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Multiple-Scale Analysis for Higher-Order Painlevé Equations

By

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Abstract

Instanton-type solutions to the second member of the first Painlevé hierarchy are constructed by using multiple-scale analysis. Some leading terms are explicitly given.

§ 1. Introduction

The purpose of this paper is to construct a family of formal solutions to the second member of the first Painlevé hierarchy with a large parameter. Formal solutions containing arbitrary constants to a nonlinear differential equation which has a large parameter in an appropriate manner necessarily admit exponential terms. Such a family of formal solutions was first constructed in [1], [2], [3] for the classical Painlevé equations (P_J) $(J = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{IV})$ with a large parameter and the solutions are called instanton-type solutions. The main tool of the construction was the multiple-scale analysis. On the other hand, Takei [11] found another method of the construction which was based on the Hamiltonian structure of the Painlevé equations. His method is quite universal and it can be applied to the Painlevé hierarchies since they have Hamiltonian structures. Although his method is general, actual computation by his method requires careful analysis. In this paper, we show that the multiple-scale analysis works also for the second member of the first Painlevé hierarchy to construct formal solutions containing four free constants. One of the advantages of our method is that the computation can be implemented easily by using computer algebras such as *Mathematica*. Instanton-type solutions play a role in the reduction theory of Painlevé hierarchies near a simple P-turning point (cf. [7]).

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§ 2. The First Painlevé Hierarchy with a Large Parameter

The first Painlevé hierarchy was introduced by several mathematicians from various viewpoints [8], [9], [5], [10]. Here we employ the following expression due to [12] which contains a large parameter η:

$$(P_{\mathrm{I}})_m \qquad \begin{cases} \dfrac{du_j}{dt} = 2\eta v_j, \\ \dfrac{dv_j}{dt} = 2\eta(u_{j+1} + u_1 u_j + w_j), \end{cases} \qquad (j = 1, 2, \ldots, m),$$

where t is the independent variable, u_j, v_j are the dependent variables with the conventional assumption $u_{m+1} \equiv 0$ and w_j denotes a polynomial of $\{u_k, v_l\}$ defined by the recurrence relation

$$(2.1) \qquad w_j = \frac{1}{2}\sum_{k=1}^{j} u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2}\sum_{k=1}^{j-1} v_k v_{j-k} + c_j + \delta_{jm} t.$$

Here c_j is a constant and δ_{jm} designates the Kronecker delta. If $m = 1$, $(P_{\mathrm{I}})_m$ recovers the classical first Painlevé equation. We are interested in the second member, namely, the case where $m = 2$. Eliminating u_2, v_j, w_j $(j = 1, 2)$ and taking $t + c_2$ as a new variable t, we obtain the following fourth order differential equation for u_1, hereafter in this article referred to as $(P_{\mathrm{I}})_2$:

$$(P_{\mathrm{I}})_2 \qquad \frac{d^4 u_1}{dt^4} = \eta^2\Big(20\, u_1 \frac{d^2 u_1}{dt^2} + 10\Big(\frac{du_1}{dt}\Big)^2\Big) + \eta^4(-40\, u_1^3 - 16\, c\, u_1 + 16\, t),$$

where $c = c_1$ is a complex constant.

§ 3. Zero-Parameter Solutions

Firstly we consider a formal solution of $(P_{\mathrm{I}})_2$ which has an expansion in the negative powers of the large parameter:

$$(3.1) \qquad u_1 = u_{1,0} + \eta^{-1} u_{1,1} + \eta^{-2} u_{1,2} + \cdots .$$

Putting this expression into $(P_{\mathrm{I}})_2$ and comparing the coefficients of the like powers in η, we find that the leading term $u_{1,0}$ should satisfy

$$(3.2) \qquad -40\, u_{1,0}^3 - 16\, c\, u_{1,0} + 16\, t = 0$$

and the higher-order terms $u_{1,j}$ $(j > 0)$ can be determined uniquely once $u_{1,0}$ is given. There are three roots of Eq. (3.2) for general c and t. Hence we have three formal solutions of $(P_{\mathrm{I}})_2$ which have the form (3.1). We call them the zero-parameter solutions of $(P_{\rm I})_2$. These solutions do not have any free parameter and hence they are very special formal solutions: General formal solution of $(P_{\rm I})_2$ should contain four free parameters, for the equation is of fourth order.

§ 4. Multiple-Scale Analysis

Let θ_0 be a root of the cubic equation $40\theta_0^3 + 16c\theta_0 - 16t = 0$ and set

$$u_1 = \theta_0 + \eta^{-1/2}\Theta. \tag{4.1}$$

We regard Θ as a new dependent variable. Then Θ should satisfy the following differential equation which is equivalent to $(P_{\rm I})_2$:

$$\begin{aligned} \frac{d^4\Theta}{dt^4} &= \eta^2\, 20\,\theta_0 \frac{d^2\Theta}{dt^2} + \eta^4\,(-120\,\theta_0^2 - 16\,c)\Theta \\ &\quad + \eta^{7/2}(-120\,\theta_0\Theta^2 + 20\,\theta_0\theta_0'' + 10\,(\theta_0')^2) + \eta^3(-40\Theta^3) \\ &\quad + \eta^2(20\,\theta_0'\frac{d\Theta}{dt} + 20\,\theta_0''\Theta) + \eta^{3/2}\Big(20\,\Theta\frac{d^2\Theta}{dt^2} + 10\Big(\frac{d\Theta}{dt}\Big)^2\Big) - \eta^{1/2}\theta_0''''. \end{aligned} \tag{4.2}$$

Multiple-scale analysis seeks for a solution to (4.2) of the form

$$\Theta(t) = \widetilde{\Theta}(t, \tau_1, \tau_2)\Big|_{\tau_2 = \eta\phi_2}^{\tau_1 = \eta\phi_1}, \tag{4.3}$$

where $\phi_j = \displaystyle\int \nu_j dt$ $(j = 1, 2)$ with

$$\begin{cases} \nu_1^2 = 10\theta_0 + 2\sqrt{\Delta}, \\ \nu_2^2 = 10\theta_0 - 2\sqrt{\Delta}, \end{cases} \qquad (\Delta = -5\theta_0^2 - 4c). \tag{4.4}$$

Note that $\pm\nu_1$, $\pm\nu_2$ are the characteristic roots of the leading part of (4.2), that is, $\nu = \pm\nu_j$ solve

$$\nu^4 - 20\theta_0\nu^2 + 120\theta_0^2 + 16c = 0. \tag{4.5}$$

For a function $\Theta(t)$ of the form (4.3), we have

$$\frac{d\Theta}{dt} = \frac{\partial\widetilde{\Theta}}{\partial t} + \eta\nu_1\frac{\partial\widetilde{\Theta}}{\partial\tau_1} + \eta\nu_2\frac{\partial\widetilde{\Theta}}{\partial\tau_2}. \tag{4.6}$$

where τ_1 and τ_2 are restricted respectively to $\eta\phi_1$ and $\eta\phi_2$ in the right-hand side. If $\widetilde{\Theta}$ solves the equation which is obtained by replacing $\dfrac{d}{dt}$ in (4.2) by $\dfrac{\partial}{\partial t} + \eta\nu_1\dfrac{\partial}{\partial\tau_1} + \eta\nu_2\dfrac{\partial}{\partial\tau_2}$, then $\Theta(t) = \widetilde{\Theta}(t, \tau_1, \tau_2)\Big|_{\tau_2 = \eta\phi_2}^{\tau_1 = \eta\phi_1}$ is a solution to (4.2). To describe the equation for $\widetilde{\Theta}$, we set $P = P_1 + P_2 + P_3$ with

$$P_1 = \nu_1^4\frac{\partial^4}{\partial\tau_1^4} - 20\theta_0\nu_1^2\frac{\partial^2}{\partial\tau_1^2} + 120\theta_0^2 + 16c, \tag{4.7}$$

$$P_2 = \nu_2^4 \frac{\partial^4}{\partial \tau_2^4} - 20\theta_0 \nu_2^2 \frac{\partial^2}{\partial \tau_2^2} + 120\theta_0^2 + 16c, \tag{4.8}$$

and

$$\begin{aligned} P_3 = {} & 4\nu_1^3\nu_2 \frac{\partial^4}{\partial \tau_1^3 \partial \tau_2} + 6\nu_1^2\nu_2^2 \frac{\partial^4}{\partial \tau_1^2 \partial \tau_2^2} + 4\nu_1\nu_2^3 \frac{\partial^4}{\partial \tau_1 \partial \tau_2^3} \\ & - 40\theta_0\nu_1\nu_2 \frac{\partial^2}{\partial \tau_1 \partial \tau_2} - (120\theta_0^2 + 16c). \end{aligned} \tag{4.9}$$

By the definition, P_1 and P_2 are factorized as follows:

$$P_1 = \nu_1^2 \Big(\nu_1^2 \frac{\partial^2}{\partial \tau_1^2} - \nu_2^2\Big)\Big(\frac{\partial^2}{\partial \tau_1^2} - 1\Big), \tag{4.10}$$

$$P_2 = \nu_2^2 \Big(\nu_2^2 \frac{\partial^2}{\partial \tau_2^2} - \nu_1^2\Big)\Big(\frac{\partial^2}{\partial \tau_2^2} - 1\Big). \tag{4.11}$$

Since the equation for $\widetilde{\Theta}$ has quite a complicated form, we do not give the full form but some of higher-order terms with respect to η:

$$\begin{aligned} P\widetilde{\Theta} = {} & \eta^{-1/2}\Big(-120\theta_0\widetilde{\Theta}^2 + 10\nu_1^2\Big(\frac{\partial \widetilde{\Theta}}{\partial \tau_1}\Big)^2 + 20\nu_1\nu_2 \frac{\partial \widetilde{\Theta}}{\partial \tau_1}\frac{\partial \widetilde{\Theta}}{\partial \tau_2} + 10\nu_2^2\Big(\frac{\partial \widetilde{\Theta}}{\partial \tau_2}\Big)^2 \\ & \qquad + 20\nu_2^2\widetilde{\Theta}\frac{\partial^2 \widetilde{\Theta}}{\partial \tau_2^2} + 40\nu_1\nu_2\widetilde{\Theta}\frac{\partial^2 \widetilde{\Theta}}{\partial \tau_1 \partial \tau_2} + 20\nu_1^2\widetilde{\Theta}\frac{\partial^2 \widetilde{\Theta}}{\partial \tau_1^2}\Big) \\ & + \eta^{-1}\Big(-40\widetilde{\Theta}^3 + 20(\theta_0'\nu_1 + \theta_0\nu_1')\frac{\partial \widetilde{\Theta}}{\partial \tau_1} + 20(\theta_0'\nu_2 + \theta_0\nu_2')\frac{\partial \widetilde{\Theta}}{\partial \tau_2} \\ & \qquad + 40\theta_0\nu_1\frac{\partial^2 \widetilde{\Theta}}{\partial t \partial \tau_1} + 40\theta_0\nu_2\frac{\partial^2 \widetilde{\Theta}}{\partial t \partial \tau_2} - 6\nu_1^2\nu_1'\frac{\partial^3 \widetilde{\Theta}}{\partial \tau_1^3} - 6\nu_2^2\nu_2'\frac{\partial^3 \widetilde{\Theta}}{\partial \tau_2^3} \\ & \qquad - 6\nu_1(2\nu_1'\nu_2 + \nu_1\nu_2')\frac{\partial^3 \widetilde{\Theta}}{\partial \tau_1^2 \partial \tau_2} - 6\nu_2(2\nu_1\nu_2' + \nu_1'\nu_2)\frac{\partial^3 \widetilde{\Theta}}{\partial \tau_1 \partial \tau_2^2} \\ & \qquad - 4\nu_1^3\frac{\partial^4 \widetilde{\Theta}}{\partial t \partial \tau_1^3} - 12\nu_1^2\nu_2\frac{\partial^4 \widetilde{\Theta}}{\partial t \partial \tau_1^2 \partial \tau_2} - 12\nu_1\nu_2^2\frac{\partial^4 \widetilde{\Theta}}{\partial t \partial \tau_1 \partial \tau_2^2} - 4\nu_2^3\frac{\partial^4 \widetilde{\Theta}}{\partial t \partial \tau_2^3}\Big) \\ & + O(\eta^{-3/2}). \end{aligned} \tag{4.12}$$

We assume that $\widetilde{\Theta}$ has an expansion of the form

$$\widetilde{\Theta} = \widetilde{\Theta}_0 + \eta^{-1/2}\widetilde{\Theta}_{1/2} + \eta^{-1}\widetilde{\Theta}_1 + \cdots. \tag{4.13}$$

Putting this into Eq. (4.12) and comparing the coefficients of the like powers of η, we find a system of differential equations for $\widetilde{\Theta}_{k/2}$ $(k = 0, 1, 2, \dots)$. First three are

$$P\widetilde{\Theta}_0 = 0, \tag{4.14}$$

$$P\widetilde{\Theta}_{1/2} = -120\theta_0\widetilde{\Theta}_0^2 + 10\nu_1^2\Big(\frac{\partial\widetilde{\Theta}_0}{\partial\tau_1}\Big)^2 + 20\nu_1\nu_2\frac{\partial\widetilde{\Theta}_0}{\partial\tau_1}\frac{\partial\widetilde{\Theta}_0}{\partial\tau_2} + 10\nu_2^2\Big(\frac{\partial\widetilde{\Theta}_0}{\partial\tau_2}\Big)^2$$
$$+ 20\nu_1^2\widetilde{\Theta}_0\frac{\partial^2\widetilde{\Theta}_0}{\partial\tau_1^2} + 40\nu_1\nu_2\widetilde{\Theta}_0\frac{\partial^2\widetilde{\Theta}_0}{\partial\tau_1\partial\tau_2} + 20\nu_2^2\widetilde{\Theta}_0\frac{\partial^2\widetilde{\Theta}_0}{\partial\tau_2^2} \tag{4.15}$$

and

(4.16)

$$\begin{aligned}
P\widetilde{\Theta}_1 = &-240\theta_0\widetilde{\Theta}_0\widetilde{\Theta}_{1/2} + 20\nu_1^2\frac{\partial\widetilde{\Theta}_0}{\partial\tau_1}\frac{\partial\widetilde{\Theta}_{1/2}}{\partial\tau_1} + 20\nu_1\nu_2\frac{\partial\widetilde{\Theta}_0}{\partial\tau_1}\frac{\partial\widetilde{\Theta}_{1/2}}{\partial\tau_2} + 20\nu_1\nu_2\frac{\partial\widetilde{\Theta}_{1/2}}{\partial\tau_1}\frac{\partial\widetilde{\Theta}_0}{\partial\tau_2}\\
&+ 20\nu_2^2\frac{\partial\widetilde{\Theta}_0}{\partial\tau_2}\frac{\partial\widetilde{\Theta}_{1/2}}{\partial\tau_2} + 20\nu_2^2\widetilde{\Theta}_0\frac{\partial^2\widetilde{\Theta}_{1/2}}{\partial\tau_2^2} + 20\nu_2^2\widetilde{\Theta}_{1/2}\frac{\partial^2\widetilde{\Theta}_0}{\partial\tau_2^2} + 40\nu_1\nu_2\widetilde{\Theta}_0\frac{\partial^2\widetilde{\Theta}_{1/2}}{\partial\tau_1\partial\tau_2}\\
&+ 40\nu_1\nu_2\widetilde{\Theta}_{1/2}\frac{\partial^2\widetilde{\Theta}_0}{\partial\tau_1\partial\tau_2} + 20\nu_1^2\widetilde{\Theta}_0\frac{\partial^2\widetilde{\Theta}_{1/2}}{\partial\tau_1^2} + 20\nu_1^2\widetilde{\Theta}_{1/2}\frac{\partial^2\widetilde{\Theta}_0}{\partial\tau_1^2} - 40\widetilde{\Theta}_0^3\\
&+ 20(\theta_0'\nu_1 + \theta_0\nu_1')\frac{\partial\widetilde{\Theta}_0}{\partial\tau_1} + 20(\theta_0'\nu_2 + \theta_0\nu_2')\frac{\partial\widetilde{\Theta}_0}{\partial\tau_2} + 40\theta_0\nu_1\frac{\partial^2\widetilde{\Theta}_0}{\partial t\partial\tau_1}\\
&+ 40\theta_0\nu_2\frac{\partial^2\widetilde{\Theta}_0}{\partial t\partial\tau_2} - 6\nu_1^2\nu_1'\frac{\partial^3\widetilde{\Theta}_0}{\partial\tau_1^3} - 6\nu_2^2\nu_2'\frac{\partial^3\widetilde{\Theta}_0}{\partial\tau_2^3}\\
&- 6\nu_1(2\nu_1'\nu_2 + \nu_1\nu_2')\frac{\partial^3\widetilde{\Theta}_0}{\partial\tau_1^2\partial\tau_2} - 6\nu_2(2\nu_1\nu_2' + \nu_1'\nu_2)\frac{\partial^3\widetilde{\Theta}_0}{\partial\tau_1\partial\tau_2^2}\\
&- 4\nu_1^3\frac{\partial^4\widetilde{\Theta}_0}{\partial t\partial\tau_1^3} - 4\nu_2^3\frac{\partial^4\widetilde{\Theta}_0}{\partial t\partial\tau_2^3} - 12\nu_1^2\nu_2\frac{\partial^4\widetilde{\Theta}_0}{\partial t\partial\tau_1^2\partial\tau_2} - 12\nu_1\nu_2^2\frac{\partial^4\widetilde{\Theta}_0}{\partial t\partial\tau_1\partial\tau_2^2}.
\end{aligned}$$

It follows from Eqs. (4.10), (4.11) and the definition of P that

$$\widetilde{\Theta}_0 = a_0^{(1,0)}(t)e^{\tau_1} + a_0^{(-1,0)}(t)e^{-\tau_1} + a_0^{(0,1)}(t)e^{\tau_2} + a_0^{(0,-1)}(t)e^{-\tau_2} \tag{4.17}$$

is a solution to (4.14), where $a_0^{(\pm1,0)}(t)$ and $a_0^{(0,\pm1)}(t)$ are arbitrary functions of t. They will be determined later. Put Eq. (4.17) into the right-hand side of Eq. (4.15) and try to find a solution to the inhomogeneous differential equation. For every $j,\ k\in\mathbb{Z}$, we have

$$Pe^{j\tau_1+k\tau_2} = p(j\nu_1 + k\nu_2)e^{j\tau_1+k\tau_2},$$

where we set

$$p(\nu) = \nu^4 - 20\theta_0\nu^2 + 120\theta_0^2 + 16c.$$

Hence $p(j\nu_1 + k\nu_2)$ never vanishes for general t if $(j,k)\neq(\pm1,0),\ (0,\pm1)$. Thus we can take a solution to (4.15) of the form

$$\begin{aligned}
\widetilde{\Theta}_{1/2} = &\ a_{1/2}^{(2,0)}(t)e^{2\tau_1} + a_{1/2}^{(-2,0)}(t)e^{-2\tau_1} + a_{1/2}^{(0,2)}(t)e^{2\tau_2} + a_{1/2}^{(0,-2)}(t)e^{-2\tau_2}\\
&+ a_{1/2}^{(1,1)}(t)e^{\tau_1+\tau_2} + a_{1/2}^{(1,-1)}(t)e^{\tau_1-\tau_2} + a_{1/2}^{(-1,1)}(t)e^{-\tau_1+\tau_2}\\
&+ a_{1/2}^{(-1,-1)}(t)e^{-\tau_1-\tau_2} + a_{1/2}^{(0,0)}(t).
\end{aligned} \tag{4.18}$$

The coefficients $a_{1/2}^{(j,k)}(t)$ are uniquely determined and they can be written in terms of $a_0^{(j,k)}(t)$. For example, we have

$$a_{1/2}^{(2,0)}(t) = -\frac{10(4\theta_0 - \nu_1^2)a_0^{(1,0)}(t)^2}{\nu_1^2(4\nu_1^2 - \nu_2^2)}. \tag{4.19}$$

Since we have, by the definition of ν_j,

$$4\theta_0 - \nu_1^2 = -2(3\theta_0 + \sqrt{\Delta}),$$

and

$$4\nu_1^2 - \nu_2^2 = 10(3\theta_0 + \sqrt{\Delta}).$$

Equation (4.19) is reduced to

$$a_{1/2}^{(2,0)}(t) = \frac{2a_0^{(1,0)}(t)^2}{\nu_1^2}. \tag{4.20}$$

We put thus obtained $\widetilde{\Theta}_{1/2}$ and (4.17) into the right-hand side of (4.16), which will be a sum of exponential terms $e^{j\tau_1 + k\tau_2}$ ($|j| + |k| \le 3$) with coefficients being functions of t expressed in terms of $a_0^{(j,k)}$, $a_{1/2}^{(j,k)}$, ν_j, θ_0 and Δ. The conditions for the absence of secular behavior (cf. [4]) are that the coefficients of $e^{\pm\tau_1}$ and $e^{\pm\tau_2}$ vanish. After some computations, we can write down the conditions as follows:

$$\frac{da_0^{(1,0)}}{dt} = -\frac{1}{2}\Big(\frac{\nu_1'}{\nu_1} + \frac{(\sqrt{\Delta})'}{\sqrt{\Delta}}\Big)a_0^{(1,0)} - f_1(t)(a_0^{(1,0)})^2 a_0^{(-1,0)} - f_2(t)a_0^{(1,0)}a_0^{(0,1)}a_0^{(0,-1)}, \tag{4.21}$$

$$\begin{aligned}\frac{da_0^{(-1,0)}}{dt} &= -\frac{1}{2}\Big(\frac{\nu_1'}{\nu_1} + \frac{(\sqrt{\Delta})'}{\sqrt{\Delta}}\Big)a_0^{(-1,0)} + f_1(t)a_0^{(1,0)}(a_0^{(-1,0)})^2 \\ &\quad + f_2(t)a_0^{(-1,0)}a_0^{(0,1)}a_0^{(0,-1)},\end{aligned} \tag{4.22}$$

$$\frac{da_0^{(0,1)}}{dt} = -\frac{1}{2}\Big(\frac{\nu_2'}{\nu_2} + \frac{(\sqrt{\Delta})'}{\sqrt{\Delta}}\Big)a_0^{(0,1)} - f_3(t)(a_0^{(0,1)})^2 a_0^{(0,-1)} - f_4(t)a_0^{(1,0)}a_0^{(-1,0)}a_0^{(0,1)}, \tag{4.23}$$

$$\begin{aligned}\frac{da_0^{(0,-1)}}{dt} &= -\frac{1}{2}\Big(\frac{\nu_2'}{\nu_2} + \frac{(\sqrt{\Delta})'}{\sqrt{\Delta}}\Big)a_0^{(0,-1)} + f_3(t)a_0^{(0,1)}(a_0^{(0,-1)})^2 \\ &\quad + f_4(t)a_0^{(1,0)}a_0^{(-1,0)}a_0^{(0,-1)}.\end{aligned} \tag{4.24}$$

Here $f_j(t)$ ($j = 1, 2, 3, 4$) are given by

$$f_1(t) = 40\frac{10\theta_0^2 + 7\theta_0\sqrt{\Delta} - 5\Delta}{\nu_1^3\nu_2^2\sqrt{\Delta}}, \tag{4.25}$$

$$f_2(t) = 160\frac{5\theta_0^2+\Delta}{\nu_1^3\nu_2^2\sqrt{\Delta}}, \tag{4.26}$$

$$f_3(t) = -40\frac{10\theta_0^2-7\theta_0\sqrt{\Delta}-5\Delta}{\nu_1^2\nu_2^3\sqrt{\Delta}}, \tag{4.27}$$

$$f_4(t) = -160\frac{5\theta_0^2+\Delta}{\nu_1^2\nu_2^3\sqrt{\Delta}}. \tag{4.28}$$

The following relation holds between f_2 and f_4:

$$f_2\nu_1 + f_4\nu_2 = 0. \tag{4.29}$$

Let us solve the system of differential equations (4.21)–(4.24). Multiplying Eqs. (4.21) and (4.22) respectively by $a_0^{(-1,0)}$ and $a_0^{(1,0)}$ and adding them up, we have a simple differential equation for the product $a_0^{(1,0)}a_0^{(-1,0)}$:

$$\frac{d}{dt}(a_0^{(1,0)}a_0^{(-1,0)}) = -\Big(\frac{\nu_1'}{\nu_1}+\frac{(\sqrt{\Delta})'}{\sqrt{\Delta}}\Big)a_0^{(1,0)}a_0^{(-1,0)}. \tag{4.30}$$

This can be integrated easily and we have

$$a_0^{(1,0)}a_0^{(-1,0)} = C_1(\nu_1\sqrt{\Delta})^{-1}, \tag{4.31}$$

where C_1 is a constant. Similarly, it follows from Eqs. (4.23) and (4.24) that

$$a_0^{(0,1)}a_0^{(0,-1)} = C_2(\nu_2\sqrt{\Delta})^{-1}, \tag{4.32}$$

where C_2 is a constant. Going back to Eqs. (4.21)–(4.24), we obtain first-order linear homogeneous differential equations for $a_0^{(1,0)}$, $a_0^{(-1,0)}$, $a_0^{(0,1)}$, $a_0^{(-1,0)}$:

$$\frac{da_0^{(1,0)}}{dt} + \Big\{\frac{1}{2}\Big(\frac{\nu_1'}{\nu_1}+\frac{(\sqrt{\Delta})'}{\sqrt{\Delta}}\Big)+\frac{C_1f_1(t)}{\nu_1\sqrt{\Delta}}+\frac{C_2f_2(t)}{\nu_2\sqrt{\Delta}}\Big\}a_0^{(1,0)} = 0, \tag{4.33}$$

$$\frac{da_0^{(-1,0)}}{dt} + \Big\{\frac{1}{2}\Big(\frac{\nu_1'}{\nu_1}+\frac{(\sqrt{\Delta})'}{\sqrt{\Delta}}\Big)-\frac{C_1f_1(t)}{\nu_1\sqrt{\Delta}}-\frac{C_2f_2(t)}{\nu_2\sqrt{\Delta}}\Big\}a_0^{(-1,0)} = 0, \tag{4.34}$$

$$\frac{da_0^{(0,1)}}{dt} + \Big\{\frac{1}{2}\Big(\frac{\nu_2'}{\nu_2}+\frac{(\sqrt{\Delta})'}{\sqrt{\Delta}}\Big)+\frac{C_2f_3(t)}{\nu_2\sqrt{\Delta}}+\frac{C_1f_4(t)}{\nu_1\sqrt{\Delta}}\Big\}a_0^{(0,1)} = 0, \tag{4.35}$$

$$\frac{da_0^{(0,-1)}}{dt} + \Big\{\frac{1}{2}\Big(\frac{\nu_2'}{\nu_2}+\frac{(\sqrt{\Delta})'}{\sqrt{\Delta}}\Big)-\frac{C_2f_3(t)}{\nu_2\sqrt{\Delta}}-\frac{C_1f_4(t)}{\nu_1\sqrt{\Delta}}\Big\}a_0^{(0,-1)} = 0. \tag{4.36}$$

Therefore we have

$$a_0^{(1,0)} = \alpha_1(\nu_1\sqrt{\Delta})^{-1/2}\theta_{11}^{C_1}\theta_{12}^{C_2}, \tag{4.37}$$

$$a_0^{(-1,0)} = \beta_1(\nu_1\sqrt{\Delta})^{-1/2}\theta_{11}^{-C_1}\theta_{12}^{-C_2}, \tag{4.38}$$

$$a_0^{(0,1)} = \alpha_2(\nu_2\sqrt{\Delta})^{-1/2}\theta_{21}^{C_1}\theta_{22}^{C_2}, \tag{4.39}$$

$$a_0^{(0,-1)} = \beta_2(\nu_2\sqrt{\Delta})^{-1/2}\theta_{21}^{-C_1}\theta_{22}^{-C_2}. \tag{4.40}$$

Here α_1, α_2, β_1, β_2 are constants and here we set

$$\theta_{11} = \exp\Big(-\int \frac{f_1}{\nu_1\sqrt{\Delta}}dt\Big) = \exp\Big(-40\int \frac{10\theta_0^2 + 7\theta_0\sqrt{\Delta} - 5\Delta}{\nu_1^4\nu_2^2\Delta}dt\Big), \tag{4.41}$$

$$\theta_{12} = \exp\Big(-\int \frac{f_2}{\nu_2\sqrt{\Delta}}dt\Big) = \exp\Big(-160\int \frac{5\theta_0^2 + \Delta}{\nu_1^3\nu_2^3\Delta}dt\Big), \tag{4.42}$$

$$\theta_{21} = \exp\Big(-\int \frac{f_4}{\nu_1\sqrt{\Delta}}dt\Big) = \exp\Big(160\int \frac{5\theta_0^2 + \Delta}{\nu_1^3\nu_2^3\Delta}dt\Big) = \theta_{12}^{-1}, \tag{4.43}$$

$$\theta_{22} = \exp\Big(-\int \frac{f_3}{\nu_2\sqrt{\Delta}}dt\Big) = \exp\Big(40\int \frac{10\theta_0^2 - 7\theta_0\sqrt{\Delta} - 5\Delta}{\nu_1^2\nu_2^4\Delta}dt\Big). \tag{4.44}$$

Since the products $a_0^{(1,0)}a_0^{(-1,0)}$ and $a_0^{(0,1)}a_0^{(0,-1)}$ satisfy respectively Eqs. (4.31) and (4.32), we find that $\alpha_1\beta_1 = C_1$, $\alpha_2\beta_2 = C_2$ should hold. Thus we have

$$a_0^{(1,0)} = \alpha_1(\nu_1\sqrt{\Delta})^{-1/2}\theta_{11}^{\alpha_1\beta_1}\theta_{12}^{\alpha_2\beta_2}, \tag{4.45}$$

$$a_0^{(-1,0)} = \beta_1(\nu_1\sqrt{\Delta})^{-1/2}\theta_{11}^{-\alpha_1\beta_1}\theta_{12}^{-\alpha_2\beta_2}, \tag{4.46}$$

$$a_0^{(0,1)} = \alpha_2(\nu_2\sqrt{\Delta})^{-1/2}\theta_{21}^{\alpha_1\beta_1}\theta_{22}^{\alpha_2\beta_2}, \tag{4.47}$$

$$a_0^{(0,-1)} = \beta_2(\nu_2\sqrt{\Delta})^{-1/2}\theta_{21}^{-\alpha_1\beta_1}\theta_{22}^{-\alpha_2\beta_2}. \tag{4.48}$$

Hence we have obtained the leading terms of Θ which agree with results given in [12] without computations.

Theorem 4.1. *Let $\Theta_{k/2}(t)$ denote the restriction of $\widetilde{\Theta}_{k/2}(t,\tau_1,\tau_2)$ to $\tau_1 = \eta\phi_1$ and $\tau_2 = \eta\phi_2$ ($k = 0, 1, 2, \dots$). Under the notation given above, we have*

$$\begin{aligned}\Theta_0 = {} & \frac{1}{(\nu_1\sqrt{\Delta})^{1/2}}(\alpha_1\theta_{11}^{\alpha_1\beta_1}\theta_{12}^{\alpha_2\beta_2}e^{\eta\phi_1} + \beta_1\theta_{11}^{-\alpha_1\beta_1}\theta_{12}^{-\alpha_2\beta_2}e^{-\eta\phi_1}) \\ & + \frac{1}{(\nu_2\sqrt{\Delta})^{1/2}}(\alpha_2\theta_{21}^{\alpha_1\beta_1}\theta_{22}^{\alpha_2\beta_2}e^{\eta\phi_2} + \beta_2\theta_{21}^{-\alpha_1\beta_1}\theta_{22}^{-\alpha_2\beta_2}e^{-\eta\phi_2}).\end{aligned} \tag{4.49}$$

In other words, we obtain first two terms of a formal solution u_1 to $(P_{\mathrm{I}})_2$:

$$\begin{aligned}u_1 = \theta_0 + \eta^{-1/2}\Big\{ & \frac{1}{(\nu_1\sqrt{\Delta})^{1/2}}(\alpha_1\theta_{11}^{\alpha_1\beta_1}\theta_{12}^{\alpha_2\beta_2}e^{\eta\phi_1} + \beta_1\theta_{11}^{-\alpha_1\beta_1}\theta_{12}^{-\alpha_2\beta_2}e^{-\eta\phi_1}) \\ & + \frac{1}{(\nu_2\sqrt{\Delta})^{1/2}}(\alpha_2\theta_{21}^{\alpha_1\beta_1}\theta_{22}^{\alpha_2\beta_2}e^{\eta\phi_2} + \beta_2\theta_{21}^{-\alpha_1\beta_1}\theta_{22}^{-\alpha_2\beta_2}e^{-\eta\phi_2})\Big\} + O(\eta^{-1}).\end{aligned} \tag{4.50}$$

By using (4.19) and similar equations for other $a_{1/2}^{(j,k)}$ which have not been given explicitly there, we obtain the subleading terms of Θ. Coefficients of $\eta^{-1}e^{j\eta\phi_1+k\eta\phi_2}$

$(|j| + |k| = 2)$ of Θ (or u_1) are given as follows:

$$\begin{aligned}
a_{1/2}^{(2,0)} &= \frac{2\alpha_1^2}{\nu_1^3\sqrt{\Delta}}\theta_{11}^{2\alpha_1\beta_1}\theta_{12}^{2\alpha_2\beta_2},\\
a_{1/2}^{(-2,0)} &= \frac{2\beta_1^2}{\nu_1^3\sqrt{\Delta}}\theta_{11}^{-2\alpha_1\beta_1}\theta_{12}^{-2\alpha_2\beta_2},\\
a_{1/2}^{(0,2)} &= \frac{2\alpha_2^2}{\nu_2^3\sqrt{\Delta}}\theta_{21}^{2\alpha_1\beta_1}\theta_{22}^{2\alpha_2\beta_2},\\
a_{1/2}^{(0,-2)} &= \frac{2\beta_2^2}{\nu_2^3\sqrt{\Delta}}\theta_{21}^{-2\alpha_1\beta_1}\theta_{22}^{-2\alpha_2\beta_2},\\
a_{1/2}^{(1,1)} &= \frac{4\,\alpha_1\alpha_2}{(\nu_1\nu_2)^{3/2}\Delta}\theta_{11}^{\alpha_1\beta_1}\theta_{12}^{\alpha_2\beta_2}\theta_{21}^{\alpha_1\beta_1}\theta_{22}^{\alpha_2\beta_2},\\
a_{1/2}^{(1,-1)} &= -\frac{4\,\alpha_1\beta_2}{(\nu_1\nu_2)^{3/2}\Delta}\theta_{11}^{\alpha_1\beta_1}\theta_{12}^{\alpha_2\beta_2}\theta_{21}^{-\alpha_1\beta_1}\theta_{22}^{-\alpha_2\beta_2},\\
a_{1/2}^{(-1,1)} &= -\frac{4\,\alpha_2\beta_1}{(\nu_1\nu_2)^{3/2}\Delta}\theta_{11}^{-\alpha_1\beta_1}\theta_{12}^{-\alpha_2\beta_2}\theta_{21}^{\alpha_1\beta_1}\theta_{22}^{\alpha_2\beta_2},\\
a_{1/2}^{(-1,-1)} &= \frac{4\,\beta_1\beta_2}{(\nu_1\nu_2)^{3/2}\Delta}\theta_{11}^{-\alpha_1\beta_1}\theta_{12}^{-\alpha_2\beta_2}\theta_{21}^{-\alpha_1\beta_1}\theta_{22}^{-\alpha_2\beta_2},\\
a_{1/2}^{(0,0)} &= \frac{40\alpha_1\beta_1(\sqrt{\Delta}-\theta_0)}{\nu_1^3\nu_2^2\sqrt{\Delta}} - \frac{40\alpha_2\beta_2(\sqrt{\Delta}+\theta_0)}{\nu_1^2\nu_2^3\sqrt{\Delta}}.
\end{aligned}$$

Remark. In Eq. (4.50), exponential terms should be considered to be bounded (i.e. the phases are pure imaginary) if u_1 is regarded as an asymptotic solution as $\eta \to \infty$. We treat, however, Eq. (4.50) as leading terms of a formal object which has formal power series expansion in $\eta^{-1/2}$ with coefficients including exponential terms $e^{\eta\phi_1}$, $e^{\eta\phi_2}$ as "transcendental" elements satisfying the differential relations $\dfrac{de^{\eta\phi_j}}{dt} = \eta\nu_j e^{\eta\phi_j}$.

Higher-order terms $\widetilde{\Theta}_{k/2}$ $(k > 1)$ can be constructed in a similar way as the case of the classical Painlevé equations with a large parameter ([1], [2], [3], [6]). For example, the coefficient $a_1^{(j,k)}$ of $e^{\eta(j\phi_1+k\phi_2)}$ in $\widetilde{\Theta}_1$ can be determined uniquely except for $(j,k) = (\pm 1, 0)$, $(0, \pm 1)$ and $a_1^{(\pm 1,0)}$, $a_1^{(0,\pm 1)}$ are obtained by the non-secularity condition for $\widetilde{\Theta}_2$. The coefficients of exponentials of $\widetilde{\Theta}_{3/2}$ are determined uniquely and they can be written down in terms of known quantities and of $a_1^{(\pm 1,0)}$, $a_1^{(0,\pm 1)}$. The non-secularity condition for $\widetilde{\Theta}_2$ turns out to be a system of first-order linear inhomogeneous differential equations for $a_1^{(\pm 1,0)}$, $a_1^{(0,\pm 1)}$ and the inhomogeneous terms are known. This system can be solved easily. Hence we have $\widetilde{\Theta}_1$ and $\widetilde{\Theta}_{3/2}$. These procedures can be carried out successively.

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