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Degenerate Stokes Geometry and Some Geometric Structure Underlying a Virtual Turning Point

By

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§ 1. Introduction

The Stokes geometry associated with a higher order linear differential equation is quite different from that of the second order equation. Ordinary turning points are not enough to describe the complete Stokes geometry, and a new object should appear in the geometry, that is a "virtual turning point" ([BNR], [AKT1]).

Although such a point is essential and indispensable for the description of the Stokes geometry, some difficulties are involved. One of the difficulties is that too many virtual turning points appear, and hence the Stokes geometry becomes formidably complicated if we will draw all new Stokes curves, i.e. a Stokes curve emanates from a virtual turning point (see Fig. 1). Fortunately, almost all portions of a new Stokes curve are apparent, in the sense that on such portions Stokes phenomena never occur. To distinguish an apparent portion of a Stokes curve, we draw it by a dotted line instead of a solid one, or even more drastically, we omit a Stokes curve whose entire portion is a dotted line, that makes the Stokes geometry understandable with the naked eye (see Fig. 2). Now the following question naturally arises for the description of the Stokes geometry:

How can we determine solid or dotted line portions of a Stokes curve?

An answer was first given by Aoki-Kawai-Takei [AKT]. They introduced an algorithm to determine solid or dotted line portions of Stokes curves, although it does not cover whole situations, it is still a useful tool in studying the complete Stokes geometry. Later the author extended the algorithm to deal with the case where the equation has a deformation parameter. In view of geometrical deformation, the key feature of the algorithm is that the Stokes geometry has a continuous deformation property, that is, each solid (or dotted) line portion of a Stokes curve is also continuously deformed under

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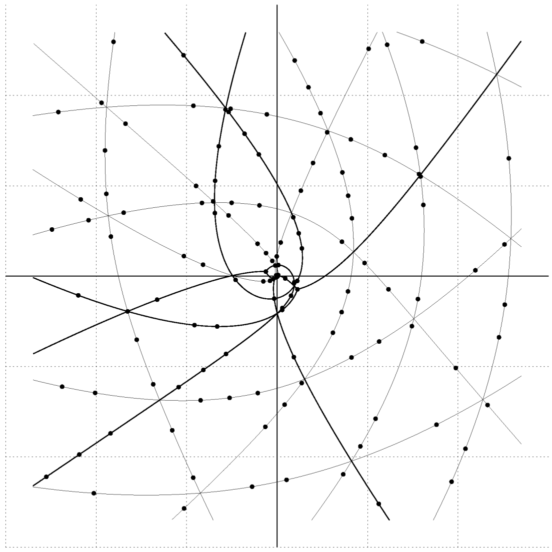

Figure 1. The Stokes geometry of NYL_2.

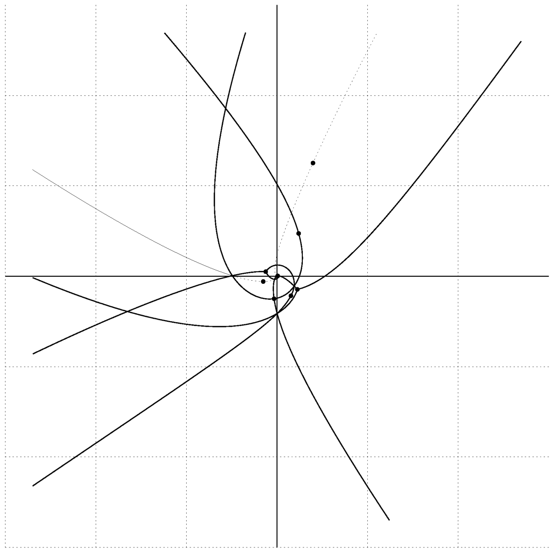

Figure 2. The algorithm has been applied to Fig. 1.

the stability condition of the Stokes geometry (see [H2, Definition 6.4] for the stability condition). For a generic parameter, we can obtain the correct Stokes geometry by simply applying our algorithm. However some care is needed in applying the algorithm to the situations where the Stokes geometry has the geometrical degeneration of the following kind:

Case 1. (geometrical degeneration between turning points) Different turning points accidentally coincide.

Case 2. (geometrical degeneration between Stokes curves and turning points) A turning point hits a Stokes curve.

Case 3. (geometrical degeneration between Stokes curves) An intersection point of Stokes curves collides with the other one, or Stokes curves become tangent each other.

It is certainly desirable to make the algorithm applicable to the geometrically degenerate situations. The principal aim of this paper is an improvement of the algorithm so that it may be applicable to Cases 1 and 3 above. See [AKSST] and [H2] for Case 2.

The plan of this paper is as follows: Section 2 gives the basic algorithm that determines solid or dotted line portions of a Stokes curve for a generic parameter.

In Section 3, we will study Case 3. It was recently investigated by Y. Umeta [U], and we will review her results. This section is also useful for the reader to understand how to apply the algorithm to the concrete problems.

In Section 4, the main part of the paper, we will study Case 1. To distinguish turning points that accidentally coincide, a Riemann manifold $\mathcal{R}_{\mathrm{sym}}$ underlying a virtual turning point will be introduced. The manifold $\mathcal{R}_{\mathrm{sym}}$ would be a fundamental geometric object in studying not only our problems in this paper but also other ones such as the "existence and uniqueness" of solid line portions in the Stokes geometry or the "finiteness" of effective virtual turning points, that is, virtual turning points other than those contained in a finite number of Riemann sheets of $\mathcal{R}_{\mathrm{sym}}$ are apparent.

The author is deeply indebted to the members of the Kawai-Takei Seminar. The paper is based on discussions in the seminar and their previous works. Especially Prof. T. Kawai gave me many valuable ideas and suggestions. Without them, the paper would not have been completed.

§ 2. Preparations

§ 2.1. Basic Facts about a Virtual Turning Point

Let T denote a parameter space, which is a complex manifold in this paper. Let $\mathscr{O}(T)$ be the set of holomorphic functions on T, and $\mathscr{O}(T)[x]$ designate the set of polynomials of the variable x with coefficients in $\mathscr{O}(T)$. We will consider a linear differential equation with a deformation parameter $t \in T$ and a large parameter η of the following form:

$$Pu = \Big(\frac{1}{\eta}\frac{d}{dx} + \frac{1}{p(x)}A(x;t,\eta)\Big)u = 0. \tag{2.1.1}$$

Here A designates an $n \times n$ matrix of formal power series of η^{-1} such that:

$$A(x;t,\eta) = A_0(x;t) + A_1(x;t)\frac{1}{\eta} + A_2(x;t)\frac{1}{\eta^2} + \cdots, \quad A_j \in \mathrm{gl}(n;\mathscr{O}(T)[x]),$$

and $p(x) \in \mathbb{C}[x]$ is a nonzero polynomial of x. A characteristic polynomial $\Lambda_t(\lambda, x)$ of λ is by the definition $\det(\lambda I - A_0(x;t))$, and let $D_t(x)$ denote the discriminant of $\Lambda_t(\lambda, x) = 0$. We denote by Z_t (resp. E_{sing}) the set of ordinary turning points (resp. singular points) of the equation, i.e. the zero set of $D_t(x)$ (resp. $p(x)$). Hereafter we always assume the following conditions:

(LA-1) $Z_t \cap E_{\mathrm{sing}} = \emptyset$ for any $t \in T$.
(LA-2) All roots of $D_t(x) = 0$ are simple for any $t \in T$; that is, the equation has only simple turning points that never merge each other when t moves.

On the complex plane $\mathbb{C}$ equipped with appropriate cut lines, let holomorphic functions $\lambda_{t,1}(x)$, $\lambda_{t,2}(x)$, $\ldots$, $\lambda_{t,n}(x)$ of x denote the roots of the algebraic equation $\Lambda_t(\lambda, x) = 0$ of λ.

Let us recall the definition of a virtual turning point, often abbreviated as a VTP. For the moment, we fix the parameter t to $t_0 \in T$, and the suffix t of Λ_t and so on, will be omitted.

Definition 2.1 ([T1]). A point $x_0 \in \mathbb{C} \setminus E_{\rm sing}$ is called a virtual turning point of type (i,j) $(i \neq j)$ if there exist a piecewise smooth closed path C_{x_0} in $\mathbb{C} \setminus E_{\rm sing}$ starting from x_0, and a continuous function $\mu(x)$ on C_{x_0} that satisfy the following conditions.

1. For any $x \in C_{x_0}$, $\mu(x)$ is a root of the equation $\Lambda(\mu, x) = 0$, and near the starting (resp. ending) point of C_{x_0}, $\mu(x) = \lambda_i(x)$ (resp. $\mu(x) = \lambda_j(x)$) holds.
2. The equality $\displaystyle\int_{C_{x_0}} \frac{\mu(x)}{p(x)}\, dx = 0$ is satisfied.

Note that an ordinary turning point is, from the logical viewpoint, a virtual turning point in the sense above. However, for the sake of convenience, we exclude ordinary turning points from the definition of virtual turning points. In what follows, a turning point means either an ordinary turning point or a virtual turning point. We can define a Stokes curve that emanates from a virtual turning point in the same way as in the case of an ordinary turning point. A Stokes curve emanating from a virtual turning point is often called a new Stokes curve.

We can successively obtain virtual turning points thanks to the following theorem. Let x_0 and x_1 be turning points, and s_0 (resp. s_1) a Stokes curve emanating from x_0 (resp. x_1). We assume that s_0 intersects with s_1 at a point x and the types of s_0 and s_1 at x are (i,j) and (j,k) respectively. Note that the index j is common in both types. Let l denote the integral curve of the real differential 1-form $\mathrm{Im}\Big(\dfrac{\lambda_i(x) - \lambda_k(x)}{p(x)} dx\Big)$ passing through x.

Theorem 2.2 (The Algorithm for Locating VTP's [AKKSST]). *If a point v in the curve l satisfies the following integral relation*

$$\int_x^{x_0} \frac{\lambda_i(x) - \lambda_j(x)}{p(x)} dx + \int_x^{x_1} \frac{\lambda_j(x) - \lambda_k(x)}{p(x)} dx + \int_x^{v} \frac{\lambda_k(x) - \lambda_i(x)}{p(x)} dx = 0,$$

then v is a VTP, i.e. a virtual turning point. Here each integration is performed along the integral curve designated above.

§ 2.2. The Solid or Dotted Line Condition

We review the algorithm that determines solid or dotted line portions of a Stokes curve for a generic parameter. Let V be a subset of the set of turning points when $t = t_0$, and let S denote the set of all Stokes curves that emanate from some point of V. We designate by $G(V)$ the Stokes geometry consisting of S and V.

We first note the following "separation rule" of Stokes curves that is important in employing the algorithm. Let v_0, $v_1 \in V$ be turning points, and let s_0 (resp. s_1) denote a Stokes curve that emanates from v_0 (resp. v_1) respectively.

Definition 2.3 (The Separation Rule)**.** If the turning points v_0 and v_1 are located at different positions, then we always consider the Stokes curves s_0 and s_1 to be different even if they coincide set-theoretically.

The rule above means that a Stokes curve s is regarded as a pair $\{v, l\}$ of a turning point v and an integral curve l which emanates from v. We denote by $[\{v, l\}]$ the underlying integral curve l of $\{v, l\}$, and we also note that "a point" in the Stokes curve $\{v, l\}$ implies one in the integral curve l. Let v, v_0, and v_1 be three turning points and s, s_0 and s_1 their Stokes curves.

Definition 2.4. We say that s is **combined** with s_0 and s_1 at x if Conditions 1, 2 and 3 below are satisfied:

1. $[s]$, $[s_0]$ and $[s_1]$ intersect at x.
2. The types of s_0, s_1 and s at x are (i, j), (j, k) and (i, k) respectively for mutually different indices i, j and k.
3. The same integral relation as in Theorem 2.2 holds, that is,

$$\int_x^{v_0} \frac{\lambda_i(x) - \lambda_j(x)}{p(x)} dx + \int_x^{v_1} \frac{\lambda_j(x) - \lambda_k(x)}{p(x)} dx + \int_x^{v} \frac{\lambda_k(x) - \lambda_i(x)}{p(x)} dx = 0.$$

Remark. In Section 4, we will modify the definition above to deal with accidental coincidence of turning points.

Definition 2.5. We say that s is **coherent** at x with respect to s_0 and s_1 if the following conditions are fulfilled:

1. s is combined with s_0 and s_1 at x,
2. s_0 and s_1 form an ordered crossing at x, that is, either $i < j < k$ or $i > j > k$ holds.

Now we are ready to introduce the algorithm for a generic parameter.

Definition 2.6 (The Solid or Dotted Line Condition)**.** For each Stokes curve $s \in G(V)$ which emanates from $v \in V$, the state of some portion of s is defined to be solid or dotted so that the following two conditions are satisfied:

1. The state of the curve s in a neighborhood of v is
   (a) solid if v is an ordinary turning point.
   (b) dotted if v is a virtual turning point.

2. The state of s should be converted at a point x in s if and only if there are Stokes curves s_0 and $s_1 \in G(V)$ satisfying Conditions (a) and (b) below:
   (a) s is coherent at x with respect to s_0 and s_1.
   (b) s_0 and s_1 are solid lines near x.

Several examples of the Stokes geometry are given in [H1].

§ 3. Geometrical degeneration between Stokes curves

In this section, we will study the case where an intersection point of Stokes curves coincides with the other one or Stokes curves become tangent each other.

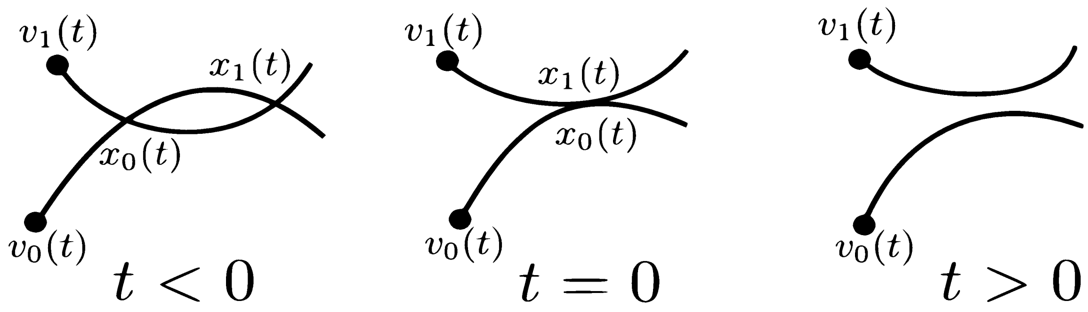

Figure 3. Stokes curves become tangent each other.

Let us first consider the following simplified example which has been observed in the Stokes geometry associated with the equation NYL_4 of the underlying Lax pair of the Noumi-Yamada system (for the Noumi-Yamada system, see [NY] and [T2]). Let $v_0(t)$ and $v_1(t)$ be ordinary turning points that depend holomorphically on a parameter t, and let $s_0(t)$ (resp. $s_1(t)$) denote a Stokes curve emanating from $v_0(t)$ (resp. $v_1(t)$). The parameter space T is assumed to be $\mathbb{C}$, and we will move a parameter t on the real axis. The configuration of the Stokes geometry is as follows (see Fig. 3).

1. If $t < 0$, the Stokes curves $s_0(t)$ and $s_1(t)$ intersect transversally at two points $x_0(t)$ and $x_1(t)$. Here $x_0(t)$ denotes the nearest intersection point from the turning point $v_0(t)$ along $s_0(t)$.
2. If $t = 0$, the intersection points $x_0(t)$ and $x_1(t)$ merge; that is, the Stokes curves become tangent at $x_0(0) = x_1(0)$ with an even order.
3. If $t > 0$, the Stokes curves are disjoint.

Let us introduce a virtual turning point $v(t)$ that is located by Theorem 2.2, and let $s(t)$ denote a new Stokes curve emanating from $v(t)$ (see Fig. 4). We suppose that when

$t \leq 0$ the curve $s(t)$ passes through both $x_0(t)$ and $x_1(t)$, and at those points $s(t)$ is coherent with respect to $s_0(t)$ and $s_1(t)$. Solid or dotted line portions of $s(t)$ will be determined in the following way.

- If $t < 0$, by Condition 1 of Definition 2.6 the state of $s(t)$ near $v(t)$ must be dotted. Thus the portion between $v(t)$ and $x_0(t)$ is a dotted line. On the other hand, since Condition 2 of Definition 2.6 is satisfied at both $x_0(t)$ and $x_1(t)$, the state of the curve should be converted there. Therefore the portion between $x_0(t)$ and $x_1(t)$ is a solid line, and the state of the curve after $x_1(t)$ is again dotted.
- If $t = 0$, in the same way as above, we conclude that the portion between $v(t)$ and $x_0(t) = x_1(t)$ is a dotted line, and the state of the curve after $x_1(t)$ becomes solid.
- If $t > 0$, the state of the entire curve is dotted.

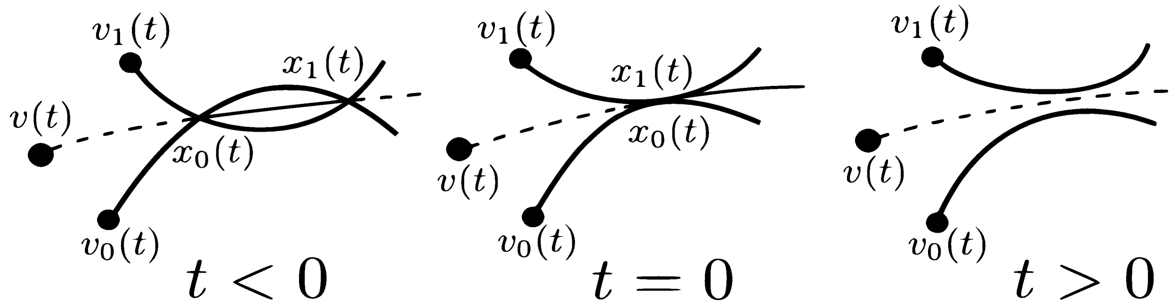

Figure 4. Apply the algorithm to Stokes curves.

In view of the continuous deformation property remarked in Section 1, the changes of the state of $s(t)$ seem a little bit strange since the state of the portion after $x_1(t)$ becomes solid only when $t = 0$ and remains dotted otherwise. Hence we might expect the portion to be always dotted. In fact, no Stokes phenomena occur on the portion even if $t = 0$ because the tangency of the curves is even order and exact WKB solutions of the equation are single valued near the tangent point.

Y. Umeta recently extended the algorithm so that the continuous deformation property still holds for this case. Let us recall her extended algorithm (see [U] for details). We denote by $\mathscr{A}$ the sheaf of real analytic functions in the underlying Euclidean space $\mathbb{R}^2$ of $\mathbb{C}$. Let l_0 (resp. l_1) be a real analytic curve in $\mathbb{C}$ defined by a real analytic function f_0 (resp. f_1) near x, and let us assume that l_0 and l_1 intersect properly at x.

Definition 3.1. The intersection multiplicity $\mathrm{mul}_x(l_0, l_1)$ of l_0 and l_1 at x is defined by

$$\mathrm{mul}_x(l_0,\, l_1) = \dim_{\mathbb{R}} \frac{\mathscr{A}_x}{\mathscr{A}_x(f_0, f_1)}.$$

Note that if l_0 and l_1 intersect transversally, then we have $\mathrm{mul}_x(l_0, l_1) = 1$. The extended algorithm is as follows:

Definition 3.2 (The Solid or Dotted Line Condition)**.** For each Stokes curve $s \in G(V)$ which emanates from $v \in V$, the state of some portion of s is defined to be solid or dotted so that the following two conditions are satisfied:

1. The state of the curve s in a neighborhood of v is
   (a) solid if v is an ordinary turning point.
   (b) dotted if v is a virtual turning point.
2. The state of s should be converted at a point x in s if and only if the number of pairs (s_0, s_1) (s_0, $s_1 \in G(V)$) of Stokes curves that satisfy Condition (a), (b) and (c) below is an odd integer:
   (a) s is coherent with respect to s_0 and s_1 at x.
   (b) s_0 and s_1 are solid lines near x.
   (c) $\mathrm{mul}_x(s_0, s_1)$ is an odd number.

Generally the behavior of a Stokes curve near a tangent point is not so simple on the contrary to the example above. She investigated all possible configurations of Stokes curves and obtained the following theorem. Let $v_0(t)$, $v_1(t)$ and $v_2(t)$ be turning points and let $s_i(t)$ $(i = 0, 1, 2)$ designate a Stokes curve emanating from $v_i(t)$. We assume that three Stokes curves intersect at a point x when $t = t_0$. Let C be a sufficiently small circle with the center x that is independent of t, and let us suppose that each Stokes curve $s_i(t)$ intersects with the circle C at only two points $x_{i,s}(t)$ and $x_{i,e}(t)$, where $x_{i,s}(t)$ designates the nearest intersection point from $v_i(t)$ along $s_i(t)$.

Theorem 3.3 ([U])**.** *There exists a neighborhood $U \subset T$ of t_0 that satisfies the following. If the state of $s_i(t)$ near $x_{i,s}(t)$ remains unchanged for any $t \in U$ $(i = 0, 1, 2)$, then the state of $s_i(t)$ near $x_{i,e}(t)$ is also unchanged for any $t \in U$.*

Roughly speaking, the theorem above implies that the continuous deformation property still holds outside C.

§ 4. Geometric Degeneration Between Turning Points

We will consider the case where different turning points accidentally coincide.

§ 4.1. An Example of Geometric Degeneration Between Turning Points

The following example was first found and studied by Aoki-Koike-Takei ([AKoT]). Let $v_0(t)$, $v_1(t)$ and $v_2(t)$ be ordinary turning points, and let $s_i(t)$ $(i = 0, 1, 2)$ designate

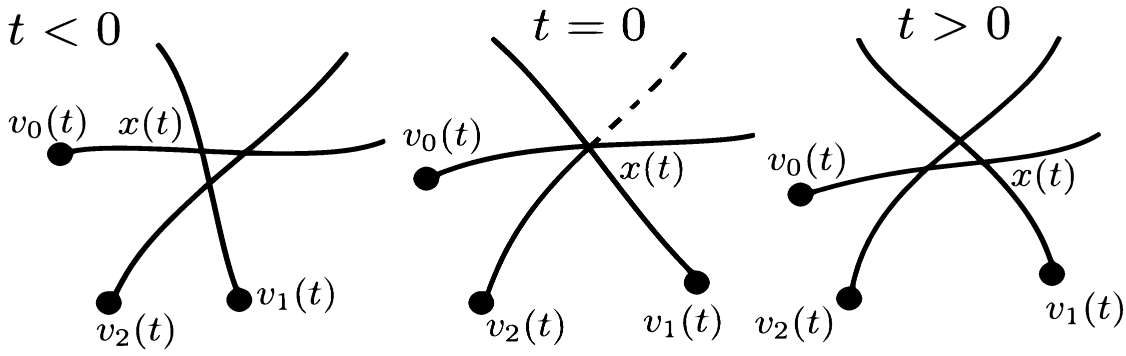

Figure 5. The example found by Aoki-Koike-Takei.

a Stokes curve that emanates from $v_i(t)$. The parameter space of the equation is assume to be $\mathbb{C}$, and let us move t along the real axis.
The characteristic feature of the example is summarized as follows (see Fig. 5):

1. The curve $s_0(t)$ intersects transversally with $s_1(t)$ at a point $x(t)$ for any t.
2. When $t = 0$, the Stokes curve $s_2(0)$ passes through $x(0)$, and at that point $s_2(0)$ is coherent with respect to $s_0(0)$ and $s_1(0)$. On the other hand, $s_2(t)$ does not pass through $x(t)$ when $t \neq 0$.

For the state of some portion of $s_2(t)$ Definition 3.2 entails:

- If $t \neq 0$, the entire portion of $s_2(t)$ is a solid line.
- If $t = 0$, the portion between $v_2(0)$ and $x(0)$ is a solid line, however, the state of the portion after $x(0)$ becomes dotted.

If we take the continuous deformation property into account, the changes of the state of $s_2(t)$ seem again strange because of the same reason as in Section 3, that is, the dotted line portion of $s_2(t)$ only exists when $t = 0$. In fact, Aoki, Koike and Takei in their paper confirmed that Stokes phenomena occur on the entire portion of $s_2(t)$ even if $t = 0$.

Why do we arrive at an erroneous conclusion? When $t \neq 0$, if we apply Theorem 2.2 to the Stokes curves $s_0(t)$ and $s_1(t)$, we can find another virtual turning point $v(t)$ which is located quite close to $v_2(t)$ and a new Stokes curve $s(t)$ emanating from $v(t)$ that passes through $x(t)$ always (see Fig. 6). Note that $s(t)$ is combined with $s_0(t)$ and $s_1(t)$ at $x(t)$ for any t. When t tends to 0, the turning points $v_2(t)$ and $v(t)$ merge and the Stokes curves $s_2(t)$ and $s(t)$ coincide. Therefore when $t = 0$, the curve which is really combined with $s_0(0)$ and $s_1(0)$ at $x(0)$ is considered to be $s(0)$. Since virtual turning points are defined in the complex plane $\mathbb{C}$, we could not distinguish $v(0)$ from $v_2(0)$ that is located at the same geometrical position, and we accidentally regarded $s_2(0)$ instead

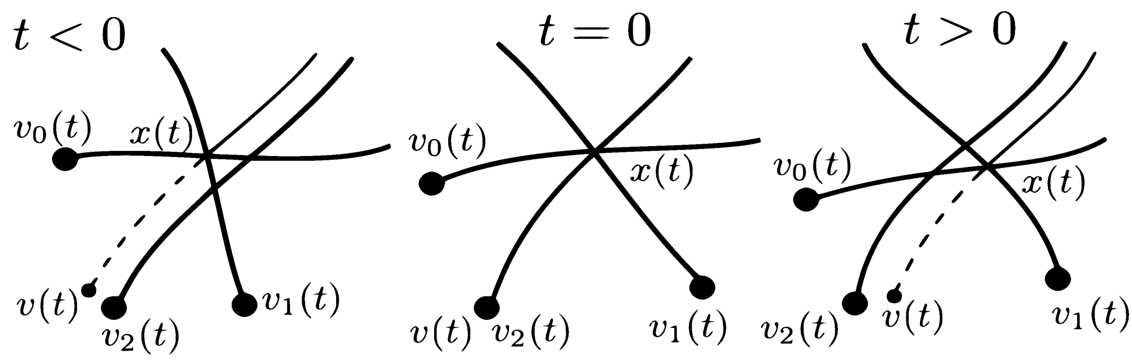

Figure 6. Two turning points coincide.

of $s(0)$ as a curve that is combined with $s_0(0)$ and $s_1(0)$ at $x(0)$. This is the reason why the algorithm leads to the incorrect conclusion. By introducing an appropriate Riemann manifold (instead of the complex plane $\mathbb{C}$) we can clarify the geometric situation even in such a degenerate case. This is what we will do in what follows.

§ 4.2. The Independent One-Cycle Condition

From now on, we consider the $n \times n$ equation given in Section 2, and we always assume Conditions (LA-1) and (LA-2). Let $\mathbb{P}^2$ be the projective space with a system of homogeneous coordinates $(\lambda, x; \mu)$, and let $W_t \subset \mathbb{C}^2$ denote the algebraic set

(4.2.1) $$\{(\lambda, x) \in \mathbb{C}^2;\ \Lambda_t(\lambda, x) = 0\}.$$

We designate by $\widehat{W}_t$ the closure of W_t in $\mathbb{P}^2$, where $\mathbb{C}^2$ is identified with $\mathbb{P}^2 \setminus \{\mu = 0\}$. Then it follows from the assumptions (LA-1) and (LA-2) that $W_t \subset \mathbb{C}^2$ is a smooth manifold for any t and depends holomorphically on a parameter t. We will also suppose the following condition (LA-3) for the simplicity.

(LA-3) The manifold W_t is connected and $\hat{W}_t$ is a topological manifold for any t.

Let $\pi_{W_t}: W_t \to \mathbb{C}$ designate the natural projection with respect to the variable x. By the assumption (LA-3), π_{W_t} has a continuous extension $\widehat{\pi}_{\widehat{W}_t}: \widehat{W}_t \to \mathbb{P}^1$. Let $\widehat{Z}_t \subset \widehat{W}_t$ denote the set of ramification points contained in W_t with respect to $\widehat{\pi}_{\widehat{W}_t}$. We also define a subset $\widehat{E}_{t,\infty}$ (resp. $\widehat{E}_{t,\text{sing}}$) of $\widehat{W}_t$ by $\widehat{W}_t \cap \widehat{\pi}^{-1}_{\hat{W}_t}(\infty)$ (resp. $\widehat{W}_t \cap \widehat{\pi}^{-1}_{\widehat{W}_t}(E_{\text{sing}})$) respectively, and set

(4.2.2) $$\widehat{E}_t = \widehat{E}_{t,\infty} \cup \widehat{E}_{t,\text{sing}}.$$

Now we study the first homology group $H_1(\widehat{W}_t \setminus \widehat{E}_t; \mathbb{Z})$ to define the index space of turning points.

Lemma 4.1. *The group $H_1(\widehat{W}_t \setminus \widehat{E}_t;\mathbb{Z})$ is a free $\mathbb{Z}$-module, and we have*

$$\mathrm{Rank}_{\mathbb{Z}} H_1(\widehat{W}_t \setminus \widehat{E}_t;\mathbb{Z}) = 1 + \#Z_t + n(\#E_{\mathrm{sing}} - 1).$$

Here $\#Z$ denotes the number of elements of a set Z.

Proof. It is well known that the group $H_1(\widehat{W}_t;\mathbb{Z})$ is a free $\mathbb{Z}$-module, and by the Riemann-Hurwitz theorem we obtain

$$\mathrm{Rank}_{\mathbb{Z}} H_1(\widehat{W}_t;\mathbb{Z}) = 2 - n + \#Z_t - \#\widehat{E}_{t,\infty}.$$

Since $\widehat{W}_t \setminus \widehat{E}_t$ is a non-compact connected manifold, we have $H_2(\widehat{W}_t \setminus \widehat{E}_t;\mathbb{Z}) = 0$. It follows from (LA-3) that $H_1(\widehat{W}_t, \widehat{W}_t \setminus \widehat{E}_t;\mathbb{Z}) = 0$ and $H_2(\widehat{W}_t, \widehat{W}_t \setminus \widehat{E}_t;\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $\#\widehat{E}_{t,\infty} + \#\widehat{E}_{t,\mathrm{sing}}$. Thus we get an exact sequence of homology groups:

$$0 \leftarrow H_1(\widehat{W}_t;\mathbb{Z}) \leftarrow H_1(\widehat{W}_t \setminus \widehat{E}_t;\mathbb{Z}) \leftarrow H_2(\widehat{W}_t, \widehat{W}_t \setminus \widehat{E}_t;\mathbb{Z}) \overset{\phi_*}{\longleftarrow} H_2(\widehat{W}_t;\mathbb{Z}) \leftarrow 0.$$

Since the morphism $\phi_*\colon H_2(\widehat{W}_t;\mathbb{Z}) \to H_2(\widehat{W}_t, \widehat{W}_t \setminus \widehat{E}_t;\mathbb{Z})$ is isomorphic to the diagonal embedding $i_\Delta\colon \mathbb{Z} \to \mathbb{Z}^{\#\widehat{E}_{t,\infty}+\#\widehat{E}_{t,\mathrm{sing}}}$ (i.e. $i_\Delta(p) = (p, p, \ldots, p)$ for any $p \in \mathbb{Z}$), $\operatorname{coker}\phi_*$ is a free $\mathbb{Z}$-module of rank $\#\widehat{E}_{t,\infty} + \#\widehat{E}_{t,\mathrm{sing}} - 1$. Therefore we can conclude that $H_1(\widehat{W}_t \setminus \widehat{E}_t;\mathbb{Z})$ is a free $\mathbb{Z}$-module, and

$$\begin{aligned}\mathrm{Rank}_{\mathbb{Z}} H_1(\widehat{W}_t \setminus \widehat{E}_t;\mathbb{Z}) &= (2 - n + \#Z_t - \#\widehat{E}_{t,\infty}) + (\#\widehat{E}_{t,\infty} + \#\widehat{E}_{t,\mathrm{sing}} - 1)\\ &= 1 + \#Z_t + n(\#E_{\mathrm{sing}} - 1).\end{aligned}$$

□

We set $\kappa = 1 + \#Z_t + n(\#E_{\mathrm{sing}} - 1)$. Let t_0 be a point in T, and $\{\sigma_1, \ldots, \sigma_\kappa\}$ a family of closed paths in $\widehat{W}_{t_0} \setminus \widehat{E}_{t_0}$ that generates the group $H_1(\widehat{W}_{t_0} \setminus \widehat{E}_{t_0};\mathbb{Z})$ over $\mathbb{Z}$. We designate by $\sigma_{t,i}$ a closed path in $\widehat{W}_t \setminus \widehat{E}_t$ that is a continuous deformation of σ_i for each t near t_0 $(i = 1, 2, \ldots, \kappa)$. For a given holomorphic 1-form ω_t on $\widehat{W}_t \setminus \widehat{E}_t$ which depends holomorphically on t, it is clear that $\displaystyle\int_{\sigma_{t,i}} \omega_t$ is a holomorphic function of t. We say that **a 1-form ω_t satisfies the independent 1-cycle condition at** t_0 if the germs of holomorphic functions $\displaystyle\int_{\sigma_{t,1}} \omega_t, \int_{\sigma_{t,2}} \omega_t, \ldots, \int_{\sigma_{t,\kappa}} \omega_t$ at t_0 are independent over $\mathbb{Z}$. Note that this definition does not depend on the choice of $\{\sigma_i\}$ at $t = t_0$.

Remark. By the theory of the existence of a meromorphic 1-form and that of period integrals, a 1-form ω_t satisfying the independent 1-cycle condition always exists.

Now we introduce the following 1-form ω that already appeared in the definition of a virtual turning point:

$$\omega = \frac{\lambda}{\mu p\left(\dfrac{x}{\mu}\right)} d\left(\frac{x}{\mu}\right), \qquad \left(\text{cf.} \quad \omega = \frac{\lambda_i(x)}{p(x)} dx\right). \tag{4.2.3}$$

Definition 4.2. If the 1-form ω defined above satisfies the independent 1-cycle condition at t_0, we simply say that the equation satisfies the independent 1-cycle condition at t_0,

Example 4.3 (NYL_2 of the Underlying Lax Pair of the Noumi-Yamada System)**.** The equation has the form:

$$\frac{1}{\eta}\frac{du}{dx} = \frac{1}{x}\begin{pmatrix} e_0 & v_1 & 1 \\ x & e_1 & v_2 \\ v_0 x & x & e_2 \end{pmatrix} u \tag{4.2.4}$$

where $(e_0, e_1, e_2, v_0, v_1, v_2)$ is a parameter. Then we have

$$\begin{aligned}\Lambda(\lambda, x) = \lambda^3 &- (e_0 + e_1 + e_2)\lambda^2 + ((u_0 + u_1 + u_2)x - (e_0 e_1 + e_1 e_2 + e_0 e_2))\lambda \\ &- (x^2 - (u_0 e_1 + u_2 e_0 + u_1 e_2 - u_0 u_1 u_2)x + e_0 e_1 e_2).\end{aligned}$$

For a generic parameter, the following facts are observed:

- $\widehat{W}$ is a complex manifold whose genus is 1.
- $\widehat{E}_\infty$ consists of only one ramification point of degree 3, and the equation has 4 simple turning points.

By Lemma 4.1 we have $H_1(\widehat{W} \setminus \widehat{E}; \mathbb{Z}) = \mathbb{Z}^5$, and the equation satisfies the independent 1-cycle condition for a generic parameter.

§ 4.3. The Type Diagram and Virtual Turning Points

Definition 2.1 suggests that a virtual turning point might be understood as a point in $\mathbb{C}$ accompanied by a kind of 1-cycle in $\widehat{W}$. To describe and calculate such a 1-cycle concretely we will introduce some graph associated with the equation which is called "the type diagram".

The (abstract) directional graph consists of two sets: a finite set whose element is called a "node" and the set of "edges" where each edge is an ordered pair of nodes.

Definition 4.4. The type diagram associated with the equation is a directional graph as follows:

1. Each node is an integer $1, 2, \ldots, n$.
2. Each edge is indexed by an ordinary turning point. If the type of $v \in Z_t$ is (i, j), then the edge indexed by v is one of the ordered pair $\{i, \{j\}\}$ or $\{j, \{i\}\}$. Note that the choice of $\{i, \{j\}\}$ or $\{j, \{i\}\}$ is arbitrary, and such a choice determines the direction of the edge.

Remark. Aoki-Kawai-Takei ([AKT2]) also introduced a similar graphical notion called a bicharacteristic graph, that is, in a sense, a "dual" notion of the type diagram.

From now on, we denote by the symbol $i \xrightarrow{v} j$ (or $j \xrightarrow{v} i$) the edge indexed by an ordinary turning point v of type (i,j). Let $L_{t,0}$ (resp. $L_{t,1}$) denote the free $\mathbb{Z}$-module generated by the nodes (resp. the edges) of the type diagram. We consider a complex $\dot{L}_t$ as

$$\dot{L}_t: \quad 0 \leftarrow L_{t,0} \overset{\partial}{\leftarrow} L_{t,1} \leftarrow 0, \tag{4.3.1}$$

where the morphism ∂ is defined by

$$\partial(i \xrightarrow{v} j) = \{j\} - \{i\}, \quad i \xrightarrow{v} j \in L_{t,1}. \tag{4.3.2}$$

Lemma 4.5. *The homology group $H_1(\dot{L}_t)$ is a free $\mathbb{Z}$-module of rank $1+\#Z_t-n$ for any t.*

Proof. We remark that by the assumption (LA-3) the underlying non-directional graph of the type diagram is also connected. Thus the conclusion immediately follows from the following exact sequence:

$$0 \leftarrow \mathbb{Z} \leftarrow L_{t,0} \overset{\partial}{\leftarrow} L_{t,1} \leftarrow H_1(\dot{L}_t) \leftarrow 0.$$

□

Remark. Although the type diagram depends on "cuts" of the complex plane $\mathbb{C}$, we can choose the cuts so that the type diagram does not change for any t. In fact, it is enough for an ordinary turning point to refrain from crossing a cut, and that is always possible by deforming each cut continuously because ordinary turning points never merge by the assumption. Therefore, in what follows, we assume that the type diagram remains unchanged for any t, and the suffix t of $\dot{L}_t$ etc. will be omitted.

We often need a basis of $H_1(\dot{L})$ over $\mathbb{Z}$ to calculate the index of a turning point. If the type diagram can be realized as a plane graph, that is, if it can be drawn in $\mathbb{R}^2$ without any intersection between edges, then we can easily obtain a basis of $H_1(\dot{L})$ in the following way. Let D be a plane graph that represents the type diagram. Then $\mathbb{R}^2 \setminus D$ consists of bounded connected components $U_1, \ldots, U_l$ and an unbounded connected component U_∞.

Definition 4.6. Let U be a bounded connected component of $\mathbb{R}^2 \setminus D$. A walking path around U is the closed path of D generated by tracing the following walking:

1. We start from a node belonging to the boundary of U.

2. We proceed on edges so that our left hands alway touch U. Here we ignore the direction of an edge.
3. Our walking turns around U only once, and we come back to the starting node.

Let D_i be the walking path around U_i $(i = 1, 2, \dots, l)$.

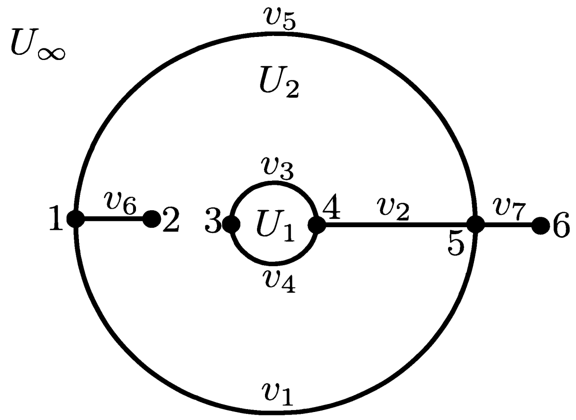

Figure 7. An example of the type diagram.

Example 4.7. Fig. 7 above is an example of the type diagram that consists of 6 nodes $\{1, 2, \dots, 6\}$ and 7 edges indexed by $\{v_1, v_2, \dots, v_7\}$. The edge indexed by v_1 (resp. $v_2, \dots, v_7$) corresponds to a turning point v_1 (resp. $v_2, \dots, v_7$) of type $(1, 5)$ (resp. $(4, 5), \dots, (5, 6)$). $\mathbb{R}^2 \setminus D$ consists of two bounded connected components, and the walking paths D_1 and D_2 are given by

$$D_1 : 4 \xrightarrow{v_3} 3 \xrightarrow{v_4} 4,$$

and

$$D_2 : 1 \xrightarrow{v_1} 5 \xrightarrow{v_2} 4 \xrightarrow{v_4} 3 \xrightarrow{v_3} 4 \xrightarrow{v_2} 5 \xrightarrow{v_5} 1 \xrightarrow{v_6} 2 \xrightarrow{v_6} 1.$$

Let $[D_i]$ denote the image of D_i in L_1 $(i = 1, \dots, l)$. Apparently each $[D_i]$ is a 1-cycle of the complex $\dot{L}$. Now we have:

Lemma 4.8. *$([D_1], [D_2], \dots, [D_l])$ is a basis of $H_1(\dot{L})$.*

Proof. Let e_k $(k = 1, 2, \dots, \#Z_t)$ denote an edge of the type diagram. Then each $[D_i]$ can be written in the form

$$[D_i] = \sum_{k=1}^{\#Z_t} d_{ik} e_k, \qquad d_{ik} \in \mathbb{Z}.$$

We first show that $[D_1], [D_2], \ldots, [D_l]$ are independent over $\mathbb{Z}$. Let us consider the equation

$$a_1[D_1] + a_2[D_2] + \cdots + a_l[D_l] = 0, \quad a_i \in \mathbb{Z},$$

and let U_j be a connected component such that D_j and the boundary of U_∞ have a common edge e_k for some k. Conditions 2 of Definition 4.6 implies that $d_{jk} \neq 0$, otherwise U_j is on the both sides of e_k. Conditions 3 implies $|d_{jk}| < 2$ because the walking path D_j already turned around U_j before the second e_k appears in the path. Thus we get $d_{jk} = \pm 1$. Moreover the edge e_k never appears in any path D_i $(i \neq j)$ since an edge is shared with at most two connected components. Thus we get $a_j = 0$. By repeating the similar arguments (the next step is to consider a connected component whose walking path has a common edge with U_∞ or the U_j above), we have $a_i = 0$ for any i.

Let $M \subset H_1(\dot{L})$ be the free $\mathbb{Z}$-module generated by $[D_i]$'s. Due to Euler's theorem for a plane graph (i.e. the number of nodes $-$ the number of edges $+$ the number of connected components $= 2$) and Lemma 4.5, we have $\mathrm{Rank}_{\mathbb{Z}} M = \mathrm{Rank}_{\mathbb{Z}} H_1(\dot{L})$. Thus for any $u \in H_1(\dot{L})$ we can find an integer $p \neq 0$ such that $pu \in M$, that is,

$$pu = a_1[D_1] + a_2[D_2] + \cdots + a_l[D_l], \quad a_i \in \mathbb{Z}$$

holds. Employing the same argument as above we conclude that each a_i can be divided by p, and thus $u \in M$. □

Note that all type diagrams that we have encountered so far are realized by plane graphs. However, in what follows, we do not necessarily assume the type diagram a plane graph.

We will establish an isomorphism of the homology groups $H_1(\dot{L})$ and $H_1(\widehat{W}_t \setminus \widehat{E}_t; \mathbb{Z})$. For that purpose, we first prepare an appropriate cut space H_t. Let $x_0 \in \mathbb{C}$ satisfy $x_0 \notin Z_t \cup E_{\mathrm{sing}}$ for any t, and for a point $p(t) \in Z_t \cup E_{\mathrm{sing}}$ we set

$$h_{p(t)} = (p(t), \infty) \subset \{\text{the half line starting from } x_0 \text{ that passes through } p(t)\} \tag{4.3.3}$$

(see Fig. 8). We assume $(Z_t \cup E_{\mathrm{sing}}) \cap h_{p(t)} = \emptyset$ for any t near t_0.

Definition 4.9. The cut space H_t is $\mathbb{C} \setminus E_{\mathrm{sing}}$ equipped with the cut lines $\{h_{p(t)}\}$ $(p(t) \in Z_t \cup E_{\mathrm{sing}})$.

Remark. The detailed form of H_t will be given by (4.4.6) in Subsection 4.4.

In what follows, we fix the cut space H_t. Note that H_t has a cut line emanating not only from a ramification point in Z_t but also from a singular point in E_{sing}, and hence that emanating from E_{sing} is ignored in considering the type diagram.

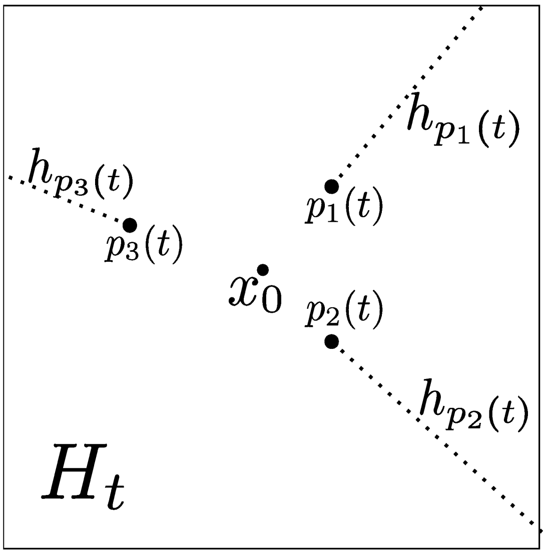

Figure 8. The cut space H_t.

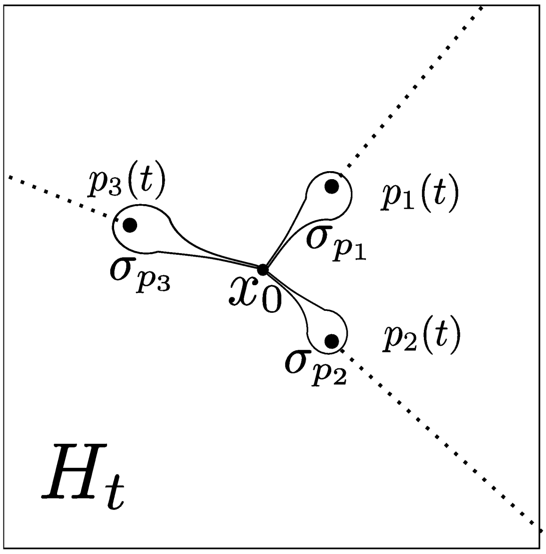

Figure 9. The path σ_p.

Let t_0 be a point in T. For a point $p \in Z_{t_0} \cup E_{\rm sing}$ we denote by σ_p a closed smooth path in $\mathbb{C} \setminus \{Z_{t_0} \cup E_{\rm sing}\}$ that satisfies the conditions below (see Fig. 9 also):

1. The path σ_p starts from x_0 and ends at the same point, and σ_p crosses the cut that emanates from p only once and never crosses any other cut. If p is a singular point, the orientation of the path is taken to be anti-clockwise around p.
2. The closure of the domain surrounded by the path does not contain either any turning point or any singular point other than p.

Let $\mathbb{Z}_n$ denote the set $\{1, 2, \ldots, n\}$. We also introduce a path $\widehat{\sigma}_t$ in $\widehat{W}_t \setminus \widehat{E}_t$ as follows:

- For an edge $e = i \xrightarrow{v} j$ of the type diagram,

$$\widehat{\sigma}_{t,e} = \text{the lift of } \sigma_v \text{ by } \widehat{\pi}_{\widehat{W}_t} \text{ starting from } (\lambda_{t,i}(x_0), x_0; 1). \tag{4.3.4}$$

- For any $(k;p) \in \mathbb{Z}_n \times E_{\rm sing}$,

$$\widehat{\sigma}_{t,(k;p)} = \text{the lift of } \sigma_p \text{ by } \widehat{\pi}_{\widehat{W}_t} \text{ starting from } (\lambda_{t,k}(x_0), x_0; 1). \tag{4.3.5}$$

Since the degree of a ramification point is 2, the following lemma is easy to prove; still it is a key for the subsequent argument.

Lemma 4.10. *Let $e = i \xrightarrow{v} j$ be an edge of the type diagram. The end point of the path $\widehat{\sigma}_{t,e}$ is $(\lambda_{t,j}(x_0), x_0; 1)$, and the lift $\widehat{\sigma}_{t,e}$ does not depend on the choice of σ_v up to a homotopic equivalence in $\widehat{W}_t \setminus \widehat{E}_t$.*

Remark. The lemma above implies, in particular, that σ_v with a different orientation (clockwise or anti-clockwise) gives the same lift $\widehat{\sigma}_{t,e}$ up to a homotopic equivalence in $\widehat{W}_t \setminus \widehat{E}_t$.

Let $C_1(\widehat{W}_t \setminus \widehat{E}_t; \mathbb{Z})$ designate the set of 1-singular chains of $\widehat{W}_t \setminus \widehat{E}_t$. We define a morphism

$$\Phi_t \colon L_1 \oplus \mathbb{Z}^{\mathbb{Z}_n \times E_{\mathrm{sing}}} \to C_1(\widehat{W}_t \setminus \widehat{E}_t; \mathbb{Z}) \tag{4.3.6}$$

of $\mathbb{Z}$-modules by

$$\begin{aligned} \Phi_t(e) &= \widehat{\sigma}_{t,e}, && \text{for} \quad e = i \to vj \in L_1, \\ \Phi_t((k;p)) &= \widehat{\sigma}_{t,(k;p)}, && \text{for} \quad (k;p) \in \mathbb{Z}_n \times E_{\mathrm{sing}}. \end{aligned} \tag{4.3.7}$$

Then the morphism Φ_t induces a morphism

$$\Psi_t \colon H_1(\dot{L}) \oplus \mathbb{Z}^{\mathbb{Z}_n \times E_{\mathrm{sing}}} \to H_1(\widehat{W}_t \setminus \widehat{E}_t; \mathbb{Z}) \tag{4.3.8}$$

which is independent of the choice of paths. For any closed path $\widehat{\sigma}$ in $\widehat{W}_t \setminus \widehat{E}_t$ which does not contain a point in $\widehat{Z}_t$, $\widehat{\pi}_{\widehat{W}_t *}(\widehat{\sigma})$ can be homotopically deformed in $\mathbb{C} \setminus (Z_t \cup E_{\mathrm{sing}})$ to a path that is a combination of $\pm\sigma_p$'s ($p \in Z_t \cup E_{\mathrm{sing}}$). Therefore the map Ψ_t is surjective, as the lifts of σ_v and $-\sigma_v$ ($v \in Z_t$) by $\widehat{\pi}_{\widehat{W}_t}$ give homotopically equivalent paths in $\widehat{W}_t \setminus \widehat{E}_t$ by Lemma 4.10. Moreover by Lemmata 4.1 and 4.5 we have

$$\mathrm{Rank}_{\mathbb{Z}} H_1(\dot{L}) \oplus \mathbb{Z}^{\mathbb{Z}_n \times E_{\mathrm{sing}}} = \mathrm{Rank}_{\mathbb{Z}} H_1(\widehat{W}_t \setminus \widehat{E}_t; \mathbb{Z}) = 1 + \#Z_t + n(\#E_{\mathrm{sing}} - 1).$$

Hence the map Ψ_t is injective, and we have obtained the following proposition.

Proposition 4.11. *The morphism Ψ_t defined above gives an isomorphism of $\mathbb{Z}$-modules $H_1(\dot{L}) \oplus \mathbb{Z}^{\mathbb{Z}_n \times E_{\mathrm{sing}}}$ and $H_1(\widehat{W}_t \setminus \widehat{E}_t; \mathbb{Z})$.*

Let $\mathbb{Z}^2_{n,<}$ (resp. $\mathbb{Z}^2_{n,\neq}$) denote the set $\{(i,j) \in \mathbb{Z}^2_n;\ i < j\}$ (resp. $\{(i,j) \in \mathbb{Z}^2_n;\ i \neq j\}$). For any $(i,j) \in \mathbb{Z}^2_{n,\neq}$, we define a subset $L_1(i,j)$ of the set L_1 by

$$L_1(i,j) = \{\sigma \in L_1;\ \partial\sigma = \{j\} - \{i\}\}, \tag{4.3.9}$$

that is, $L_1(i,j)$ is the set of paths form the node i to the node j of the type diagram. Let $\{\alpha_{ij}\}_{(i,j) \in \mathbb{Z}^2_{n,\neq}}$ be a family of paths in the type diagram with $\alpha_{ij} \in L_1(i,j)$.

Definition 4.12. We say that $\{\alpha_{ij}\}_{(i,j) \in \mathbb{Z}^2_{n,\neq}}$ satisfies the 1-cocycle condition in the type diagram if Conditions 1 and 2 below hold:

1. (anti-symmetric) For any $(i,j) \in \mathbb{Z}^2_{n,\neq}$, $a_{ij} \in L_1(i,j)$ and

$$\alpha_{ij} = -\alpha_{ji}. \tag{4.3.10}$$

2. (1-cocycle condition) For mutually different indices $i, j, k \in \mathbb{Z}_n$

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0. \tag{4.3.11}$$

We fix a family $\{\alpha_{ij}\}$ that satisfies the 1-cocycle condition in the type diagram. Note that such a family $\{\alpha_{ij}\}$ always exists. If paths $\alpha_{12}, \alpha_{23}, \ldots, \alpha_{n-1,n}$ ($\alpha_{i,i+1} \in L_1(i, i+1)$) are given, then for $(i,j) \in \mathbb{Z}^2_{n,<}$ we can determine α_{ij} and α_{ji} uniquely by

$$\alpha_{ij} = \alpha_{i,i+1} + \alpha_{i+1,i+2} + \cdots + \alpha_{j-1,j} \quad \text{and} \quad \alpha_{ji} = -\alpha_{ij}. \tag{4.3.12}$$

Let v be a virtual turning point of type (i,j) at $t = t_0$. It then follows from the definition that we can find a closed smooth curve C_v and a continuous function $\mu(x)$ on C_v that satisfy the conditions of Definition 2.1. Noticing $C_v = (-l) + (l + C_v - l) + l$ where l is a path from x_0 to v in H_{t_0}, we have

$$\int_{x_0}^{v} \frac{\lambda_{t_0,j}(x) - \lambda_{t_0,i}(x)}{p(x)} dx + \int_{C_{x_0}} \widetilde{\mu}(x) dx = 0. \tag{4.3.13}$$

Here the path of the first integration is taken in H_{t_0}, C_{x_0} is the closed path $l + C_v - l$, and $\widetilde{\mu}(x)$ is a continuous extension of $\mu(x)$ so that $\Lambda(\widetilde{\mu}(x), x) = 0$ still holds for $x \in C_{x_0}$. We may suppose that C_{x_0} is written by a continuous function $c(s)\colon [0,1] \to \mathbb{C} \setminus E_{\text{sing}}$ with $c(0) = c(1) = x_0$, and let $\widehat{C}_{x_0}$ be the path in $\widehat{W}_{t_0} \setminus \widehat{E}_{t_0}$ defined by $(\widetilde{\mu}(c(s)), c(s); 1)$ $(0 \le s \le 1)$. Then (4.3.13) becomes

$$\int_{x_0}^{v} \frac{\lambda_{t_0,j}(x) - \lambda_{t_0,i}(x)}{p(x)} dx + \int_{\widehat{C}_{x_0}} \omega = 0 \tag{4.3.14}$$

where the 1-form ω is given by (4.2.3). Since $\widehat{\sigma} = \widehat{C}_{x_0} + \Phi_{t_0}(\alpha_{ji})$ is a closed path in $\widehat{W}_{t_0} \setminus \widehat{E}_{t_0}$, $\widehat{\sigma}$ defines the 1-cycle $[\widehat{\sigma}] \in H_1(\widehat{W}_{t_0} \setminus \widehat{E}_{t_0}; \mathbb{Z})$, and we obtain

$$\int_{x_0}^{v} \frac{\lambda_{t_0,j}(x) - \lambda_{t_0,i}(x)}{p(x)} dx + \int_{\Phi_{t_0}(\alpha_{ij})} \omega + \int_{[\widehat{\sigma}]} \omega = 0. \tag{4.3.15}$$

We will introduce a morphism $I_t\colon L_1 \oplus \mathbb{Z}^{\mathbb{Z}_n \times E_{\text{sing}}} \to \mathbb{C}$ of $\mathbb{Z}$ modules to link the second and the third terms of (4.3.15) to the type diagram.

Definition 4.13. The morphism $I_t\colon L_1 \oplus \mathbb{Z}^{\mathbb{Z}_n \times E_{\text{sing}}} \to \mathbb{C}$ is defined in the following way:

1. For any edge $k \xrightarrow{w} l \in L_1$ with $w \in Z_t$ being of type (k,l), we set

$$I_t(k \xrightarrow{w} l) = \int_{x_0}^{w} \frac{\lambda_{t,k}(x) - \lambda_{t,l}(x)}{p(x)} dx. \tag{4.3.16}$$

Here the path of integration is the segment from x_0 to w in H_t.

2. For $(k;q) \in \mathbb{Z}_n \times E_{\mathrm{sing}}$, we set

$$I_t((k;q)) = 2\pi\sqrt{-1}\,\mathrm{Res}_q\Big(\frac{\lambda_{t,k}(x)}{p(x)}\Big) \tag{4.3.17}$$

where $\mathrm{Res}_x(f)$ designate the residue of a holomorphic function f at x.

Modifying the path of integration (see Fig. 9 also), we can easily show the lemma below.

Lemma 4.14. *For any edge $e = k \xrightarrow{w} l \in L_1$ (resp. $(k;q) \in \mathbb{Z}_n \times E_{\mathrm{sing}}$), one has*

$$\int_{\widehat{\sigma}_{t,e}} \omega = I_t(k \xrightarrow{w} l), \qquad \Big(\text{resp.}\ \int_{\widehat{\sigma}_{t,(k;q)}} \omega = I_t((k;q))\Big) \tag{4.3.18}$$

where $\widehat{\sigma}_{t,e}$ (resp. $\widehat{\sigma}_{t,(k;q)}$) is given by (4.3.4) (*resp.* (4.3.5)).

We fix a basis of $H_1(\dot{L})$ and denote it by

$$(g_1, g_2, \ldots, g_\kappa), \qquad g_k \in H_1(\dot{L}), \tag{4.3.19}$$

and what follows, $H_1(\dot{L})$ is identified with $\mathbb{Z}^\kappa$ by this basis. Then by Proposition 4.11 and Lemma 4.14 we find an index

$$\{\alpha_k\}_{k=1}^{\kappa} \oplus \{\beta_{k,p}\}_{(k;p)\in\mathbb{Z}_n\times E_{\mathrm{sing}}} \in H_1(\dot{L}) \oplus \mathbb{Z}^{\mathbb{Z}_n\times E_{\mathrm{sing}}}, \qquad \alpha_k,\ \beta_{k,p} \in \mathbb{Z},$$

so that we have

$$\int_{[\widehat{\sigma}]} \omega = I_{t_0}(\{\alpha_k\} \oplus \{\beta_{k,p}\}) = \sum_{k=1}^{\kappa} \alpha_k I_{t_0}(g_k) + \sum_{(k;p)\in\mathbb{Z}_n\times E_{\mathrm{sing}}} \beta_{k,p} I_{t_0}((k;p)). \tag{4.3.20}$$

Let $F_{t,i,j}(x)$ denote the function

$$F_{t,i,j}(x) = \int_{x_0}^{x} \frac{\lambda_{t,j}(x) - \lambda_{t,i}(x)}{p(x)} dx \tag{4.3.21}$$

where the path of integration is taken in H_t. Note that $F_{t,i,j}(x)$ is always regarded as a (single valued) holomorphic function in the cut space H_t. Then noticing (4.3.15) and (4.3.20), we obtain finally the following proposition:

Proposition 4.15. *For any turning point v of type $(i,j) \in \mathbb{Z}_{n,\neq}^2$ at $t = t_0$, there exists an index $\alpha \oplus \beta \in H_1(\dot{L}) \oplus \mathbb{Z}^{\mathbb{Z}_n\times E_{\mathrm{sing}}}$ such that v is a root of the equation*

$$F_{t_0,i,j}(x) + I_{t_0}(\alpha_{ij}) + I_{t_0}(\alpha \oplus \beta) = 0. \tag{4.3.22}$$

Conversely for any index $\alpha \oplus \beta \in H_1(\dot{L}) \oplus \mathbb{Z}^{\mathbb{Z}_n\times E_{\mathrm{sing}}}$ each root of (4.3.22) *determines a turning point.*

We define a holomorphic function $f_{t,i,j,\alpha\oplus\beta}$ in H_t with a holomorphic parameter t by

$$f_{t,i,j,\alpha\oplus\beta}(x) = F_{t,i,j}(x) + I_t(\alpha_{ij}) + I_t(\alpha \oplus \beta) \tag{4.3.23}$$

for any $(i, j, \alpha \oplus \beta) \in \mathbb{Z}^2_{n,\neq} \times (H_1(\dot{L}) \oplus \mathbb{Z}^{\mathbb{Z}_n \times E_{\mathrm{sing}}})$.

§ 4.4. The Riemann Manifold $\mathcal{R}_{\mathrm{sym},t}$

Let v be a turning point that is a root of $f_{t_0,i,j,\,\alpha\oplus\beta} = 0$. A Stokes curve emanating from v is, in our formulation, a smooth locus of the analytic set defined by the equation

$$\operatorname{Im} f_{t_0,i,j,\alpha\oplus\beta}(x) = 0 \tag{4.4.1}$$

that emanates from v. Hence we often need an analytic continuation of $f_{t_0,i,j,\alpha\oplus\beta}$ when the Stokes curve crosses a cut of H_{t_0}. The following vectors $r_{k\xrightarrow{w}l}$ and $r_{(k;p)}$ play an important role to describe an analytic continuation of $f_{t,i,j,\alpha\oplus\beta}$.

Definition 4.16. We set the vectors $r_{k\xrightarrow{w}l}$ and $r_{(k;p)}$ as follows:

1. For any edge $k \xrightarrow{w} l$ with $w \in Z_t$ being of type (k, l),

   $$r_{k\xrightarrow{w}l} \in \mathbb{Z}^{\kappa} = [k \xrightarrow{w} l + \alpha_{lk}] \in H_1(\dot{L}). \tag{4.4.2}$$

   Here we identify $H_1(\dot{L})$ with $\mathbb{Z}^{\kappa}$ by the basis (4.3.19). Note that $r_{k\xrightarrow{w}l} = -r_{l\xrightarrow{w}k}$ holds.
2. For $(k; p) \in \mathbb{Z}_n \times E_{\mathrm{sing}}$,

   $$r_{(k;p)} = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^{\mathbb{Z}_n \times E_{\mathrm{sing}}} \tag{4.4.3}$$

   where the element indexed by $(k; p)$ is 1.

Let $c\colon [0, 1] \to \mathbb{C}$ be a continuous curve in $\mathbb{C} \setminus E_{\mathrm{sing}}$.

Proposition 4.17. *Let $p \in Z_t \cup E_{\mathrm{sing}}$ and assume that the curve c crosses the cut h_p only once and never crosses any other cut. Then an analytic continuation of $f_{t,i,j,\,\alpha\oplus\beta}$ along c has the same form $f_{t,\,\xi}$. Here the index $\xi \in \mathbb{Z}^2_{n,\neq} \times (H_1(\dot{L}) \oplus \mathbb{Z}^{\mathbb{Z}_n \times E_{\mathrm{sing}}})$ is given as follows*:

1. *Suppose that p is an ordinary turning point $v \in Z_t$.*
   (a) *If the type of v is (i, j), then $\xi = (j, i, \alpha - 2r_{i\xrightarrow{v}j} \oplus \beta)$.*
   (b) *If that is (j, k) (resp. (i, k)) for $k \notin \{i, j\}$, then $\xi = (i, k, \alpha + r_{j\xrightarrow{v}k} \oplus \beta)$ (resp. $\xi = (k, j, \alpha - r_{i\xrightarrow{v}k} \oplus \beta)$) respectively.*

2. *Suppose that* $p \in E_{\text{sing}}$ *is a singular point.*

   (a) *If the curve* c *crosses the cut* h_p *anti-clockwise, then* $\xi = (i, j, \alpha \oplus \beta + r_{(j;p)} - r_{(i;p)})$;

   (b) *otherwise* $\xi = (i, j, \alpha \oplus \beta + r_{(i;p)} - r_{(j;p)})$.

Proof. We designate by $\mathrm{A}_c(f)$ an analytic continuation of f along c.

1. (a) By a modification of the integration path, $\mathrm{A}_c(F_{t,i,j})$ is given by

$$\int_{x_0}^{x} \frac{\lambda_{t,i}(x) - \lambda_{t,j}(x)}{p(x)} dx + I_t(j \xrightarrow{v} i) - I_t(i \xrightarrow{v} j) = F_{t,j,i}(x) - 2I_t(i \xrightarrow{v} j).$$

Thus we have:

$$\begin{aligned} \mathrm{A}_c(f_{t,i,j,\alpha\oplus\beta}) &= F_{t,j,i}(x) - 2I_t(i \xrightarrow{v} j) + I_t(\alpha_{ij}) + I_t(\alpha \oplus \beta) \\ &= F_{t,j,i}(x) + I_t(\alpha_{ji}) - 2(I_t(i \xrightarrow{v} j) + I_t(\alpha_{ji})) + I_t(\alpha \oplus \beta) \\ &= F_{t,j,i}(x) + I_t(\alpha_{ji}) + I_t(\alpha - 2r_{i \xrightarrow{v} j} \oplus \beta) \\ &= f_{t,j,i,\alpha - 2r_{i \xrightarrow{v} j} \oplus \beta}. \end{aligned}$$

1. (b) If the type of v is (j, k), then we have

$$\begin{aligned} \mathrm{A}_c(f_{t,i,j,\,\alpha\oplus\beta}) &= F_{t,i,k}(x) + I_t(j \xrightarrow{v} k) + I_t(\alpha_{ij}) + I_t(\alpha \oplus \beta) \\ &= F_{t,i,k}(x) + I_t(j \xrightarrow{v} k) + I_t(\alpha_{ik} + \alpha_{kj}) + I_t(\alpha \oplus \beta) \\ &= F_{t,i,k}(x) + I_t(\alpha_{ik}) + I_t(\alpha + r_{j \xrightarrow{v} k} \oplus \beta) \\ &= f_{t,i,k,\alpha + r_{j \xrightarrow{v} k} \oplus \beta}, \end{aligned}$$

and if v is of type (k, i), then

$$\mathrm{A}_c(f_{t,i,j,\alpha\oplus\beta}) = \mathrm{A}_c(-f_{t,j,i,-\alpha\oplus-\beta}) = -f_{t,j,k,-\alpha + r_{i \xrightarrow{v} k} \oplus -\beta} = f_{t,k,j,\alpha - r_{i \xrightarrow{v} k} \oplus \beta}.$$

2. The proof is similar. □

We denote by Ξ the index space

$$\mathbb{Z}^2_{n,\neq} \times (H_1(\dot{L}) \oplus \mathbb{Z}^{\mathbb{Z}_n \times E_{\text{sing}}}), \tag{4.4.4}$$

and set

$$X_t = \mathbb{C} \setminus \Big(\bigcup_{p(t) \in Z_t \cup E_{\text{sing}}} h_{p(t)} \cup E_{\text{sing}} \Big). \tag{4.4.5}$$

Let us now consider the cut space H_t to be the set

$$H_t = X_t \bigsqcup \Big(\bigcup_{p(t) \in Z_t \cup E_{\text{sing}}} h^R_{p(t)} \Big) \bigsqcup \Big(\bigcup_{p(t) \in Z_t \cup E_{\text{sing}}} h^L_{p(t)} \Big), \tag{4.4.6}$$

where $h^R_{p(t)}$ and $h^L_{p(t)}$ are copies of the open half line $h_{p(t)}$ that is defined by (4.3.3). We make H_t a topological space so that $h^R_{p(t)}$ (resp. $h^L_{p(t)}$) becomes the right (resp. left) side boundary of X_t on $h_{p(t)}$. Let $\pi_{H_t}\colon H_t \to \mathbb{C} \setminus E_{\mathrm{sing}}$ denote the canonical projection. For any point $x \in H_t$, let x^* denote the opposite point in $\pi^{-1}_{H_t}\pi_{H_t}(x)$, that is, if $x \in h^R_{p(t)}$, then x^* is the point in $h^L_{p(t)}$ with $\pi_{H_t}(x^*) = \pi_{H_t}(x)$. Note that $p^* = p$ for any $p \in Z_t$. Let $H_{t,\xi}$ designate a copy of H_t for $\xi = (i, j, \alpha \oplus \beta) \in \Xi$. We set

$$H_{t,\,\Xi} = \bigsqcup_{\xi \in \Xi} H_{t,\xi}. \tag{4.4.7}$$

and let $\pi_{H_{t,\Xi}}\colon H_{t,\Xi} \to H_t$ designate the canonical projection. Taking Proposition 4.17 into account, we will construct a Riemann manifold $\mathcal{R}_t$ over $\mathbb{C} \setminus E_{\mathrm{sing}}$ by gluing $H_{t,\xi}$'s. We first define a map $\mathcal{J}_t\colon H_{t,\Xi} \to H_{t,\Xi}$ as follows.

Definition 4.18. The map $\mathcal{J}_t$ is defined by the following way.

1. The map is the identity on the fiber $\pi^{-1}_{H_{t,\Xi}}(x)$ for any $x \in X_t \setminus Z_t$.
2. If $x \in h^R_p \cup h^L_p \cup \{p\}$ with $p \in Z_t$ being of type (i, j), then the map is defined on the fiber $\pi^{-1}_{H_{t,\Xi}}(x)$ as:
   - $\mathcal{J}_t(x, i, j, \alpha \oplus \beta) = (x^*, j, i, \alpha - 2r_{i \overset{v}{\to} j} \oplus \beta)$ and
     $\mathcal{J}_t(x, j, i, \alpha \oplus \beta) = (x^*, i, j, \alpha - 2r_{j \overset{v}{\to} i} \oplus \beta)$ for any $\alpha \oplus \beta$,
   - $\mathcal{J}_t(x, k, j, \alpha \oplus \beta) = (x^*, k, i, \alpha + r_{j \overset{v}{\to} i} \oplus \beta)$ and
     $\mathcal{J}_t(x, k, i, \alpha \oplus \beta) = (x^*, k, j, \alpha + r_{i \overset{v}{\to} j} \oplus \beta)$ for any $\alpha \oplus \beta$ and $k \notin \{i, j\}$,
   - $\mathcal{J}_t(x, i, k, \alpha \oplus \beta) = (x^*, j, k, \alpha - r_{i \overset{v}{\to} j} \oplus \beta)$ and
     $\mathcal{J}_t(x, j, k, \alpha \oplus \beta) = (x^*, i, k, \alpha - r_{j \overset{v}{\to} i} \oplus \beta)$ for each $\alpha \oplus \beta$ and $k \notin \{i, j\}$,
   - $\mathcal{J}_t(\widetilde{x}) = \widetilde{x}$ for any other point $\tilde{x} \in \pi^{-1}_{H_{t,\Xi}}(x)$.
3. If $x^R \in h^R_p$ with $p \in E_{\mathrm{sing}}$, then for any k, l and $\alpha \oplus \beta$
   $$\mathcal{J}_t(x^R, k, l, \alpha \oplus \beta) = ((x^R)^*, k, l, \alpha \oplus \beta + r_{(j;p)} - r_{(i;p)}),$$
   and if $x^L \in h^L_p$ with $p \in E_{\mathrm{sing}}$, then for any k, l and $\alpha \oplus \beta$
   $$\mathcal{J}_t(x^L, k, l, \alpha \oplus \beta) = ((x^L)^*, k, l, \alpha \oplus \beta - r_{(j;p)} + r_{(i;p)}).$$

Since $\mathcal{J}_t$ is an involution map in $H_{t,\Xi}$ (i.e. $\mathcal{J}_t \circ \mathcal{J}_t = \mathrm{Id}_{H_{t,\Xi}}$), we can define an equivalence relation $\overset{\mathcal{J}_t}{\sim}$ in the following way:

$$\widetilde{x} \overset{\mathcal{J}_t}{\sim} \widetilde{y} \quad \text{if} \quad \widetilde{x} = \mathcal{J}_t(\widetilde{y}) \quad \text{or} \quad \widetilde{x} = \widetilde{y}. \tag{4.4.8}$$

Then $\mathcal{R}_t$ and a function $f_{t,\Xi}$ in $\mathcal{R}_t$ are introduced as:

Definition 4.19. The Riemann manifold $\mathcal{R}_t$ over $\mathbb{C} \setminus E_{\mathrm{sing}}$ is the set of equivalence classes $H_{t,\Xi}/\overset{\mathcal{J}_t}{\sim}$, and the single valued function $f_{t,\Xi}$ in $\mathcal{R}_t$ is determined by the family of holomorphic functions $\{f_{t,i,j,\alpha\oplus\beta}\}_{(i,j,\alpha\oplus\beta)\in\Xi}$.

Let $\pi_{\mathcal{R}_t}\colon \mathcal{R}_t \to \mathbb{C} \setminus E_{\mathrm{sing}}$ (resp. $\rho_{H_{t,\Xi}}\colon H_{t,\Xi} \to \mathcal{R}_t$) denote the canonical projection (resp. surjection). We can readily confirm the following properties of $\mathcal{R}_t$.

- The set $\mathcal{R}_t$ can be regarded as a smooth complex manifold that depends holomorphically on a parameter t, and $f_{t,\Xi}$ is a single valued holomorphic function in $\mathcal{R}_t$.
- If $x \in Z_t$ is an ordinary turning point of type (i,j), then we have $(x,i,j,\alpha\oplus\beta) \overset{\mathcal{J}_t}{\sim} (x,j,i,\alpha - 2r_{i\overset{x}{\to}j} \oplus \beta)$, and they give the same point $\widetilde{x}$ in $\mathcal{R}_t$. Therefore $\mathcal{R}_t$ is locally a double covering space with respect to the map $\pi_{\mathcal{R}_t}$ near $\widetilde{x}$, and $\widetilde{x}$ is a ramification point of degree 2. In the same way, for any point $\widetilde{x} = \rho_{H_{t,\Xi}}(x,k,l,\alpha\oplus\beta)$ with $\{k,l\} \cap \{i,j\} \neq \emptyset$ and $x \in Z_t$, $\mathcal{R}_t$ has the same topological structure near $\widetilde{x}$.
- If $x \in E_{\mathrm{sing}}$ is a singular point, then locally $\mathcal{R}_t$ is a finite disjoint union of Log type covering spaces over $V \setminus \{x\}$ for a small neighborhood $V \subset \mathbb{C}$ of x.

Let us define another involution map $\mathcal{I}_t\colon H_{t,\Xi} \to H_{t,\Xi}$ by

$$\mathcal{I}_t(x,i,j,\alpha\oplus\beta) = (x,j,i,-\alpha\oplus-\beta). \tag{4.4.9}$$

Then it follows from the commutativity of $\mathcal{I}_t$ and $\mathcal{J}_t$, i.e.

$$\mathcal{I}_t \circ \mathcal{J}_t = \mathcal{J}_t \circ \mathcal{I}_t, \tag{4.4.10}$$

that the map $\mathcal{I}_t$ induces an involution map $\mathcal{I}_{\mathcal{R}_t}\colon \mathcal{R}_t \to \mathcal{R}_t$. Since the definitions of turning points and Stokes curves are symmetric with respect to the change of indices, that is, the equations

$$\operatorname{Im} f_{t,i,j,\alpha\oplus\beta}(x) = 0 \quad \text{and} \quad \operatorname{Im} f_{t,j,i,-\alpha\oplus-\beta}(x) = 0$$

define the same Stokes curve, every point $\widetilde{x} \in \mathcal{R}_t$ is required to be identified with $\mathcal{I}_{\mathcal{R}_t}(\widetilde{x}) \in \mathcal{R}_t$. Hence we will introduce the following Riemann manifold:

Definition 4.20. The Riemann manifold $\mathcal{R}_{\mathrm{sym},t}$ is defined as

$$\mathcal{R}_{\mathrm{sym},t} = \mathcal{R}_t/\sim \tag{4.4.11}$$

where $\widetilde{x} \sim \widetilde{y}$ if $\mathcal{I}_{\mathcal{R}_t}(\widetilde{x}) = \widetilde{y}$ or $\widetilde{x} = \widetilde{y}$.

We designate by $\pi_{\mathcal{R}_{\mathrm{sym},t}}\colon \mathcal{R}_{\mathrm{sym},t} \to \mathbb{C} \setminus E_{\mathrm{sing}}$ (resp. $\rho_{H_{t,\Xi}}\colon H_{t,\Xi} \to \mathcal{R}_{\mathrm{sym},t}$) the canonical projection (resp. surjection). Note that the equivalence class of $\widetilde{x} \in \mathcal{R}_{\mathrm{sym},t}$ is given by the set of (possibly duplicated) points in $H_{t,\Xi}$

$$\{\widetilde{y}, \mathcal{I}_t(\widetilde{y}),\ \mathcal{J}_t(\widetilde{y}),\ (\mathcal{I}_t \circ \mathcal{J}_t)(\widetilde{y})\} \tag{4.4.12}$$

for some point $\widetilde{y} \in H_{t,\Xi}$ with $\rho_{H_{t,\Xi}}(\widetilde{y}) = \widetilde{x}$ because of the commutativity (4.4.10). Now we can define the most basic objects, i.e. a turning point and a Stokes curve, in $\mathcal{R}_{\mathrm{sym},t}$ using $f_{t,\Xi}$.

Definition 4.21. A **turning point** in $\mathcal{R}_{\mathrm{sym},t}$ is a point in the zero set of $f_{t,\Xi}$, and a **Stokes curve** in $\mathcal{R}_{\mathrm{sym},t}$ emanating from a turning point $\widetilde{v} \in \mathcal{R}_{\mathrm{sym},t}$ is a smooth locus of the zero set of $\mathrm{Im}\, f_{t,\Xi}$ that emanates from $\widetilde{v}$.

These notions are well-defined on $\mathcal{R}_{\mathrm{sym},t}$ because $f_{t,\Xi}(\mathcal{I}_{\mathcal{R}_t}(\widetilde{x})) = -f_{t,\Xi}(\widetilde{x})$ holds for any $\widetilde{x} \in \mathcal{R}_t$. Let $\widetilde{Z}_t \subset \mathcal{R}_{\mathrm{sym},t}$ be the image of the set of ramification points in $\mathcal{R}_t$ by the canonical surjection, i.e.

$$\widetilde{Z}_t = \rho_{H_{t,\Xi}}(\{(x,i,j,\alpha\oplus\beta) \in H_{t,\Xi};\, x \in Z_t,\, (\{\text{the type of } x\} \cap \{i,j\}) \neq \emptyset\}). \tag{4.4.13}$$

We investigate the local structure of $\mathcal{R}_{\mathrm{sym},t}$ near a point in $\widetilde{Z}_t$. If $v(t) \in Z_t$ is of type (i,j), $\mathcal{R}_t$ is locally a double covering space near $\widetilde{x} = \rho_{H_{t,\Xi}}(v(t),i,j,\alpha\oplus\beta)$ with respect to $\pi_{\mathcal{R}_t}$, as we explained before. Let us denote by $\mathcal{S}_0$ (resp. $\mathcal{S}_1$) a local double covering space near $\widetilde{x}$ (resp. $\mathcal{I}_{\mathcal{R}_t}(\widetilde{x})$). If $(i,j,\alpha\oplus\beta) \neq (i,j,r_{i \xrightarrow{v(t)} j} \oplus 0)$, then since $\widetilde{x}$ and $\mathcal{I}_{\mathcal{R}_t}(\widetilde{x})$ are different points in $\mathcal{R}_t$, to identify $\mathcal{S}_0$ with $\mathcal{S}_1$ by the map $\mathcal{I}_{\mathcal{R}_t}$ means just to forget $\mathcal{S}_0$ or $\mathcal{S}_1$. Hence $\mathcal{R}_{\mathrm{sym},t}$ near $\widetilde{x}$ is still a double covering space with respect to $\pi_{\mathcal{R}_t}$, and $\widetilde{x}$ is a ramification point of degree 2. On the other hand, if $\widetilde{x} = \rho_{H_{t,\Xi}}(v(t),i,j,r_{i \xrightarrow{v(t)} j} \oplus 0)$, then $\widetilde{x}$ and $\mathcal{I}_{\mathcal{R}_t}(\widetilde{x})$ are the same point in $\mathcal{R}_t$, and $\mathcal{R}_{\mathrm{sym},t}$ near $\widetilde{x}$ is locally isomorphic to $\mathbb{C}$ by $\pi_{\mathcal{R}_{\mathrm{sym},t}}$. Therefore, in view of topological structures, the point $(v(t),i,j,r_{i \xrightarrow{v(t)} j} \oplus 0)$ should have some specific feature. In fact, it corresponds to an ordinary turning point in $\mathcal{R}_{\mathrm{sym},t}$ because of the equalities

$$\begin{aligned}\int_{v(t)}^{x} \frac{\lambda_{t,j}-\lambda_{t,i}}{p(x)}dx &= \Big(\int_{v(t)}^{x} \frac{\lambda_{t,j}-\lambda_{t,i}}{p(x)}dx + I_t(j \xrightarrow{v(t)} i)\Big) + I_t(i \xrightarrow{v(t)} j) \\ &= F_{t,i,j}(x) + I_t(\alpha_{ij}) + I_t(i \xrightarrow{v(t)} j + \alpha_{ji}) \\ &= f_{t,i,j,r_{i \xrightarrow{v} j}\oplus 0}(x)\end{aligned} \tag{4.4.14}$$

(see also Corollary 4.29 below). Therefore $\widetilde{x}$ is called **an ordinary turning point in** $\mathcal{R}_{\mathrm{sym},t}$. Summing up, we have:

Lemma 4.22. *For any point $\widetilde{x} \in \widetilde{Z}_t$ except for an ordinary turning point in $\mathcal{R}_{\mathrm{sym},t}$, the Riemann manifold $\mathcal{R}_{\mathrm{sym},t}$ over $\mathbb{C} \setminus E_{\mathrm{sing}}$ is a double covering space in a neighborhood of $\widetilde{x}$, and is ramified at $\widetilde{x}$ with respect to $\pi_{\mathcal{R}_{\mathrm{sym},t}}$. On the contrary $\mathcal{R}_{\mathrm{sym},t}$ is not ramified at an ordinary turning point in $\mathcal{R}_{\mathrm{sym},t}$.*

Now we confirm that several important notions for the Stokes geometry (an ordered crossing, the type of a Stokes curve, etc.) are well-defined on $\mathcal{R}_{\mathrm{sym},t}$.

Definition 4.23. Let $\widetilde{x}_1$, $\widetilde{x}_2$ and $\widetilde{x}_3$ be points in $\mathcal{R}_{\mathrm{sym},t} \setminus \widetilde{Z}_t$.

1. A pair $\widetilde{x}_1$ and $\widetilde{x}_2$ is said to have **a hinged index** if it satisfies
$$\widetilde{x}_1 = \rho_{H_{t,\Xi}}(x, i, j, \alpha_1 \oplus \beta_1) \quad \text{and} \quad \widetilde{x}_2 = \rho_{H_{t,\Xi}}(x, j, k, \alpha_2 \oplus \beta_2)$$
for mutually different indices $i, j, k \in \mathbb{Z}_n$ and some point $x \in H_t$. Such a pair of points $(x, i, j, \alpha_1 \oplus \beta_1)$ and $(x, j, k, \alpha_2 \oplus \beta_2)$ in $H_{t,\Xi}$ is called **an ordered representative** of the hinged index pair.
2. We say that $\widetilde{x}_1$, $\widetilde{x}_2$ and $\widetilde{x}_3$ form **a circuit index triplet** if there exist mutually different indices $i, j, k \in \mathbb{Z}_n$ and a point $x \in H_t$ satisfying
$$\widetilde{x}_1 = \rho_{H_{t,\Xi}}(x, i, j, \alpha_1 \oplus \beta_1), \quad \widetilde{x}_2 = \rho_{H_{t,\Xi}}(x, j, k, \alpha_2 \oplus \beta_2), \quad \widetilde{x}_3 = \rho_{H_{t,\Xi}}(x, k, i, \alpha_3 \oplus \beta_3).$$
An ordered representative of the triplet is also defined in the same way as above.

These notions are symmetric with respect to a permutation. For example, if $\widetilde{x}_1$, $\widetilde{x}_2$ and $\widetilde{x}_3$ form a circuit index triple, then $\widetilde{x}_2$, $\widetilde{x}_1$ and $\widetilde{x}_3$ do also, and if $\{\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3\}$ is an ordered representative of the former triplet, then that of the latter is given by

$$\mathcal{I}_t(\{\widetilde{y}_2, \widetilde{y}_1, \widetilde{y}_3\}) \, (= \{\mathcal{I}_t(\widetilde{y}_2), \mathcal{I}_t(\widetilde{y}_1), \mathcal{I}_t(\widetilde{y}_3)\}). \tag{4.4.15}$$

We also note that an ordered representative is not necessarily unique. If $\mathcal{Y} = \{\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3\}$ is an ordered representative of a circuit index triplet, then by taking (4.4.12) and $\rho_{H_t,\Xi}(\widetilde{y}_l) \notin \widetilde{Z}_t$ $(l = 1, 2, 3)$ into account, possible ordered representatives of the circuit index triplet are given by

$$\mathcal{Y} \quad \text{and} \quad \mathcal{J}_t(\mathcal{Y}) \, (= \{\mathcal{J}_t(\widetilde{y}_1), \, \mathcal{J}_t(\widetilde{y}_2), \, \mathcal{J}_t(\widetilde{y}_3)\}). \tag{4.4.16}$$

Let M be a subset of $H^3_{t,\Xi}$ that is stable by an action of $\mathcal{I}_t$ and $\mathcal{J}_t$, and that contains every point $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \in H^3_{t,\Xi}$ for $\widetilde{x}_1 = (x, i, j, \alpha_1 \oplus \beta_1)$, $\widetilde{x}_2 = (x, j, k, \alpha_2 \oplus \beta_2)$ and $\widetilde{x}_3 = (x, k, i, \alpha_3 \oplus \beta_3)$ with mutually different indices i, j and k. Let $Q(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$ be a symmetric property on M (i.e. $Q(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \Longleftrightarrow Q(\widetilde{x}_{l_1}, \widetilde{x}_{l_2}, \widetilde{x}_{l_3})$ for a permutation $\{l_1, l_2, l_3\}$ of $\{1, 2, 3\}$). The following lemma follows from (4.4.15) and (4.4.16).

Lemma 4.24. *If the property Q is stable under an action of $\mathcal{I}_t$ and $\mathcal{J}_t$, that is, if for $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \in M$*

$$Q(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \Longleftrightarrow Q(\mathcal{I}_t(\widetilde{x}_1), \mathcal{I}_t(\widetilde{x}_2), \mathcal{I}_t(\widetilde{x}_3)) \Longleftrightarrow Q(\mathcal{J}_t(\widetilde{x}_1), \mathcal{J}_t(\widetilde{x}_2), \mathcal{J}_t(\widetilde{x}_3)) \tag{4.4.17}$$

hold, then Q induces a symmetric property defined on the set of circuit index triplets.

Remark. We can obtain the same lemma for a symmetric property $Q(\widetilde{x}_1, \widetilde{x}_2)$ of two variables.

The most important symmetric property which is well-defined on the set of hinged index pairs is that of an “ordered crossing”.

Definition 4.25. Let $\widetilde{x}_1$ and $\widetilde{x}_2$ be points in $\mathcal{R}_{\mathrm{sym},t} \setminus \widetilde{Z}_t$. We say that $\widetilde{x}_1$ and $\widetilde{x}_2$ are located at **an ordered crossing position** if Conditions 1 and 2 below are satisfied:

1. $\widetilde{x}_1$ and $\widetilde{x}_2$ form a hinged index pair.
2. For an ordered representative $\widetilde{y}_1$ and $\widetilde{y}_2$ of the pair,

$$\operatorname{Re} f_{t,\Xi}(\rho_{H_{t,\Xi}}(\widetilde{y}_1)) \quad \text{and} \quad \operatorname{Re} f_{t,\Xi}(\rho_{H_{t,\Xi}}(\widetilde{y}_2))$$

are not zero and have the same signature.

Another important and related notion “combined” on the set of circuit index triplets will be later introduced (cf. Theorem 4.31 and Definition 4.32).

The type of a Stokes curve is another important notion. Let us construct the space of type $\mathcal{T}_{\mathrm{sym},t}$ in the same way as $\mathcal{R}_{\mathrm{sym},t}$. We prepare copies $\{H_{t,i,j}\}_{(i,j)\in\mathbb{Z}^2_{n,\neq}}$ of H_t, and set

$$H_{t,\mathbb{Z}^2_{n,\neq}} = \bigsqcup_{(i,j)\in\mathbb{Z}^2_{n,\neq}} H_{t,i,j}. \tag{4.4.18}$$

Let $\mathcal{J}_{H_{t,\mathbb{Z}^2_{n,\neq}}} : H_{t,\mathbb{Z}^2_{n,\neq}} \to H_{t,\mathbb{Z}^2_{n,\neq}}$ designate the pushout of $\mathcal{J}_t$, i.e.

$$\mathcal{J}_{H_{t,\mathbb{Z}^2_{n,\neq}}}(x,i,j) = p(\mathcal{J}_t(x,i,j,0\oplus 0)) \tag{4.4.19}$$

with $p(x,i,j,\alpha\oplus\beta) = (x,i,j)$, and we set

$$\mathcal{T}_t = H_{t,\mathbb{Z}^2_{n,\neq}} / \sim \tag{4.4.20}$$

where the equivalence relation is given by $\mathcal{J}_{H_{t,\mathbb{Z}^2_{n,\neq}}}$. Note that $\mathcal{T}_t$ is nothing but a Riemann surface associated with analytic continuations of $\{\lambda_{t,j} - \lambda_{t,i}\}_{(i,j)\in\mathbb{Z}^2_{n,\neq}}$.

Definition 4.26. The space of type $\mathcal{T}_{\mathrm{sym},t}$ is the set of equivalence classes $\mathcal{T}_t/\sim$ with the equivalence relation being “$(x,i,j) \sim (x,j,i)$ or $(x,i,j) \sim (x,i,j)$”. For a point $\widetilde{x} \in \mathcal{R}_{\mathrm{sym},t}$, the image of $\widetilde{x}$ by the canonical projection $\pi_{t,\mathcal{R},\mathcal{T}} : \mathcal{R}_{\mathrm{sym},t} \to \mathcal{T}_{\mathrm{sym},t}$ is called **the type of** $\widetilde{x}$.

Remark. Let $\widetilde{x}_1$, $\widetilde{x}_2$ be points in $\pi^{-1}_{\mathcal{R}_{\mathrm{sym},t}}(x)$. Then we often say that $\widetilde{x}_1$ and $\widetilde{x}_2$ have **the same type** if they give the same point in $\mathcal{T}_{\mathrm{sym},t}$, and we also say that $\widetilde{x}_1$ and $\widetilde{x}_2$ have **a common index** if there exist mutually different indices i, j, k satisfying $\pi_{t,\mathcal{R},\mathcal{T}}(\widetilde{x}_1) = (x,i,j)$ and $\pi_{t,\mathcal{R},\mathcal{T}}(\widetilde{x}_2) = (x,j,k)$.

Let $\widetilde{v} \in \mathcal{R}_{\mathrm{sym},t_0}$ be a turning point at $t = t_0$. Since $f_{t,\Xi}(x)$ is a holomorphic function of t, when t moves near t_0, the equation

$$f_{t,\Xi}(x) = 0 \tag{4.4.21}$$

has a root $\widetilde{v}(t)$ near t_0 with $\widetilde{v}(t_0) = \widetilde{v}$ where $\widetilde{v}(t)\colon T \to \mathcal{R}_{\mathrm{sym},t}$ is a (possibly multivalued) holomorphic map in a neighborhood of t_0. The map $\widetilde{v}(t)$ of t is called **a holomorphic germ of a turning point** in $\mathcal{R}_{\mathrm{sym},t}$ at t_0.

Lemma 4.27. *We have the following.*

(i) *If $\widetilde{v}(t_0) \notin \widetilde{Z}_{t_0}$, then $\widetilde{v}(t)$ is a single valued holomorphic map near t_0.*
(ii) *If $\widetilde{v}(t_0)$ and an ordinary turning point in $\mathcal{R}_{\mathrm{sym},t_0}$ have the same type (resp. a common index), then the number of branches of $\widetilde{v}(t)$ is at most* 3 *(resp.* 2*).*

Proof. We first remark that

$$\frac{\partial}{\partial x} f_{t,i,j,\alpha\oplus\beta}(x) = \lambda_{t,j}(x) - \lambda_{t,i}(x). \tag{4.4.22}$$

Hence (i) is clear. For (ii), by putting the Puiseux expansions of $\lambda_{t,i}(x)$ and $\lambda_{t,j}(x)$ into $F_{t,i,j}(x)$, we can easily obtain the result. □

We denote by $\mathcal{B}(\widetilde{v})(t) \subset \mathcal{T}_{\mathrm{sym},t}$ the set of values evaluated at t of all branches of $\pi_{t,\mathcal{R},\mathcal{T}}(\widetilde{v}(t))$, that is,

$$\mathcal{B}(\widetilde{v})(t) = \bigcup_{\widetilde{w} \in \{\text{branches of } \widetilde{v} \text{ near } t\}} \pi_{t,\mathcal{R},\mathcal{T}}(\widetilde{w}(t)). \tag{4.4.23}$$

Let $\widetilde{v}_0(t)$ and $\widetilde{v}_1(t)$ be holomorphic germs of turning points at t_0. We say that $\widetilde{v}_0(t)$ and $\widetilde{v}_1(t)$ give **different germs in the type space** at t_0 if there exists a neighborhood U of t_0 such that the set $\{t \in U;\, \mathcal{B}(\widetilde{v}_0)(t) \cap \mathcal{B}(\widetilde{v}_1)(t) = \emptyset\}$ is an open dense subset of U.

Theorem 4.28. *Assume that the equation satisfies the independent 1-cycle condition at t_0. If $\widetilde{v}_0(t_0)$ and $\widetilde{v}_1(t_0)$ are different points in $\mathcal{R}_{\mathrm{sym},t_0}$, then $\widetilde{v}_0(t)$ and $\widetilde{v}_1(t)$ give different germs in the type space at t_0.*

Proof. We consider the case where the type of $\widetilde{v}_0(t_0)$ is the same as that of $\widetilde{v}_1(t_0)$ and there exist an ordinary turning point $\widetilde{v}(t)$ of type $(v(t), i, j) \in \mathcal{T}_{\mathrm{sym},t}$ satisfying either Cases 1 or 2 at $t = t_0$ below:

Case 1. The type of $\widetilde{v}(t_0)$ coincides with that of $\widetilde{v}_0(t_0)$.
Case 2. The type of $\widetilde{v}(t_0)$ has a common index with that of $\widetilde{v}_0(t_0)$.

Other cases are trivial or proved in the same way. Let $\mathcal{S}_{t,k}$ $(k = 0, 1)$ designate a local covering space near $\widetilde{v}_k(t)$ with respect to $\pi_{\mathcal{R}_{\mathrm{sym},t}}$. We may suppose that $\mathcal{S}_{t,k}$ consists of sheets H_{t,ξ_k^+} and H_{t,ξ_k^-} for indices ξ_k^+, $\xi_k^- \in \Xi$ where

1. $\xi_k^+ = (i, j, \alpha_k \oplus \beta_k)$ and $\xi_k^- = (i, j, -\alpha_k + 2r_{i \xrightarrow{v} j} \oplus -\beta_k)$ if we consider Case 1. Note that $\xi_k^- = \xi_k^+$ is allowed.
2. $\xi_k^+ = (i, q, \alpha_k \oplus \beta_k)$ and $\xi_k^- = (q, j, -\alpha_k + r_{i \xrightarrow{v} j} \oplus -\beta_k)$ $(q \notin \{i, j\})$ if we consider Case 2.

We remark that we have

$$\rho_{H_{t,\Xi}}(v(t), \xi_k^+) = \rho_{H_{t,\Xi}}(v(t), \xi_k^-) \in \mathcal{R}_{\mathrm{sym},t} \quad \text{and} \quad f_{t,\,\xi_k^+}(v(t)) = -f_{t,\,\xi_k^-}(v(t)) \tag{4.4.24}$$

$(k = 0,\ 1)$, and for any $w \in \mathcal{B}(\widetilde{v}_k)(t)$ the equality either

$$f_{t,\,\xi_k^+}(\pi_{\mathcal{T}_{\mathrm{sym},t}}(w)) = 0 \qquad \text{or} \qquad f_{t,\,\xi_k^-}(\pi_{\mathcal{T}_{\mathrm{sym},t}}(w)) = 0 \tag{4.4.25}$$

holds by the definition of a holomorphic germ $(k = 0,\ 1)$.

Let us now assume that the conclusion of the theorem were false. Then we should find an open set V in a sufficiently small neighborhood of t_0 that satisfies

$$\mathcal{B}(\widetilde{v}_0)(t) \cap \mathcal{B}(\widetilde{v}_1)(t) \neq \emptyset$$

for $t \in V$. Set

$$V_0 = \{t \in V;\ \pi_{\mathcal{T}_{\mathrm{sym},t}}(\mathcal{B}(\widetilde{v}_0)(t) \cap \mathcal{B}(\widetilde{v}_1)(t)) \cap \{v(t)\} \neq \emptyset\}$$

and

$$V_1 = \{t \in V;\ \pi_{\mathcal{T}_{\mathrm{sym},t}}(\mathcal{B}(\widetilde{v}_0)(t) \cap \mathcal{B}(\widetilde{v}_1)(t)) \setminus \{v(t)\} \neq \emptyset\},$$

then we have $V = V_0 \cup V_1$.

First consider the case where V_0 has an interior point t_1. Then noticing (4.4.24) and (4.4.25), the equalities

$$f_{t,\,\xi_0^+}(v(t)) = f_{t,\,\xi_1^+}(v(t)) = 0 \tag{4.4.26}$$

are satisfied near t_1 for Cases 1 and 2. Hence we get $I_t(\alpha_0 \oplus \beta_0) = I_t(\alpha_1 \oplus \beta_1)$ in a neighborhood of t_1. It follows from the independent 1-cycle condition and Proposition 4.11 that for a basis $\{g_1, \ldots, g_\kappa\}$ of $H_1(\dot{L})$ and $(k; p) \in \mathbb{Z}_n \times E_{\mathrm{sing}}$ the holomorphic functions of t

$$I_t(g_1), l_t(g_2), \ldots, I_t(g_\kappa),\ \{I_t((k;\, p))\}_{(k;\, p) \in \mathbb{Z}_n \times E_{\mathrm{sing}}}$$

are also independent over $\mathbb{Z}$, thus we have $\alpha_0 = \alpha_1$ and $\beta_0 = \beta_1$, in particular, $\widetilde{v}_0(t_0) = \widetilde{v}_1(t_0)$.

Now suppose that the set $V_1 \setminus V_0$ has an interior point t_1. Since any point in $\pi_{\mathcal{T}_{\mathrm{sym},t}}(\mathcal{B}(\widetilde{v}_0)(t) \cap \mathcal{B}(\widetilde{v}_1)(t))$ does not belong to Z_t near t_1, by Lemma 4.27 we find a single valued holomorphic map $\phi(t)\colon T \to \mathcal{T}_{\mathrm{sym},t}$ near t_1 which is a common branch of both $\pi_{t,\mathcal{R},\mathcal{T}}(\widetilde{v}_0(t))$ and $\pi_{t,\mathcal{R},\mathcal{T}}(\widetilde{v}_1(t))$. By (4.4.25), $\phi(t)$ satisfies the following equalities near t_1:

1. Either

$$f_{t,\xi_0^{\pm}}(\pi_{\mathcal{T}_{\mathrm{sym},t}}(\phi(t))) = f_{t,\xi_1^{\pm}}(\pi_{\mathcal{T}_{\mathrm{sym},t}}(\phi(t))) = 0$$

or

$$f_{t,\xi_0^{\pm}}(\pi_{\mathcal{T}_{\mathrm{sym},t}}(\phi(t))) = f_{t,\xi_1^{\mp}}(\pi_{\mathcal{T}_{\mathrm{sym},t}}(\phi(t))) = 0$$

for Case 1.

2. $f_{t,\xi_0^{\pm}}(\pi_{\mathcal{T}_{\mathrm{sym},t}}(\phi(t))) = f_{t,\xi_1^{\pm}}(\pi_{\mathcal{T}_{\mathrm{sym},t}}(\phi(t))) = 0$ for Case 2.

By employing the same argument as above, we find that either $\xi_0^{\pm} = \xi_1^{\pm}$ or $\xi_0^{\mp} = \xi_1^{\pm}$ for Case 1, and that $\xi_0^{\pm} = \xi_1^{\pm}$ for Case 2. In either case, by (4.4.24) we have $\widetilde{v}_0(t_0) = \widetilde{v}_1(t_0)$, which is a contradiction. The proof is complete. □

Let $v(t) \in Z_t$ be of type (i,j), and $\widetilde{v}_1(t)$ a holomorphic germ of a turning point in $\mathcal{R}_{\mathrm{sym},t}$ at t_0.

Corollary 4.29. *Assume that the equation in question satisfies the independent 1-cycle condition at t_0. If $(v(t), i, j) \in \mathcal{B}(\widetilde{v}_1)(t)$ holds near t_0, then $\widetilde{v}_1(t)$ is an ordinary turning point in $\mathcal{R}_{\mathrm{sym},t}$, in particular, $\widetilde{v}_1(t)$ is a single valued holomorphic map.*

For Stokes curves in $\mathcal{R}_{\mathrm{sym},t}$, we have the following result:

Lemma 4.30. *Let $\widetilde{v}$ be a turning point in $\mathcal{R}_{\mathrm{sym},t_0}$, and let us assume $\widetilde{v} \in \widetilde{Z}_{t_0}$. Then the number of Stokes curves that emanate from $\widetilde{v}$ is as follows:*

(i) *Suppose that $\widetilde{v}$ and an ordinary turning point in $\mathcal{R}_{\mathrm{sym},t_0}$ have the same type. If $\widetilde{v}$ itself is an ordinary turning point, then we have 3 Stokes curves, otherwise we have 6 Stokes curves.*

(ii) *If $\widetilde{v}$ and an ordinary turning point in $\mathcal{R}_{\mathrm{sym},t_0}$ have a common index, then we have 4 Stokes curves.*

Proof. If $\widetilde{v}$ is an ordinary turning point in $\mathcal{R}_{\mathrm{sym},t_0}$, then a neighborhood of $\widetilde{v}$ is locally isomorphic to $\mathbb{C}$ by $\pi_{\mathcal{R}_{\mathrm{sym},t_0}}$, thus the configuration of Stokes curves emanating from $\widetilde{v}$ is the same as that in $\mathbb{C}$ locally.

For other cases, $\widetilde{v}$ is a ramification point of degree 2. Therefore there exist two copies H_{t,ξ_1} and H_{t,ξ_2} of H_t that give sheets of $\mathcal{R}_{\mathrm{sym},t}$ near $\widetilde{v}$. If we consider (i) (resp. (ii)) of the lemma, then in each copy we have 3 (resp. 2) Stokes curves that emanate from $\pi_{\mathcal{R}_{\mathrm{sym},t}}(\widetilde{v})$. Hence we obtain the results. □

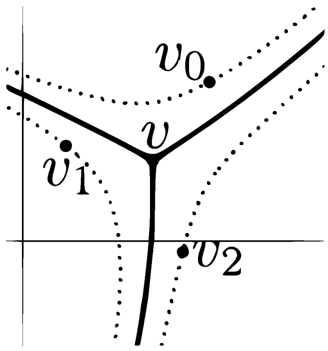

Figure 10. 6 Stokes curves emanating from branches when $t \neq t_0$

At the first glance, the fact that 6 Stokes curves emanate from a turning point seems curious. However the following example explains why 6 Stokes curves emanate. In the base space, that is $\mathbb{C} \setminus E_{\mathrm{sing}}$, if a virtual turning point $v(t)$ coincides with an ordinary turning point at $t = t_0$, then when t moves, by Lemma 4.27 $v(t)$ splits into 3 virtual turning points $v_0(t)$, $v_1(t)$, and $v_2(t)$ as branches of a holomorphic germ of a turning point. The configuration of Stokes curves that emanate from these virtual turning points becomes like Fig. 10. In the figure, we find 6 Stokes curves that emanate from the branches of $v(t)$, and these curves converge to 3 Stokes curves that emanate from the ordinary turning point in the base space. However since the convergence of each curve occurs in a different Riemann sheet of $\mathcal{R}_{\mathrm{sym},t}$ due to Corollary 4.29, we still have 6 Stokes curves in $\mathcal{R}_{\mathrm{sym},t_0}$ when $t = t_0$.

Let $\widetilde{v}_i(t)$ $(i = 0, 1, 2)$ be a single valued holomorphic germ of a turning point in $\mathcal{R}_{\mathrm{sym},t}$ at t_0, and let $\widetilde{s}_i(t)$ be a Stokes curve in $\mathcal{R}_{\mathrm{sym},t}$ emanating from $\widetilde{v}_i(t)$ such that $\widetilde{s}_i(t)$ is continuously deformed near $\widetilde{v}_i(t)$ when t moves. We denote by $v_i(t)$ (resp. $s_i(t)$) $(i = 0, 1, 2)$ the image of $\widetilde{v}_i(t)$ (resp. $\widetilde{s}_i(t)$) by the projection $\pi_{\mathcal{R}_{\mathrm{sym},t}} : \mathcal{R}_{\mathrm{sym},t} \to \mathbb{C} \setminus E_{\mathrm{sing}}$ respectively. We suppose the following situations when $t = t_0$:

- The Stokes curves $s_0(t_0)$, $s_1(t_0)$ and $s_2(t_0)$ intersect transversally at a point $y \in \mathbb{C} \setminus (Z_{t_0} \cup E_{\mathrm{sing}})$.
- We can find a point $\widetilde{y}_l \in \widetilde{s}_l(t_0)$ over y $(l = 0, 1, 2)$ such that $\widetilde{y}_0$, $\widetilde{y}_1$ and $\widetilde{y}_2$ form a circuit index triplet. Let $(y, i, j, \alpha_0 \oplus \beta_0)$, $(y, j, k, \alpha_1 \oplus \beta_1)$ and $(y, k, i, \alpha_2 \oplus \beta_2)$ denote an ordered representative of the triplet.

Under these situations at $t = t_0$ we have:

Theorem 4.31. *If the equation satisfies the independent 1-cycle condition at* t_0*, then the following conditions* (i) *and* (ii) *are equivalent.*

(i) *For any* t *near* t_0*, the Stokes curves* $s_0(t)$*,* $s_1(t)$ *and* $s_2(t)$ *mutually intersect at some point* $y(t) \in \mathbb{C}$ *where* $y(t)$ *is a continuous function of* t *with* $y(t_0) = y$*, and they are combined at* $y(t)$*.*

(ii) *The following relation of indices holds*:

$$\alpha_0 \oplus \beta_0 + \alpha_1 \oplus \beta_1 + \alpha_2 \oplus \beta_2 = 0. \tag{4.4.27}$$

Proof. By the definitions of a Stokes curve and a turning point we have for any (x, t) near (y, t_0)

$$\begin{aligned} &\int_{v_0(t)}^{x} \frac{\lambda_{t,j} - \lambda_{t,i}}{p(x)} dx = f_{t,i,j,\alpha_0 \oplus \beta_0}(x), \quad \int_{v_1(t)}^{x} \frac{\lambda_{t,k} - \lambda_{t,j}}{p(x)} dx = f_{t,j,k,\alpha_1 \oplus \beta_1}(x), \\ &\int_{v_2(t)}^{x} \frac{\lambda_{t,i} - \lambda_{t,k}}{p(x)} dx = f_{t,k,i,\alpha_2 \oplus \beta_2}(x). \end{aligned} \tag{4.4.28}$$

Here the path of each integration is composed of the projection of a portion of $\widetilde{s}_l(t)$ from $\widetilde{v}_l(t)$ to a point near $\widetilde{y}_l$ and a path in H_t to reach x ($l = 0, 1, 2$). It follows from the form of $F_{t,i,j}$ given by (4.3.21) that we have

$$F_{t,i,j}(x) + F_{t,j,k}(x) + F_{t,k,i}(x) = 0. \tag{4.4.29}$$

Thus we obtain:

$$\begin{aligned} &\int_{v_0(t)}^{x} \frac{\lambda_{t,j} - \lambda_{t,i}}{p(x)} dx + \int_{v_1(t)}^{x} \frac{\lambda_{t,k} - \lambda_{t,j}}{p(x)} dx + \int_{v_2(t)}^{x} \frac{\lambda_{t,i} - \lambda_{t,k}}{p(x)} dx \\ &\quad = f_{t,i,j,\alpha_0 \oplus \beta_0}(x) + f_{t,j,k,\alpha_1 \oplus \beta_1}(x) + f_{t,k,i,\alpha_2 \oplus \beta_2}(x) \\ &\quad = F_{t,i,j}(x) + F_{t,j,k}(x) + F_{t,k,i}(x) \\ &\qquad + I_t(\alpha_{ij} + \alpha_{jk} + \alpha_{ki}) + I_t(\alpha_0 \oplus \beta_0 + \alpha_1 \oplus \beta_1 + \alpha_2 \oplus \beta_2) \\ &\quad = I_t(\alpha_0 \oplus \beta_0 + \alpha_1 \oplus \beta_1 + \alpha_2 \oplus \beta_2). \end{aligned} \tag{4.4.30}$$

We first prove that (ii) implies (i). Thanks to (4.4.30) and the assumption we have

$$\int_{v_0(t)}^{x} \frac{\lambda_{t,j} - \lambda_{t,i}}{p(x)} dx + \int_{v_1(t)}^{x} \frac{\lambda_{t,k} - \lambda_{t,j}}{p(x)} dx + \int_{v_2(t)}^{x} \frac{\lambda_{t,i} - \lambda_{t,k}}{p(x)} dx = 0. \tag{4.4.31}$$

Let us show that Stokes curves $s_0(t)$, $s_1(t)$ and $s_2(t)$ mutually intersect at some points for any t near t_0. Since $s_0(t_0)$ and $s_1(t_0)$ intersect transversally at $t = t_0$, they always intersect at $y(t)$ where $y(t)$ is a continuous function of t with $y(t_0) = y$. Let l be a line passing through $y(t_0)$ that intersects transversally with $s_2(t_0)$ at $y(t_0)$, and set $l(t) = l + (y(t) - y(t_0))$. Then $l(t)$ and $s_2(t)$ also intersect at $w(t)$ near t_0 where $w(t)$ is a continuous function of t with $w(t_0) = y(t_0)$. Let us consider a smooth curve $\tau\colon [0,1] \to T$ with $\tau(0) = t_0$, and set $\Theta = \{\theta \in [0,1];\ y(\tau(\theta)) = w(\tau(\theta))\}$. Note that Θ is a non-empty closed set. Now we will assume $\theta_0 = \sup\{\theta \in [0,1];\ [0,\theta] \subset \Theta\} < 1$. By the definition of a Stokes curve, we obtain

$$\operatorname{Im}\int_{v_0(t)}^{y(t)} \frac{\lambda_{t,j} - \lambda_{t,i}}{p(x)}dx = \operatorname{Im}\int_{v_1(t)}^{y(t)} \frac{\lambda_{t,k} - \lambda_{t,j}}{p(x)}dx = \operatorname{Im}\int_{v_2(t)}^{w(t)} \frac{\lambda_{t,i} - \lambda_{t,k}}{p(x)}dx = 0.$$

Therefore taking (4.4.31) into account we get

$$\operatorname{Im}\int_{w(t)}^{y(t)} \frac{\lambda_{t,i} - \lambda_{t,k}}{p(x)}dx = 0$$

where the integration is performed along $l(t)$. This implies that both

$$\operatorname{Im}\Big(\frac{\lambda_{t,i} - \lambda_{t,k}}{p(x)}dx\Big|_{l(\tau(\theta_0))}\Big) \qquad \text{and} \qquad \operatorname{Im}\Big(\frac{\lambda_{t,i} - \lambda_{t,k}}{p(x)}dx\Big|_{s_2(\tau(\theta_0))}\Big)$$

are zero at $y(\tau(\theta_0))$, and that is impossible because $l(\tau(\theta_0))$ and $s_2(\tau(\theta_0))$ are transversally intersecting at $y(\tau(\theta_0))$. Hence $\theta_0 = 1$, and (i) follows from (ii).

For the converse, by (4.4.30) we get $I_t(\alpha_0 \oplus \beta_0 + \alpha_1 \oplus \beta_1 + \alpha_2 \oplus \beta_2) = 0$ for any t near t_0, and this implies $\alpha_0 \oplus \beta_0 + \alpha_1 \oplus \beta_1 + \alpha_2 \oplus \beta_2 = 0$ because of the independent 1-cycle condition. □

Remark. Employing the formula (4.4.27) of the theorem above, we can calculate the index of a virtual turning point that is located by Theorem 2.2. Note that (4.4.27) is well defined on the set of circuit index triplets. To see this, it suffices to confirm (4.4.17) of Lemma 4.24, and in this case we can show a stronger assertion that the map on the set of circuit index triplets with its value $H_1(\dot{L}) \oplus \mathbb{Z}^{\mathbb{Z}_n \times E_{\mathrm{sing}}}$ in the form

$$\alpha_1 \oplus \beta_1 + \alpha_2 \oplus \beta_2 + \alpha_3 \oplus \beta_3 \in H_1(\dot{L}) \oplus \mathbb{Z}^{\mathbb{Z}_n \times E_{\mathrm{sing}}} \tag{4.4.32}$$

is well defined up to sign. For example, suppose that $x \in H_t$ is in a cut line which emanates from $v \in Z_t$ of type (i,j), then for an ordered representative of some circuit index triplet $\widetilde{x}_1 = (x, i, j, \alpha_1 \oplus \beta_1)$, $\widetilde{x}_2 = (x, j, k, \alpha_2 \oplus \beta_2)$ and $\widetilde{x}_3 = (x, k, i, \alpha_3 \oplus \beta_3)$, another representative of the circuit index triplet is given by

$$\mathcal{J}_t(\widetilde{x}_1) = (x^*, j, i, \alpha_1 - 2r_{i \xrightarrow{v} j} \oplus \beta_1), \quad \mathcal{J}_t(\widetilde{x}_2) = (x^*, i, k, \alpha_2 - r_{j \xrightarrow{v} i} \oplus \beta_2) \quad \text{and}$$
$$\mathcal{J}_t(\widetilde{x}_3) = (x^*, k, j, \alpha_3 + r_{i \xrightarrow{v} j} \oplus \beta_3),$$

and hence we have

$$(\alpha_1 - 2r_{i\xrightarrow{v}j} \oplus \beta_1) + (\alpha_2 - r_{j\xrightarrow{v}i} \oplus \beta_2) + (\alpha_3 + r_{i\xrightarrow{v}j} \oplus \beta_3) = \alpha_1 \oplus \beta_1 + \alpha_2 \oplus \beta_2 + \alpha_3 \oplus \beta_3.$$

§ 4.5. The Algorithm at a Limiting Point

By taking into account Theorems 4.28 and 4.31, it is almost clear how to extend the algorithm to determine solid or dotted line portions of a Stokes curve when $t = t_0$. The algorithm for a generic parameter was already introduced in Subsection 2.2 with Definition 3.2 instead of Definition 2.6. The algorithm will be modified with respect to the following points (A) and (B) in the context of Subsection 2.2.

(**A**) The base space of the algorithm is now $\mathcal{R}_{\mathrm{sym},t_0}$, that is, both turning points and Stokes curves are considered to be those defined in $\mathcal{R}_{\mathrm{sym},t_0}$.
(**B**) For the notion of "combined", Definition 2.4 is replaced with Definition 4.32 below.

Let $\widetilde{s}_0$, $\widetilde{s}_1$ and $\widetilde{s}$ be Stokes curves in $\mathcal{R}_{\mathrm{sym},t_0}$ and $\widetilde{x}$ a point in the curve $[\widetilde{s}]$.

Definition 4.32. We say that $\widetilde{s}$ is combined with $\widetilde{s}_0$ and $\widetilde{s}_1$ at $\widetilde{x}$ if the following conditions are satisfied:

1. There exist points $\widetilde{x}_0 \in [\widetilde{s}_0]$ and $\widetilde{x}_1 \in [\widetilde{s}_1]$ so that $\widetilde{x}$, $\widetilde{x}_0$ and $\widetilde{x}_1$ form a circuit index triplet.
2. For an ordered representative $(x, i, j, \alpha \oplus \beta)$, $(x, j, k, \alpha_0 \oplus \beta_0)$ and $(x, k, i, \alpha_1 \oplus \beta_1)$ of the triplet, the relation below holds:

$$\alpha \oplus \beta + \alpha_0 \oplus \beta_0 + \alpha_1 \oplus \beta_1 = 0. \tag{4.5.1}$$

Remark. In practice, we need not to know the concrete shape of $\mathcal{R}_{\mathrm{sym},t_0}$. What we really need is the finite data $\{r_{i\xrightarrow{v}j}\}_{v \in Z_t}$ (Definition 4.16) and a "recipe" to obtain the index of a Stokes curve in $\mathcal{R}_{\mathrm{sym},t_0}$. The former can be calculated by the type diagram and the latter was already given in Proposition 4.17 or Definition 4.18.

Example 4.33. Let us come back to the example in Subsection 4.1. Hereafter we set $t = t_0 = 0$, and x (resp. v_0, s_0, etc.) stands for $x(0)$ (resp. $v_0(0)$, $s_0(0)$, etc.). The type diagram of the example can be realized by a plane graph that has only 1 bounded connected component (see Fig. 11). Therefore we obtain $\mathrm{Rank}_{\mathbb{Z}}\, H_1(\dot{L}) = 1$, ant its basis is given by the walking path

$$D_1 : 1 \xrightarrow{v_0} 2 \xrightarrow{v_1} 3 \xrightarrow{v_2} 1.$$

We define α_{ij} by

$$\alpha_{12} = 1 \xrightarrow{v_0} 2, \quad \alpha_{23} = 2 \xrightarrow{v_1} 3, \quad \alpha_{13} = \alpha_{12} + \alpha_{23}.$$

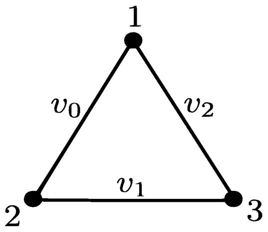

Figure 11. The type diagram of the example.

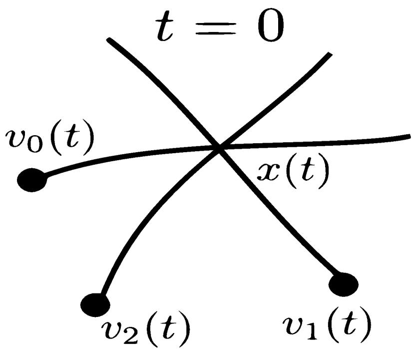

Figure 12. The example in [AKoT].

Then noticing $3 \xrightarrow{v_2} 1 + \alpha_{13} = [D_1]$, we have

$$r_{1 \xrightarrow{v_0} 2} = 0, \quad r_{2 \xrightarrow{v_1} 3} = 0, \quad r_{3 \xrightarrow{v_2} 1} = 1,$$

and since v_0, v_1 and v_2 are ordinary turning points, the index of v_0 (resp. v_1 and v_2) in $\mathcal{R}_{\mathrm{sym},t_0}$ is $(1,2,0) \in \mathbb{Z}^2_{3,\neq} \times H_1(\dot{L})$ (resp. $(2,3,0)$ and $(3,1,1)$). For example, the index of s_0 is the same as that of v_0 since $\mathcal{R}_{\mathrm{sym},t_0}$ is not ramified at an ordinary turning point with respect to $\pi_{\mathcal{R}_{\mathrm{sym},t_0}}$, and it remains $(1,2,0)$ because s_0 does not cross any cut. It is now clear that the relation (4.5.1) is not satisfied at x $(0+0+1 \neq 0)$, therefore s_2 is not combined with s_0 and s_1 at x in the sense of Definition 4.32, and the state of s_2 remains unchanged at x. Moreover Theorem 4.31 entails that the index of the virtual turning point v which was located by Theorem 2.2 is $(3,1,0)$ hence we can distinguish v from v_2 in $\mathcal{R}_{\mathrm{sym},t_0}$.

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