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Monodromy Matrices of a Second Order Fuchsian Differential Equation with Five Singular Points

By

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Abstract

Monodromy matrices are computed for an explicitly given second order Fuchsian differential equation with five singular points by using the exact WKB analysis.

§ 1. Introduction

The aim of this article is to compute the monodromy matrices of the following differential equation:

$$\left(-\frac{d^2}{dx^2}+\eta^2 Q(x)\right)\psi=0, \tag{1.1}$$

where

$$Q(x)=\frac{(x-2i)(x-(1-i))(x+(1+i))(x-(1-3i))(x+(1+3i))(x+4i)}{x^2(x-1)^2(x+1)^2(x+2i)^2(x+3i)^2} \tag{1.2}$$

and η designates a large parameter. Equation (1.1) is a second order Fuchsian differential equation with five singular points

$$b_0=0,\ b_1=1,\ b_2=-1,\ b_3=-2i,\ b_4=-3i. \tag{1.3}$$

Our computation is based on the exact WKB analysis developed by [2], [4] in which monodromy matrices are computed for a given second order Fuchsian differential equation with four regular singular points. The method of computation employed here is

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exactly the same as that given in [2], [4]: Firstly we draw the Stokes curves of (1.1) which are integral curves of the direction field $\mathrm{Im}\sqrt{Q(x)}\,dx = 0$ emanating from the turning points

$$a_0 = 2i,\ a_1 = 1 - i,\ a_2 = -1 - i,\ a_3 = 1 - 3i,\ a_4 = -1 - 3i,\ a_5 = -4i \tag{1.4}$$

On the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, the Stokes curves, the turning points and the singular points form a sphere graph with vertex 2-coloring which is called the Stokes graph of Eq. (1.1). Secondly we take the Borel sums of WKB solutions and choose them as a system of fundamental solutions of Eq. (1.1). For each singular point, we take a closed oriented path with a fixed base point encircling the singular point. Finally we take the analytic continuation of the system of fundamental solutions along the path. We use the connection formula for WKB solutions every time the path crosses the Stokes curves and multiply all of thus obtained connection matrices. Then we have the monodromy matrix of the contour with respect to the system.

Graph theoretic classification of Stokes graphs of second order Fuchsian differential equations with five regular singular points is given in [1], [3]. There are 25 different types of Stokes graphs for such differential equations and they are classified in terms of the order sequences of dual graphs. Equation (1.1) is an example given in [1] whose Stokes graph is characterized by the order sequence (4,4,4,3,3) which is called the index of the graph in [1].

§ 2. Stokes Curves and the WKB Solutions

The Stokes graph of Eq. (1.1) can be obtained by using numerical computation and it has the configuration as in Fig. 2.1 (cf. [1, Fig. A.1, (i)]):

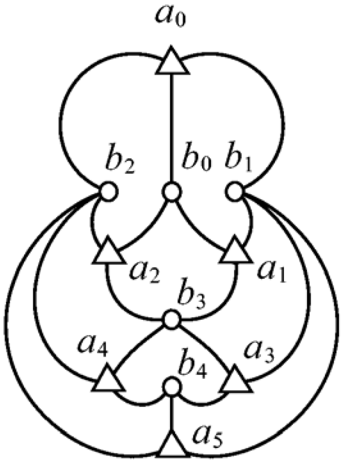

Fig. 2.1

Here triangles and small circles designate turning points and regular singular points, respectively. We fix a point x_0 outside the graph and take the WKB solutions

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\Big(\pm \int_{x_0}^{x} S_{\mathrm{odd}} dx\Big) \tag{2.1}$$

as a system of fundamental solutions at x_0. Here $S_{\mathrm{odd}} = \sum_{n=0}^{\infty} \eta^{1-2n} S_{2n-1}$ denotes the odd part of the formal solution $S = \sum \eta^{-k} S_k$ of the Riccati equation

$$\frac{dS}{dx} + S^2 = \eta^2 Q(x) \tag{2.2}$$

associated with Eq. (1.1). Let C_k ($k = 0, 1, 2, 3, 4$) be closed paths going around b_k with the base point x_0 as shown in Fig. 2.2.k:

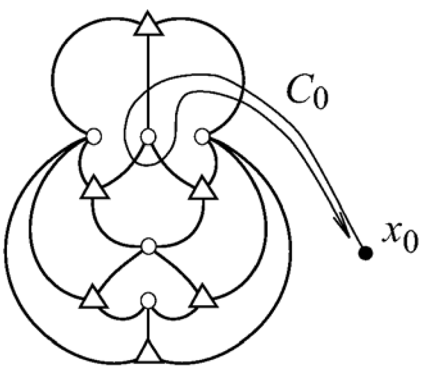

Fig. 2.2.0

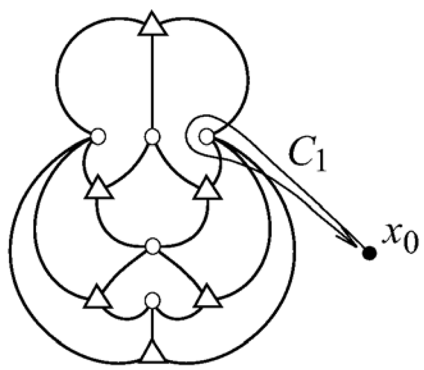

Fig. 2.2.1

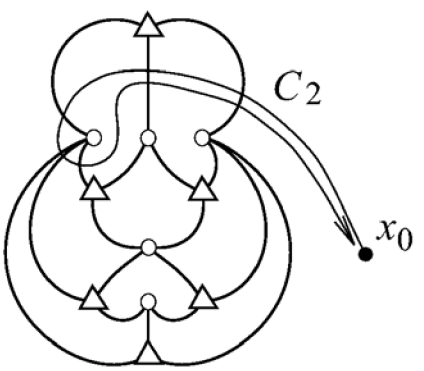

Fig. 2.2.2

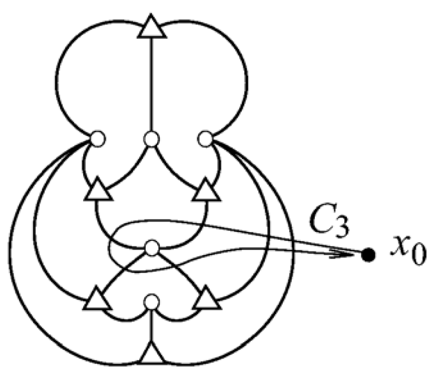

Fig. 2.2.3

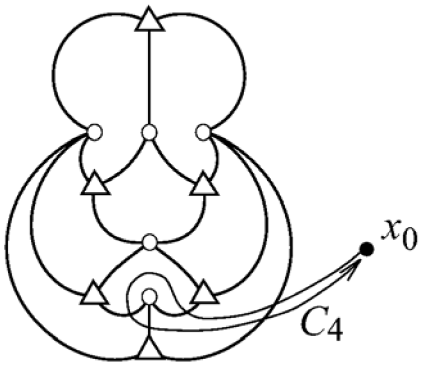

Fig. 2.2.4

We will compute the monodromy matrices of C_k ($k = 0, 1, 2, 3, 4$) with respect to the Borel sums of the WKB solutions (2.1). We place the cuts as shown by wavy curves in Fig. 2.3 and fix the branch of the leading term $S_{-1} = \sqrt{Q(x)}$ of S so that $\sqrt{Q(x)} \sim -1/x^2$ holds as $x \to +\infty$. We set

$$c_k = \operatorname*{Res}_{x=b_k} \sqrt{Q(x)}, \quad \text{and} \quad \nu_k^{\pm} = \exp\bigl[i\pi(1 \pm 2 \operatorname*{Res}_{x=b_k} S_{\text{odd}})\bigr]$$

for $k = 0, 1, 2, 3, 4$. Note that $\nu_k^{\pm} = \exp i\pi(1 \pm \sqrt{4c_k^2\eta^2 + 1})$ hold and $\dfrac{1 \pm \sqrt{4c_k^2\eta^2+1}}{2}$ are the characteristic exponents of (1.1) at b_k (cf. [2], [4]). We have

$$\text{(2.3)} \qquad \operatorname{Re} c_0 < 0,\ \operatorname{Re} c_1 > 0,\ \operatorname{Re} c_2 < 0,\ \operatorname{Re} c_3 > 0,\ \operatorname{Re} c_4 > 0.$$

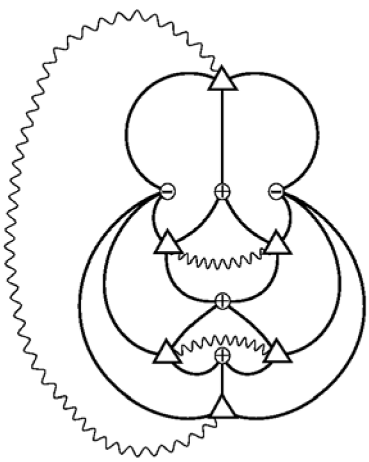

Fig. 2.3

In Fig. 2.3, we give the plus or the minus sign on each singular point b_k according to the signature of $-\operatorname{Re} c_k$ which will determine the dominance relation of WKB solutions at b_k (cf. Section 3.1).

Let γ_j be an oriented curve starting from x_0 and terminating at a_j shown by a dotted curve in Fig. 2.4.

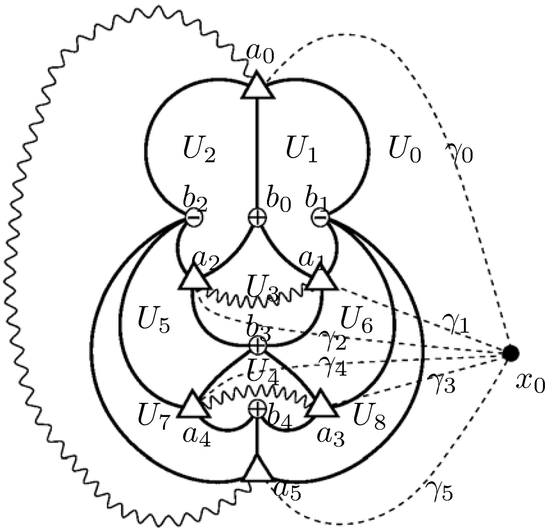

Fig. 2.4

Let U_j $(j = 0, 1, \ldots, 8)$ denote the connected components of the complement of the Stokes graph in $\mathbb{P}^1(\mathbb{C})$ as shown in Fig. 2.4.

To describe the monodromy matrices of C_k $(k = 0, 1, \ldots, 4)$, we define e_j, u_j and $u_{jj'}$ $(j, j' = 0, 1, \ldots 5)$ by

(2.4) $$e_j = \exp\Big(\int_{\gamma_j} S_{\mathrm{odd}} dx\Big),$$

(2.5) $$u_j = e_j^2 = \exp\Big(2\int_{\gamma_j} S_{\mathrm{odd}} dx\Big),$$

(2.6) $$u_{jj'} = u_j^{-1} u_{j'}$$

and set

(2.7) $$E_j = \begin{pmatrix} e_j & 0 \\ 0 & e_j^{-1} \end{pmatrix},$$

(2.8) $$D_k = \begin{pmatrix} \nu_k^+ & 0 \\ 0 & \nu_k^- \end{pmatrix},$$

(2.9)
$$U_{jj'} = \begin{pmatrix} u_{jj'} & 0 \\ 0 & u_{jj'}^{-1} \end{pmatrix},$$

(2.10)
$$V_{\pm} = \begin{pmatrix} 1 & 0 \\ \pm i & 0 \end{pmatrix}, \quad V^{\pm} = \begin{pmatrix} 1 & \pm i \\ 0 & 1 \end{pmatrix}.$$

Let $\varphi_{j,\pm}$ denote the WKB solutions normalized at a_j:

(2.11)
$$\varphi_{j,\pm} = \frac{1}{\sqrt{S_{\rm odd}}} \exp\Big(\pm \int_{a_j}^{x} S_{\rm odd} dx\Big).$$

§ 3. Computation of Monodromy Matrices

§ 3.1. The monodromy Matrix of C_0

We use the same notation as in the preceding sections. Let t_l ($l = 0, 1, 2, 3, 4$) denote the l-th crossing point of C_0 and Stokes curves as shown in Fig. 3.1.

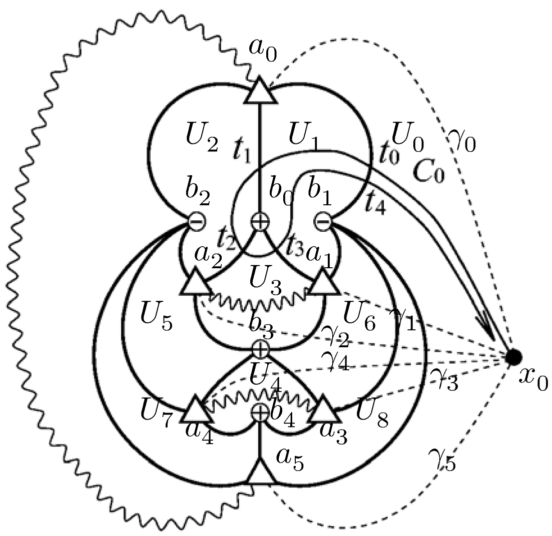

Fig. 3.1

Namely, C_0 crosses the Stokes curves in order $t_0 \to t_1 \to t_2 \to t_3 \to t_4$ and passes through the Stokes regions U_l ($l = 1, \dots, 8$) in order $U_0 \to U_1 \to U_2 \to U_3 \to U_1 \to U_0$. We compute the connection matrix at t_l for each l.

Let $\varphi^l_{j,\pm}$ and $\psi^l_{\pm}$ be the Borel sums of $\varphi_{j,\pm}$ and $\psi_{\pm}$ in the Stokes region U_l respectively. To determine $\psi^l_{\pm}$, we have to specify the path of integration in the WKB solutions, which is taken along C_0. The path C_0 passes the Stokes regions U_0 and U_1 twice. We denote by $\widetilde{\psi}^l_\pm$ the Borel sums of $\psi_\pm$ in U_l for the second pass ($l = 0, 1$). The Borel sums $\varphi^l_{j,\pm}$ are considered in a neighborhood of a_j.

Since C_0 crosses a Stokes curve emanating from a_0 at $x = t_0$ clockwise with respect to the center a_0. Since $\operatorname{Re} c_1 > 0$, we have $\operatorname{Re} \displaystyle\int_{a_0}^{x} \sqrt{Q(t)}\, dt < 0$ on the curve and φ_- is, by definition, dominant there. Hence the connection matrix between $(\varphi^0_{0,+}, \varphi^0_{0,-})$ and $(\varphi^1_{0,+}, \varphi^1_{0,-})$ is given as follows ([5]):

$$(\varphi^0_{0,+}, \varphi^0_{0,-}) = (\varphi^1_{0,+}, \varphi^1_{0,-}) V^-. \tag{3.1}$$

Two pairs of WKB solutions $(\varphi_{0,+}, \varphi_{0,-})$ and (ψ_+, ψ_-) are related by

$$(\psi_+, \psi_-) = (\varphi_{0,+}, \varphi_{0,-}) E_0 \tag{3.2}$$

near $x = t_0$ and hence we have

$$(\psi^l_+, \psi^l_-) = (\varphi^l_{0,+}, \varphi^l_{0,-}) E_0 \tag{3.3}$$

for $l = 0, 1$. Here, in Eq. (3.3), we take the Borel sum of E_0, which is expressed by the same letter. For such constant matrices as E_j, $U_{jj'}$, we employ the same convention. Namely, the Borel sums of E_j and $U_{jj'}$ are respectively denoted by the same letters. Note that they do not depend on the choice of the Stokes region. Combining (3.1) with (3.3), we obtain

$$(\psi^0_+, \psi^0_-) = (\psi^1_+, \psi^1_-) E_0^{-1} V^- E_0. \tag{3.4}$$

Thus we have the connection matrix $T^{(0)}_0 = E_0^{-1} V^- E_0$ for the Borel sums of (ψ_+, ψ_-) at $x = t_0$. In a similar way, we obtain the connection matrix $T^{(0)}_1 = E_0^{-1} V_- E_0$ at $x = t_1$. We note that $\operatorname{Re} c_0 < 0$ and that $\operatorname{Re} \displaystyle\int_{a_0}^{x} \sqrt{Q(t)}\, dt > 0$ holds on the Stokes curve connecting a_0 with b_0 and hence φ_+ is dominant there.

Next we compute the connection matrix at $x = t_2$. Let us consider a closed oriented curve consisting of the portion of C_0 from x_0 through t_2, the portion of the Stokes curve from t_2 through a_2, and γ_2^{-1}. It encircles b_0, b_1, and the cut connecting a_2 with a_1 counterclockwise. Therefore we have the relation

$$(\psi_+, \psi_-) = (\varphi_{2,+}, \varphi_{2,-}) E_2 D_0 D_1 U_{21} \tag{3.5}$$

near $x = t_2$ and hence we obtain

$$(\psi^l_+, \psi^l_-) = (\varphi^l_{2,+}, \varphi^l_{2,-}) E_2 D_0 D_1 U_{21}, \tag{3.6}$$

for $l = 2, 3$.

On the Stokes curve connecting a_2 with b_0, φ_+ is dominant and C_0 crosses the curve clockwise with respect to a_2. Thus we have the connection matrix between $(\varphi^2_{2,+}, \varphi^2_{2,-})$ and $(\varphi^3_{2,+}, \varphi^3_{2,-})$:

$$(\varphi^2_{2,+}, \varphi^2_{2,-}) = (\varphi^3_{2,+}, \varphi^3_{2,-})V_-. \tag{3.7}$$

Combining this with (3.6), we find

$$(\psi^2_+, \psi^2_-) = (\psi^3_+, \psi^3_-)(E_2D_0D_1U_{21})^{-1}V_-(E_2D_0D_1U_{21}). \tag{3.8}$$

Thus we obtain the connection matrix $T^{(0)}_2 = (E_2D_0D_1U_{21})^{-1}V_-(E_2D_0D_1U_{21})$ at t_2.

In a similar manner, we have

$$(\psi^3_+, \psi^3_-) = (\psi^4_+, \psi^4_-)T^{(0)}_3 \tag{3.9}$$

with $T^{(0)}_3 = (E_1D_0D_1)^{-1}V_-(E_1D_0D_1)$ and

$$(\psi^4_+, \psi^4_-) = (\widetilde{\psi}^0_+, \widetilde{\psi}^0_-)T^{(0)}_4 \tag{3.10}$$

with $T^{(0)}_4 = (E_0D_0)^{-1}V^+(E_0D_0)$. The Borel sum $(\widetilde{\psi}^0_+, \widetilde{\psi}^0_-)$ of (termwise) continuation of the WKB solution (ψ_+, ψ_-) along C_0 gains the local monodromy at b_0:

$$(\tilde{\psi}^0_+, \tilde{\psi}^0_-) = (\psi^0_+, \psi^0_-)D_0. \tag{3.11}$$

Hence we have the relation

$$(\psi^0_+, \psi^0_-)_{C_0} = (\psi^0_+, \psi^0_-)D_0T^{(0)}_4T^{(0)}_3T^{(0)}_2T^{(0)}_1T^{(0)}_0, \tag{3.12}$$

where $(\psi^0_+, \psi^0_-)_{C_0}$ designates the analytic continuation of (ψ^0_+, ψ^0_-) along C_0. Therefore the monodromy matrix M_0 of C_0 is given by

$$M_0 = D_0T^{(0)}_4T^{(0)}_3T^{(0)}_2T^{(0)}_1T^{(0)}_0, \tag{3.13}$$

or explicitly,

$$\begin{pmatrix} \nu^+_0 + \nu^-_0 + \dfrac{(u_2u^2_{21} + u_1)\nu^+_0\nu^+_1}{u_0\nu^-_1} & -\dfrac{i(u_2u^2_{21} + u_1)\nu^+_1\nu^+_0}{u^2_0\nu^-_1} - \dfrac{i\nu^+_0}{u_0} \\ -iu_0\nu^-_0 - \dfrac{i(u_2u^2_{21} + u_1)\nu^+_0\nu^+_1}{\nu^-_1} & -\dfrac{(u_2u^2_{21} + u_1)\nu^+_0\nu^+_1}{u_0\nu^-_1} \end{pmatrix}. \tag{3.14}$$

§ 3.2. The Monodromy Matrix of C_k for $k \geq 1$

In a similar manner, we can compute the monodromy matrix M_k of the closed curve C_k for $k \geq 1$ with respect to the Borel sums of the WKB solutions (2.1). We omit the details and only give the results:

$$M_1 = D_1T^{(1)}_3T^{(1)}_2T^{(1)}_1T^{(1)}_0 \tag{3.15}$$

with

$$T_0^{(1)} = E_0^{-1} V^- E_0, \tag{3.16}$$

$$T_1^{(1)} = (E_1 D_1)^{-1} V^- (E_1 D_1), \tag{3.17}$$

$$T_2^{(1)} = (E_3 D_1)^{-1} V^- (E_3 D_1), \tag{3.18}$$

$$T_3^{(1)} = (E_5 D_1)^{-1} V^- (E_5 D_1). \tag{3.19}$$

$$M_2 = D_2 T_7^{(2)} T_6^{(2)} T_5^{(2)} T_4^{(2)} T_3^{(2)} T_2^{(2)} T_1^{(2)} T_0^{(2)} \tag{3.20}$$

with

$$T_0^{(2)} = E_0^{-1} V^- E_0, \tag{3.21}$$

$$T_1^{(2)} = E_0^{-1} V_- E_0, \tag{3.22}$$

$$T_2^{(2)} = E_0^{-1} V^- E_0, \tag{3.23}$$

$$T_3^{(2)} = (E_5 D_0 D_1 D_2 D_3 D_4 U_{12} U_{34})^{-1} V^- (E_5 D_0 D_1 D_2 D_3 D_4 U_{12} U_{34}), \tag{3.24}$$

$$T_4^{(2)} = (E_4 D_0 D_1 D_2 D_3 U_{\ 21})^{-1} V^- (E_4 D_0 D_1 D_2 D_3 U_{21}), \tag{3.25}$$

$$T_5^{(2)} = (E_2 D_0 D_1 D_2 U_{21})^{-1} V^- (E_2 D_0 D_1 D_2 U_{21}), \tag{3.26}$$

$$T_6^{(2)} = (E_0 D_2)^{-1} V_+ (E_0 D_2), \tag{3.27}$$

$$T_6^{(2)} = (E_0 D_2)^{-1} V^+ (E_0 D_{\ 2}). \tag{3.28}$$

$$M_3 = D_3 T_7^{(3)} T_6^{(3)} T_5^{(3)} T_4^{(3)} T_3^{(3)} T_2^{(3)} T_1^{(3)} T_0^{(3)} \tag{3.29}$$

with

$$T_0^{(3)} = E_5^{-1} V^+ E_5, \tag{3.30}$$

$$T_1^{(3)} = E_3^{-1} V^+ E_3, \tag{3.31}$$

$$T_2^{(3)} = E_1^{-1} V_- E_1, \tag{3.32}$$

$$T_3^{(3)} = E_2^{-1} V_- E_2, \tag{3.33}$$

$$T_4^{(3)} = (E_4 D_3)^{-1} V_- (E_4 D_3), \tag{3.34}$$

$$T_5^{(3)} = (E_3 D_3)^{-1} V_- (E_3 D_3), \tag{3.35}$$

$$T_6^{(3)} = (E_3 D_3)^{-1} V^- (E_3 D_3), \tag{3.36}$$

$$T_7^{(3)} = (E_5 D_3)^{-1} V^- (E_5 D_3). \tag{3.37}$$

$$M_4 = D_4 T_4^{(4)} T_3^{(4)} T_2^{(4)} T_1^{(4)} T_0^{(4)} \tag{3.38}$$

with

$$T_0^{(4)} = E_5^{-1} V^+ E_5, \tag{3.39}$$

$$T_1^{(4)} = E_3^{-1} V_- E_3, \tag{3.40}$$

$$T_2^{(4)} = (E_4 U_{43})^{-1} V_- (E_4 U_{43}), \tag{3.41}$$

$$T_3^{(4)} = (E_5 D_4)^{-1} V_- (E_5 D_4), \tag{3.42}$$

$$T_4^{(4)} = (E_5 D_4)^{-1} V^- (E_5 D_4). \tag{3.43}$$

Explicit forms:

$$M_1 = \begin{pmatrix} \nu_1^+ & -\frac{i(u_1u_3 + u_5u_3 + u_1u_5)\nu_1^-}{u_1u_3u_5} - \frac{i\nu_1^+}{u_0} \\ 0 & \nu_1^- \end{pmatrix},$$

$$M_2 = \begin{pmatrix} \nu_2^- & 0 \\ -\frac{i\nu_0^-\nu_1^-\nu_2^-\nu_3^-\nu_4^- u_0^2}{u_5 u_{21}^2 u_{34}^2 \nu_0^+\nu_1^+\nu_3^+\nu_4^+} - \frac{i\nu_2^-(u_2u_4\nu_0^+\nu_1^+\nu_3^+u_{21}^2 + u_0u_2\nu_0^+\nu_1^-\nu_3^- + u_0u_4\nu_0^+\nu_1^-\nu_3^+)u_0}{u_2u_4u_{21}^2\nu_0^+\nu_1^+\nu_3^+} & \nu_2^+ \end{pmatrix},$$

$$M_3 = \begin{pmatrix} -\frac{(u_1+u_2)(u_3+u_5)\nu_3^- + (u_3^2+u_4u_3+u_4u_5)\nu_3^+}{u_3u_5} & -\frac{i(u_3+u_5)(u_1u_3+u_2u_3+u_5u_3+u_1u_5+u_2u_5)\nu_3^- + i(u_3+u_5)(u_3^2+u_4u_3+u_4u_5)\nu_3^+}{u_3^2u_5^2} \\ -i(u_1+u_2)\nu_3^- - i(u_3+u_4)\nu_3^+ & \frac{(u_1u_3+u_2u_3+u_5u_3+u_1u_5+u_2u_5)\nu_3^- + (u_3+u_4)(u_3+u_5)\nu_3^+}{u_3u_5} \end{pmatrix},$$

$$M_4 = \begin{pmatrix} -\frac{(u_3u_{34}^2 + u_4)\nu_4^-}{u_5u_{34}^2} & -\frac{i(u_3u_{34}^2 + u_5u_{34}^2 + u_4)\nu_4^-}{u_5^2u_{34}^2} \\ -\frac{i(u_3u_{34}^2 + u_4)\nu_4^-}{u_{34}^2} - iu_5\nu_4^+ & \frac{(u_3u_{34}^2 + u_5u_{34}^2 + u_4)\nu_4^-}{u_5u_{34}^2} + \nu_4^+ \end{pmatrix}.$$

Remark. The product $C_2C_0C_1C_3C_4$ is equal to the unit element in the fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{b_0, b_1, b_2, b_3, b_4\}, x_0)$. The product of M_j's in the reverse order becomes

$$M_4M_3M_1M_0M_2 = \begin{pmatrix} -u_0u_1^{-1}u_2u_3u_4^{-1}u_5^{-1}\nu_0^-\nu_1^-\nu_2^-\nu_3^-\nu_4^- & 0 \\ 0 & -u_0^{-1}u_1u_2^{-1}u_3^{-1}u_4u_5\nu_0^+\nu_1^+\nu_2^+\nu_3^+\nu_4^+ \end{pmatrix}$$

and we can confirm that this is equal to the identity matrix by using the relations

$$\nu_k^+\nu_k^- = 1 \quad \text{and} \quad u_{21}u_{34}u_{05}\nu_0^+\nu_1^+\nu_2^+\nu_3^+\nu_4^+ = -1.$$

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