.11) with (3.10), and (3.24) and (3.26) correspond to (3.12) and (3.13). □

§ 3.2. Lax Pairs of $(P_{\text{II-2}})_m$, $(P_{\mathrm{IV}})_m$ and Their Equivalent Hierarchies

In this subsection we show that the Lax pairs of $(\widetilde{P}_{\text{II-2}})_m$ and $(\widetilde{P}_{\mathrm{IV}})_m$ given in [Ko], where the labeling $(P_{\mathrm{II}})_m$ and $(P_{\mathrm{IV}})_m$ is used, can be obtained from those given by [GJP2] (see also [N1]) through the relation (3.14).

Since the arguments are not different for $(\widetilde{P}_{\text{II-2}})_m$ and $(\widetilde{P}_{\mathrm{IV}})_m$, we mainly discuss the Lax pair of $(\widetilde{P}_{\mathrm{IV}})_m$. To begin with, let us recall the following result:

Theorem 3.8 ([GJP2]). *The compatibility condition of the following equations* (3.27) *is equivalent to* (3.8):

$$\gamma x\eta^{-1}\frac{\partial\overrightarrow{\phi}}{\partial x}=A\overrightarrow{\psi},\quad \eta^{-1}\frac{\partial\overrightarrow{\phi}}{\partial t}=B\overrightarrow{\psi},\tag{3.27}$$

where

$$\begin{aligned}A=\frac{1}{4}&\begin{pmatrix}-(2x-u)(\mathcal{K}+2\gamma t)-\eta^{-1}\dfrac{d\mathcal{K}}{dt}-2\eta^{-1}\gamma & 2(\mathcal{K}+2\gamma t)\\ -2\eta^{-1}\dfrac{dL}{dt}-2v(\mathcal{K}+2\gamma t) & (2x-u)(\mathcal{K}+2\gamma t)+\eta^{-1}\dfrac{d\mathcal{K}}{dt}-2\eta^{-1}\gamma\end{pmatrix}\\ &+\frac{1}{4}\begin{pmatrix}I_1 & 0\\ 2I_2/(\mathcal{K}_m+2\gamma t) & -I_1\end{pmatrix},\\ B=&\begin{pmatrix}-x+u/2 & 1\\ v & x-u/2\end{pmatrix}.\end{aligned}$$

Here $\mathcal{K}$ and $\mathcal{L}$ are defined by

$$\mathcal{K}=\sum_{j=0}^{m}x^{m-j}\mathcal{K}_j,\quad \mathcal{L}=\sum_{j=0}^{m}x^{m-j}\mathcal{L}_j,\tag{3.28}$$

and

$$I_1 = \eta^{-1}\frac{d\mathcal{K}_m}{dt} - u(\mathcal{K}_m + 2\gamma t) - 2\mathcal{L}_m + 4\theta_1 + 2\eta^{-1}\gamma,$$
$$I_2 = \eta^{-1}(\mathcal{K}_m + 2\gamma t)\frac{d\mathcal{L}_m}{dt} + v(\mathcal{K}_m + 2\gamma t)^2 p - (\mathcal{L}_m - 2\theta_1)^2 + 4{\theta_2}^2.$$

Note that $I_1 = I_2 = 0$ is nothing but $(P_{\mathrm{IV}})_m$.

Theorem 3.9 ([Ko]). *The compatibility condition of the following equations* (3.29) *are equivalent to* (3.11):

$$\gamma x\eta^{-1}\frac{\partial\overrightarrow{\psi}}{\partial x} = \widetilde{A}\,\overrightarrow{\psi}, \quad \eta^{-1}\frac{\partial\overrightarrow{\psi}}{\partial t} = \widetilde{B}\overrightarrow{\psi}, \tag{3.29}$$

where

$$\widetilde{A} = \begin{pmatrix} -[x^{m+1} + V + xC(x) + \gamma xt - \theta_1] & U + C(x) + \gamma t \\ -2[xV + W + v_{m+1} + \gamma t v_1] & x^{m+1} + V + xC(x) + \gamma xt - \theta_1 \end{pmatrix}, \tag{3.30}$$

$$\widetilde{B} = \begin{pmatrix} -(x + u_1) & 1 \\ -2v_1 & x + u_1 \end{pmatrix}, \tag{3.31}$$

$$U(x) = x^m - \sum_{j=1}^{m} u_j x^{m-j}, \quad V(x) = \sum_{j=1}^{m} v_j x^{m-j}, \tag{3.32}$$

$$W(x) = \sum_{j=1}^{m} w_j x^{m-j}, \quad C(x) = \sum_{j=1}^{m} c_j x^{m-j}. \tag{3.33}$$

Our aim is to recover (3.29) from (3.27). First, we note that

$$\eta^{-1}\frac{d\mathcal{K}}{dt} = -(2x - u)\mathcal{K} - 2\mathcal{K}_{m+1} + 2\mathcal{L} + 4(x^{m+1} + xC(x)), \tag{3.34}$$

$$\eta^{-1}\frac{d\mathcal{L}}{dt} = -v\mathcal{K} + 2x\mathcal{L} + 2\mathcal{L}_{m+1} + 4\mathcal{W} \tag{3.35}$$

follow from (3.21) and (3.22), where $C(x)$ is defined by (3.33), and

$$\mathcal{W} = \sum_{j=1}^{m} x^{m-j}\mathcal{W}_j. \tag{3.36}$$

Next, by using the same argument to derive (3.24) and (3.26), we obtain

$$I_1 = -2\mathcal{K}_{m+1} - 2\gamma t u + 4\theta_1 + 2\eta^{-1}\gamma, \tag{3.37}$$

$$I_2 = (\mathcal{K}_m + 2\gamma t)(2\mathcal{L}_{m+1} + 4\mathcal{W}_m + 2\gamma t v) - (\mathcal{L}_m - 2\theta_1)^2 + 4{\theta_2}^2. \tag{3.38}$$

Hence

$$4A_{1,1} = -(2x - u)(\mathcal{K} + 2\gamma t) - \eta^{-1}\frac{d\mathcal{K}}{dt} - 2\eta^{-1}\gamma + I_1 \tag{3.39}$$

$$= -(2x-u)(\mathcal{K}+2\gamma t) - \eta^{-1}\frac{d\mathcal{K}}{dt} + (-2\mathcal{K}_{m+1} - 2\gamma tu + 4\theta_1).$$

Then it follows from (3.34) that

$$4A_{1,1} = -2\mathcal{L} - 4(x^{m+1} + xC(x)) - 4\gamma xtu + 4\theta_1. \tag{3.40}$$

We also obtain

(3.41)

$$\begin{aligned}
2A_{2,1} &= -\eta^{-1}\frac{d\mathcal{L}}{dt} - v(\mathcal{K}+2\gamma t) + \frac{I_2}{\mathcal{K}_m + 2\gamma t} \\
&= -\eta^{-1}\frac{d\mathcal{L}}{dt} - v(\mathcal{K}+2\gamma t) + \left\{2\mathcal{L}_{m+1} + 4\mathcal{W}_m + 2\gamma tv - \frac{(\mathcal{L}_m - 2\theta_1)^2 - 4{\theta_2}^2}{\mathcal{K}_m + 2\gamma t}\right\} \\
&= -2x\mathcal{L} - 4\mathcal{W} + 4\mathcal{W}_m - \frac{(\mathcal{L}_m - 2\theta_1)^2 - 4{\theta_2}^2}{\mathcal{K}_m + 2\gamma t}.
\end{aligned}$$

Hence by the replacement

$$\mathcal{K} \to 2U(x) + 2C(x), \quad \mathcal{L} \to 2V(x), \quad \mathcal{W} \to W(x), \tag{3.42}$$

and $(\psi_1, \psi_2) \to (\psi_1, \psi_2/4)$, we find the matrix $\widetilde{A}$ from A.

In a similar manner we obtain the Lax pair of $(\widetilde{P}_{\mathrm{II}})_m$ from the following Lax pair for $(P_{\mathrm{II}})_m$ given in [GJP2] (see also [N1], [N2]):

$$\begin{aligned}
&\gamma\eta^{-1}\frac{\partial\overrightarrow{\psi}}{\partial x} = A\overrightarrow{\psi}, \quad \eta^{-1}\frac{\partial\overrightarrow{\psi}}{\partial t} = B\overrightarrow{\psi}, \\
&A = \frac{1}{4}\begin{pmatrix} -(2x-u)\mathcal{K} - \eta^{-1}\dfrac{d\mathcal{K}}{dt} & 2\mathcal{K} \\ -2\eta^{-1}\dfrac{d\mathcal{L}}{dt} - 2v\mathcal{K} & (2x-u)\mathcal{K} + \eta^{-1}\dfrac{d\mathcal{K}}{dt} \end{pmatrix} \\
&\qquad + \frac{1}{2}\begin{pmatrix} -\mathcal{K}_{m+1} - 2\gamma t & 0 \\ 2(\mathcal{L}_{m+1} - 2\kappa) & \mathcal{K}_{m+1} + 2\gamma t \end{pmatrix}, \\
&B = \begin{pmatrix} -x + u/2 & 1 \\ v & x - u/2 \end{pmatrix}.
\end{aligned} \tag{3.43}$$

In fact, it follows from (3.34) and (3.35) that

$$A = \frac{1}{2}\begin{pmatrix} -2x^{m+1} - \mathcal{L} - 2xC(x) - 2\gamma t & \mathcal{K} \\ -\eta^{-1}\dfrac{d\mathcal{L}}{dt} - v\mathcal{K} + 2\mathcal{L}_{m+1} - 4\kappa & 2x^{m+1} + \mathcal{L} + 2xC(x) + 2\gamma t \end{pmatrix}. \tag{3.44}$$

Then the replacement (3.42) entails following Lax pair of $(P_{\mathrm{II}})_m$ (cf. [Ko]):

$$\gamma\eta^{-1}\frac{\partial\vec{\psi}}{\partial x} = \widetilde{A}\vec{\psi}, \quad \eta^{-1}\frac{\partial\vec{\psi}}{\partial t} = \widetilde{B}\vec{\psi}, \tag{3.45}$$

where

$$\widetilde{A} = \begin{pmatrix} -[x^{m+1} + V + xC(x) + \gamma t] & U + C(x) \\ -2[xV + W + \kappa] & x^{m+1} + V + xC(x) + \gamma t \end{pmatrix}, \tag{3.46}$$

$$\widetilde{B} = \begin{pmatrix} -(x + u_1) & 1 \\ -2v_1 & x + u_1 \end{pmatrix}. \tag{3.47}$$

Appendix A. First Two Members of the P_{I}-Hierarchy and the P_{34}-Hierarchy

$(P_{\mathrm{I}})_1$, $(P_{\mathrm{I}})_2$ and $(P_{34})_1$, $(P_{34})_2$ are given as follows:

$$(P_{\mathrm{I}})_1 \qquad \eta^{-2}\frac{d^2u}{dt^2} + 3u^2 + c_1u + 2\gamma t + \frac{1}{2}c_2 = 0,$$

$$\begin{aligned}(P_{\mathrm{I}})_2 \qquad & \eta^{-4}\frac{d^4u}{dt^4} + \eta^{-2}\Big[(10u + c_1)\frac{d^2u}{dt^2} + 5\Big(\frac{du}{dt}\Big)^2\Big] \\ & + 10u^3 + 3c_1u^2 + c_2u + 2\gamma t + \frac{1}{2}\,c_3 = 0.\end{aligned}$$

$$\begin{aligned}(P_{34})_1 \quad & 2(u + 2\gamma t + \frac{1}{2}c_1)\cdot \eta^{-2}\frac{d^2u}{dt^2} - (\eta^{-1}\frac{du}{dt} + 2\eta^{-1}\gamma)^2 + 4u(u + 2\gamma t + \frac{1}{2}c_1)^2 = -\kappa^2, \\ & \Longleftrightarrow 2\eta^{-2}y\frac{d^2y}{dt^2} = \Big(\eta^{-1}\frac{dy}{dt}\Big)^2 - 4y^3 + 4\gamma t y^2 - \kappa^2 \quad \text{with} \quad y = u + 2\gamma t + \frac{1}{2}c_1.\end{aligned}$$

$$\begin{aligned}(P_{34})_2 \qquad & 2(\eta^{-2}\frac{d^2u}{dt^2} + 3u^2 + c_1u + 2\gamma t + \frac{1}{2}\,c_2) \\ & \times \Big\{\eta^{-4}\frac{d^4u}{dt^4} + (6ub_1 + cb_1)\eta^{-2}\frac{d^2u}{dt^2} + 6\Big(\eta^{-1}\frac{du}{dt}\Big)^2\Big\} \\ & - (\eta^{-3}\frac{d^3u}{dt^3} + (6u_1 + c_1)\eta^{-1}\frac{du}{dt} + 2\eta^{-1}\gamma)^2 \\ & + 4u\{\eta^{-2}\frac{d^2u}{dt^2} + 3u^2 + c_1u + 2\gamma t + \frac{1}{2}\,c_2\}^2 = -\kappa^2.\end{aligned}$$

Appendix B. Large Parameter by Scalings

The Painlevé hierarchies *with a large parameter* η and their Lax pairs with the large parameter we deal with in this article are found through the scaling of unknown functions, independent variables and parameters of the corresponding Painlevé hierarchies and their Lax pairs. For example, if we change the unknown functions, the independent variable and parameters in $(P_{\mathrm{I}})_m$ with $\eta = 1$ by

$$u = \eta^{2\alpha}u^{\dagger}, \quad t = \eta^{1-\alpha}t^{\dagger}, \quad c_j = \eta^{2j\alpha}c_j^{\dagger}\ (1 \le j \le m+1), \quad \gamma = \eta^{(2m+3)\alpha - 1}\gamma^{\dagger},$$

where α is arbitrary, then $u^\dagger$ etc. satisfy $(P_{\mathrm{I}})_m$ with the large parameter η. We can also obtain $(L_{\mathrm{I}})_m$ with a large parameter from that with $\eta = 1$ by changing

$$x = \eta^{2\alpha} x^\dagger, \quad \overrightarrow{\psi} = \begin{pmatrix} 1 & 0 \\ 0 & \eta^\alpha \end{pmatrix} \overrightarrow{\psi}^{\,\dagger}.$$

In what follows we symbolically designate the procedure above as follows:

$$(P_{\mathrm{I}})_m : \qquad u \to \eta^{2\alpha} u, \quad t \to \eta^{1-\alpha} t, \quad x \to \eta^{2\alpha} x, \quad \gamma \to \eta^{(2m+3)\alpha - 1}, \quad c_j \to \eta^{2j\alpha} c_j.$$

Using this symbolic expression, we list up the concrete form of the scaling we used:

$$\begin{aligned}
(\widetilde{P}_{\mathrm{I}})_m : \quad & u_j \to \eta^{2j\alpha} u_j, && v_j \to \eta^{(2j+1)\alpha} v_j, && t \to \eta^{1-\alpha} t, && x \to \eta^{2\alpha} x, \\
& \widetilde{\gamma} \to \eta^{(2m+3)\alpha-1} \widetilde{\gamma}, && \widetilde{c}_j \to \eta^{2(j+1)\alpha} \widetilde{c}_j. \\
(P_{34})_m : \quad & u \to \eta^{2\alpha} u, && t \to \eta^{1-\alpha} t, && x \to \eta^{2\alpha} x, \\
& \widetilde{\gamma} \to \eta^{(2m+1)\alpha-1} \widetilde{\gamma}, && \widetilde{\kappa} \to \eta^{(2m+1)\alpha} \widetilde{\kappa}, && c_j \to \eta^{2j\alpha} c_j. \\
(\widetilde{P}_{34})_m : \quad & u_j \to \eta^{2j\alpha} u_j, && v_j \to \eta^{(2j+1)\alpha} v_j, && t \to \eta^{1-\alpha} t, && x \to \eta^{2\alpha} x, \\
& \widetilde{\gamma} \to \eta^{(2m+1)\alpha-1} \widetilde{\gamma}, && \widetilde{\kappa} \to \eta^{(2m+1)\alpha} \widetilde{\kappa}, && \widetilde{c}_j \to \eta^{2(j+1)\alpha} \widetilde{c}_j. \\
(P_{\mathrm{II}})_m : \quad & u \to \eta^{\alpha} u, && v \to \eta^{2\alpha} v, && t \to \eta^{1-\alpha} t, && x \to \eta^{2\alpha} x, \\
& \gamma \to \eta^{(m+2)\alpha-1} \gamma, && \kappa \to \eta^{(m+2)\alpha} \kappa, && c_j \to \eta^{j\alpha} c_j. \\
(\widetilde{P}_{\mathrm{II}})_m : \quad & u_j \to \eta^{j\alpha} u_j, && v_j \to \eta^{(j+1)\alpha} v_j, && t \to \eta^{1-\alpha} t, && x \to \eta^{2\alpha} x, \\
& \gamma \to \eta^{(m+2)\alpha-1} \gamma, && \kappa \to \eta^{(m+2)\alpha} \kappa, && c_j \to \eta^{j\alpha} c_j, \\
& (w_j \to \eta^{(j+2)\alpha} w_j). \\
(P_{\mathrm{IV}})_m : \quad & u \to \eta^{\alpha} u, && v \to \eta^{2\alpha} v, && t \to \eta^{1-\alpha} t, && x \to \eta^{2\alpha} x, \\
& \gamma \to \eta^{(m+1)\alpha-1} \gamma, && \theta_j \to \eta^{(m+1)\alpha} \theta_j, && c_j \to \eta^{j\alpha} c_j. \\
(\widetilde{P}_{\mathrm{IV}})_m : \quad & u_j \to \eta^{j\alpha} u_j, && v_j \to \eta^{(j+1)\alpha} v_j, && t \to \eta^{1-\alpha} t, && x \to \eta^{2\alpha} x, \\
& \gamma \to \eta^{(m+1)\alpha-1} \gamma, && \theta_j \to \eta^{(m+1)\alpha} \theta_j, && c_j \to \eta^{j\alpha} c_j \\
& (w_j \to \eta^{(j+2)\alpha} w_j).
\end{aligned}$$

The following are the scalings of variables and constants to introduce a large parameter for the degenerate Garnier systems. We note that the degree of the scalings is assigned through the relations between the degenerate Garnier systems and the corresponding Painlevé hierarchy.

$$\begin{aligned}
G(g + \frac{5}{2}; g) : \quad & \lambda_j \to \eta^{2/(2g+3)} \lambda_j, && \mu_j \to \eta^{(2g+1)/(2g+3)} \mu_j, \\
& t_j \to \eta^{2(g-j+2)/(2g+3)} t_j, && h_j \to \eta^{2(g+j+1)/(2g+3)} h_j, \\
& z \to \eta^{2/(2g+3)} z.
\end{aligned}$$

$$G(1, g+\frac{3}{2}; g): \quad \lambda_j \to \eta^{2/(2g+1)}\lambda_j, \qquad \mu_j \to \eta^{(2g-1)/(2g+1)}\mu_j,$$
$$t_j \to \eta^{2(g-j+1)/(2g+1)}t_j, \quad h_j \to \eta^{2(g+j)/(2g+1)}h_j,$$
$$\kappa_0 \to \eta\kappa_0, \qquad z \to \eta^{2/(2g+2)}z.$$

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