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Stochastic interacting particle systems and nonlinear partial differential equations from fluid mechanics

By

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Abstract

We study two systems of interacting particles and derive in the limit, as the number of particles tends to infinity, laws of large numbers for the empirical measures. As limit dynamics we obtain two important nonlinear partial differential equations from fluid mechanics: the three-dimensional Navier-Stokes equation and the porous medium equation. Details of the proofs can be found in [8] and [9].

§ 1. Introduction

A fluid is usually modelled as a continuous medium and described by macroscopic quantities such as density, velocity, pressure and temparature. These quantities are then related by partial differential equations. However, mechanics is a physical science that pretends to describe the behaviour of matter (solids, liquids, or gases), and therefore its mathematical formulation relies on experience and theory. In view of this the fundamental concept of a continuous medium is an abstraction which is, strictly speaking, against the universally accepted atomic theory, which describes reality at scales which are smaller than nanometers; for example, the radius of the smallest atom is about $4 \cdot 10^{-11}$m. Nevertheless, the mathematical theory of fluid mechanics is based on precisely this concept. This needs an explanation, which is as follows: the task consists in constructing a mathematical theory that serves as a model for *one part* of reality. This model must be judged from the mathematical point of view, taking into account the beauty, extension and profoundness of the involved mathematics; and from the physical point of view, taking into account how efficiently it reflects and explains the underlying reality and allows to predict its future evolution.

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In this sense, although the hypothesis of a continuum is rigorously false at microscopic levels, it turns out to be extremely efficient and adequate when one studies phenomena which occur at macroscopic scales; to fix ideas, lenghts greater than 10^{-7}m. One therefore does exact mathematics in order to describe with very good approximation phenomena which could othwerwise be described neither quantitatively nor qualitatively.

The approximation by the continuous medium turns out to be so efficient that one often forgets that it is just a model. It is nevertheless important to take into account the starting hypotheses. In this way, the consideration of the fluid as a continuous medium is based on the assumption that it consists of an aggregate of particles in chaotic motion and that the characteristic distance of this motion, the so called mean free path, is much smaller than the experimenatal lengths, so that we only observe a certain average of the individual processes between particles.

Having specified that one works on scales which are much larger than the mean free path of the particles one can forget about the fine details of their individual motion and consider around each point of space and at each time a representative elementary volume δV of mesoscopic size, i.e. much larger than the mean free path and much smaller than the macroscopic lengths. This elementary volume, also called fluid particle, is considered as a continuous and homogeneous medium; in this volume one defines a mean velocity of the motion of this element, which is then the point velocity in this point and at this time. More precisely, one supposes that there exists a limit of the averages when δV becomes very small at the intermediate scale, i.e. very small but still much above the atomic scale. In the same way, one speaks of the other macroscopic quantities, such as density, which is the mass per unit of volume in the sense of the limit described above, and pressure, which is the normal force per unit of area exerced by the fluid on an ideal surface which is immersed in it or encloses it. These three quantities are complemented by others, such as e.g. temperature, internal energy and viscosity. The existence of these average values for the fundamental quantities in each fluid particle is what is called the continuum hypothesis[1]. It is precisely this hypothesis which allows to describe the motion of a fluid by partial differential equations. For general introductions to fluid mechanics we refer to the classical book by Landau and Lifshitz [4] and to the lecture notes by Vázquez [13].

As we have said, despite its usefulness and succes, the continuum hypothesis is strictly speaking false. It is therefore desirable to find rigorous connections between the microscale and the macroscale. More precisely: suppose we know that on the macroscale the motion of a fluid is described by a certain partial differential equation, then we want to find a microscopic model which allows us, when the number of particles

[1] not to be confused with the continuum hypothesis of set theory

tends to infinity, to derive that partial differential equation as limit equation. This is a very important project in mathematics to which many people have contributed. For general introductions (and many references) to this subject we refer to the books by Kipnis and Landim [3] and Spohn [10].

§ 2. Stochastic particle approximations for the Navier-Stokes and the porous medium equation

In this paper we concentrate on the following two equations of fluid mechanics: the well-known three-dimensional Navier-Stokes equation

$$\begin{aligned}\frac{\partial u}{\partial t} + (u \cdot \nabla)u &= -\nabla p + \nu \Delta u \\ \operatorname{div} u &= 0 \\ u(t,x) &\to 0 \text{ for } |x| \to \infty\end{aligned}$$

and the less prominent, but also very important porous medium equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\, \Delta(\rho^2) \tag{2.1}$$

which describes the density of a gas flowing through a porous medium (see e.g. [12], Chapter I.1).

We start with the porous medium equation because it is simpler from the probabilistic point of view. We study the following system of interacting particles in $\mathbb{R}^d$:

$$\begin{aligned} dX_t^{N,i,\varepsilon,\delta} &= -\frac{1}{N}\sum_{j=1}^{N} \nabla V^\varepsilon (X_t^{N,i,\varepsilon,\delta} - X_t^{N,j,\varepsilon,\delta})dt + \delta dB_t^i,\ i = 1, \dots, N \\ X_0^{N,i,\varepsilon,\delta} &= \zeta^i. \end{aligned} \tag{2.2}$$

Here V^ε is a smooth interaction kernel which is obtained from a function V by the scaling

$$V^\varepsilon(x) := \frac{1}{\varepsilon^d} V(x/\varepsilon),$$

$(B^i)_{i\in\mathbb{N}}$ is a sequence of independent standard Brownian motions, and $(\zeta^i)_{i\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables, independent of the Brownian motions and whose distribution has a given smooth density ρ_0 with respect to Lebesgue measure.

The particle system (2.2) depends on three parameters: $N \in \mathbb{N}$, $\varepsilon > 0$ and $\delta > 0$. N is the number of particles, ε measures the range of interaction, and δ measures the strength of the additional diffusion caused by the Brownian motions.

Now we let $N \to \infty$, $\varepsilon \to 0$, $\delta \to 0$ in such a way that $N \gg 1/\varepsilon$ and $\varepsilon \ll \delta$. It suffices for instance to take $N \geq \exp(\varepsilon^{-2d-5})$ and $\varepsilon \leq K(\delta)^{-1}$, where $K(\delta)$ is defined in Proposition 3.2. Then we have the following theorem:

Theorem 2.1 ((Corollary 3.1 in [8])).

1. *For each $t \geq 0$ the empirical measure $\mu_t^{N,\varepsilon,\delta} := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i,\varepsilon,\delta}}$ of the particle system converges weakly to a deterministic measure P_t on $\mathbb{R}^d$ as $N \to \infty$. This measure has a density $\rho(t,\cdot)$ which solves the porous medium equation* (2.1) *with initial datum ρ_0.*
2. *The distribution of the position $X_t^{N,i,\varepsilon,\delta}$ of each particle also converges weakly to P_t.*
3. *Any fixed number of particles remains approximately independent in the course of time, in spite of the interaction.*

The third statement is known as *propagation of chaos.* In this context the word "chaotic" is used as a synonym for "independent and identically distributed". By definition the situation at time $t = 0$ is chaotic (because the initial positions ζ^i of the particles are independent and identically distributed), and we claim that at later times the situation is approximately chaotic, too: the chaos propagates. For an introduction to propagation of chaos we refer to Sznitman [11], and for an introduction to the theory of the porous medium equation to Vázquez [12].

Remark.

1. The conditions $N \gg 1/\varepsilon$ and $\varepsilon \ll \delta$ are crucial: the first one ensures that even when ε, which measures the range of interaction, is small, each particle interacts with many other particles. The second one ensures that the stochastic effects, whose strength is measured by δ, are strong enough.
2. A deterministic interacting particle system similar to the system (2.2) has been studied by Oelschläger [7] under the restrictive assumption that the initial datum is strictly positive.

It is more difficult to give a similar probabilistic interpretation of the Navier-Stokes equation, mainly because of the term ∇p in the equation and because of the incompressibility condition $\operatorname{div} u = 0$. But there is a nice trick which allows us to avoid this problem: we do not study the velocity u directly, but instead of it we study the vorticity $w := \operatorname{curl} u$. By taking the curl of the Navier-Stokes equation we obtain the *vorticity equation*

$$\frac{\partial w}{\partial t} + (u \cdot \nabla)w = (w \cdot \nabla)u + \nu \Delta w. \tag{2.3}$$

In this equation the pressure term ∇p has disappeared (because of $\operatorname{curl}\nabla = 0$). Moreover, thanks to the decay condition $u(t,x) \to 0$ for $|x| \to \infty$, the velocity, which is still present in (2.3), can be recovered from the vorticity (see e.g. [5], Proposition 2.16): Let $K(x) := -\frac{x}{4\pi|x|^3}$, then $u(t,x) = \int_{\mathbb{R}^3} K(x-y) \times w(t,y)dy$.

The initial vorticity w_0 is supposed to satisfy $\operatorname{div} w_0 = 0$ (in the sense of distributions) and $w_0 \in L^1(\mathbb{R}^3,\mathbb{R}^3) \cap L^p(\mathbb{R}^3,\mathbb{R}^3)$ for some $p \in (\frac{3}{2}, 3)$. It is known (see Lemma 4.1 below) that under these assumptions there is a $T^* > 0$ such that the vorticity equation (2.3) has a unique solution w on $[0,T^*]$.

We now approximate equation (2.3) by the following system $(X_t^{N,i,\varepsilon,R}, a_t^{N,i,\varepsilon,R})_{i=1}^N$ of interacting discrete vortices:

$$dX_t^{N,i,\varepsilon,R} = \left\{ \frac{1}{N} \sum_{j=1}^N K^\varepsilon(X_s^{N,i,\varepsilon,R} - X_s^{N,j,\varepsilon,R}) \times \chi_R(a_s^{N,j,\varepsilon,R}) \right\} dt + \sqrt{2\nu} dW_t^i$$

$$da_t^{N,i,\varepsilon,R} = \left\{ \frac{1}{N} \sum_{j=1}^N \nabla K^\varepsilon(X_s^{N,i,\varepsilon,R} - X_s^{N,j,\varepsilon,R}) \times \chi_R(a_s^{N,j,\varepsilon,R}) \right\} \chi_R(a_s^{N,i,\varepsilon,R}) dt$$

$$X_0^{N,i,\varepsilon,R} = \xi^i, \qquad a_0^{N,i,\varepsilon,R} = \alpha^i. \tag{2.4}$$

Here $X_t^{N,i,\varepsilon,R} \in \mathbb{R}^3$ represents the position and $a_t^{N,i,\varepsilon,R} \in \mathbb{R}^3$ stands for the intensity of the i-th vortex. $N \in \mathbb{N}$ is the number of vortices, $\varepsilon > 0$ is a smoothing parameter, and $R > 0$ is a cutoff parameter. K^ε is a smoothed version of the kernel K, defined by

$$K^\varepsilon := \varphi^\varepsilon * K, \qquad \text{where} \qquad \varphi^\varepsilon(x) := \frac{1}{\varepsilon^3}\varphi(x/\varepsilon)$$

for a function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ with $\varphi \geq 0$ and $\int_{\mathbb{R}^3} \varphi(x)dx = 1$. Moreover the cutoff function χ_R is defined by

$$\chi_R(x) := \begin{cases} x & \text{if } |x| \leq R \\ \frac{R}{|x|}x & \text{if } |x| > R. \end{cases}$$

Finally $(W^i)_{i=1}^N$ are independent standard Brownian motions.

We choose the initial positions ξ^i and the initial intensities α^i ($i = 1, \dots, N$) of the discrete vortices in the following way: we first decompose the initial vorticity w_0 in the form $w_0(x) = p(x)h(x)$, where p is a probability density and h is a bounded $\mathbb{R}^3$-valued weight function. This is possible thanks to our assumption that $w_0 \in L^1(\mathbb{R}^3,\mathbb{R}^3)$. For instance one can choose $p(x) := \frac{|w_0(x)|}{\|w_0\|_{L^1}}$ and $h(x) := \frac{w_0(x)}{|w_0(x)|}\|w_0\|_{L^1}$ (with the convention $\frac{0}{0} := 0$). Then we choose the ξ^i to be independent of each other and of the Brownian motions, and identically distributed with $P\big[\xi^i \in dx\big] = p(x)dx$, and we set $\alpha^i := h(\xi^i)$.

Now we choose $R > 0$ large enough (but fixed!), and we let $N \to \infty$ and $\varepsilon \to 0$ in such a way that $\varepsilon \gg 1/N$. We will show that then the following holds: For each

$t \in [0, T^*]$ the weighted empirical measure

$$\mu_t^{N,\varepsilon,R} := \frac{1}{N} \sum_{i=1}^{N} a_t^{N,i,\varepsilon,R} \delta_{X_t^{N,i,\varepsilon,R}}$$

of the system (2.4) converges to the measure with density $w(t, \cdot)$ as $N \to \infty$. More precisely, let

$$H := \{ f \in \mathcal{C}^{0,1}(\mathbb{R}^3) \cap L^{p'}(\mathbb{R}^3) \mid \|f\|_{L^\infty} \leq 1, |f|_{lip} \leq 1, \|f\|_{L^{p'}} \leq 1 \},$$

where p' is such that $1/p + 1/p' = 1$. Then we have the following theorem:

Theorem 2.2 ((Theorem 1 in [9])). *There exists a strictly positive time $T^* > 0$ and constants $R_0, A_1, A_2, A_3 < \infty$ such that for each $N \in \mathbb{N}$, each $\varepsilon \in (0, 1]$ and each $R \geq R_0$:*

$$\sup_{t \in [0,T^*]} \sup_{f \in H} E\left[|< \mu_t^{N,\varepsilon,R}, f > - < w(t, \cdot), f >|^2 \right] \leq A_1 \varepsilon^{12} \exp(A_2 \varepsilon^{-10}) \frac{1}{N} + A_3 \varepsilon.$$

Corollary 2.1. *Let $(\varepsilon_N)_{N \in \mathbb{N}}$ be a sequence converging to 0 such that $\varepsilon_N^{12} \exp(A_2 \varepsilon_N^{-10}) \frac{1}{N} \to 0$ for $N \to \infty$. Then for each $t \in [0, T^*]$ the weighted empirical measure $\mu_t^{N,\varepsilon_N,R}$ of the particle system converges to the measure with density $w(t, \cdot)$.*

Remark. Fontbona [2] studied a particle system which is similar to (2.4), but more complicated. Moreover he did not give any estimate for the speed of convergence.

In the two-dimensional case it is much easier to prove that the Navier-Stokes equation can be approximated by a system of interacting discrete vortices (because then the *vortex stretching term* $(w \cdot \nabla)u$ in (2.3) vanishes), and this problem was solved more than twenty years ago by Marchioro and Pulvirenti [6].

§ 3. Sketch of the proof of Theorem 2.1

As intermediate objects between the particle system (2.2) and the porous medium equation (2.1) we introduce *nonlinear processes* $\overline{X}^{i,\varepsilon,\delta}$ $(i \in \mathbb{N}, \varepsilon, \delta > 0)$ and $\overline{X}^{i,\delta}$ $(i \in \mathbb{N}, \delta > 0)$ defined as solutions of the following nonlinear stochastic differential equations:

$$d\overline{X}_t^{i,\varepsilon,\delta} = -(\nabla V^\varepsilon * \rho^{\varepsilon,\delta})(t, \overline{X}_t^{i,\varepsilon,\delta}) dt + \delta dB_t^i \tag{3.1}$$

$$\overline{X}_0^{i,\varepsilon,\delta} = \zeta^i \tag{3.2}$$

$$P\left[\overline{X}_t^{i,\varepsilon,\delta} \in dx \right] = \rho^{\varepsilon,\delta}(t, dx) \tag{3.3}$$

and

$$d\overline{X}_t^{i,\delta} = -\nabla\rho^\delta(t,\overline{X}_t^{i,\delta})dt + \delta dB_t^i \tag{3.4}$$

$$\overline{X}_0^{i,\delta} = \zeta^i \tag{3.5}$$

$$P\left[\overline{X}_t^{i,\delta} \in dx\right] = \rho^\delta(t,x)dx \tag{3.6}$$

$$\rho^\delta \in \mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d) \quad \forall T \geq 0. \tag{3.7}$$

Note that the processes $\overline{X}^{i,\varepsilon,\delta}$ ($i \in \mathbb{N}$) are driven by the *same* Brownian motion B^i and have the *same* initial value ζ^i as the i-th particle of the system (2.2). They are therefore independent copies of each other: their initial positions are independent, they are driven by independent Brownian motions, and they do not interact with each other. The same holds for the processes $\overline{X}^{i,\delta}$ ($i \in \mathbb{N}$).

A solution of (3.1)–(3.3) is a couple $(\overline{X}^{i,\varepsilon,\delta}, \rho^{\varepsilon,\delta})$ consisting of a stochastic process $\overline{X}^{i,\varepsilon,\delta}$ and a probability measure $\rho^{\varepsilon,\delta}$ on $\mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^d)$, the space of continuous functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}^d$, such that the stochastic differential equation (3.1)–(3.2) is satisfied and the distribution of $\overline{X}_t^{i,\varepsilon,\delta}$ is given by $\rho^{\varepsilon,\delta}(t,\cdot)$.

A solution of (3.4)–(3.7) is a couple $(\overline{X}^{i,\delta}, \rho^\delta)$ consisting of a stochastic process $\overline{X}^{i,\delta}$ and a function $\rho^\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ with $\rho^\delta \in \mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d)$ for any $T \geq 0$ such that the stochastic differential equation (3.4)–(3.5) is satisfied and the distribution of $\overline{X}_t^{i,\delta}$ is given by the measure with density $\rho^\delta(t,\cdot)$.

Remark.

1. Both nonlinear stochastic differential equations (3.1)–(3.3) and (3.4)–(3.7) have a unique solution (see [8], Propositions 5.2 and 5.3).

2. Itô's formula implies that $\rho^{\varepsilon,\delta}$ is a solution of the integro-differential equation

$$\begin{aligned} \frac{\partial\rho^{\varepsilon,\delta}}{\partial t} &= \frac{\delta^2}{2}\Delta\rho^{\varepsilon,\delta} + \operatorname{div}\left((\nabla V^\varepsilon * \rho^{\varepsilon,\delta})\rho^{\varepsilon,\delta}\right) \\ \rho^{\varepsilon,\delta}(0,\cdot) &= \rho_0, \end{aligned}$$

while ρ^δ is a solution of the *viscous porous medium equation*

$$\begin{aligned} \frac{\partial\rho^\delta}{\partial t} &= \frac{\delta^2}{2}\Delta\rho^\delta + \operatorname{div}\left(\nabla\rho^\delta\rho^\delta\right) \\ &= \frac{\delta^2}{2}\Delta\rho^\delta + \frac{1}{2}\Delta\left((\rho^\delta)^2\right) \\ \rho^\delta(0,\cdot) &= \rho_0. \end{aligned}$$

The proof of Theorem 2.1 now consists of the following parts (for details see [8]): we first show the convergence of $X^{N,i,\varepsilon,\delta}$ to $\overline{X}^{i,\varepsilon,\delta}$ as $N \to \infty$:

Proposition 3.1. *There are constants $C_1, C_2 < \infty$ (depending only on V) such that for each $T \geq 0$ and each $i \in \{1, \ldots, N\}$ we have:*

$$E\left[\sup_{0\leq s\leq T}\left|X_s^{N,i,\varepsilon,\delta} - \overline{X}_s^{i,\varepsilon,\delta}\right|^2\right] \leq C_1\varepsilon^2 \exp(C_2T^2\varepsilon^{-2d-4})\frac{1}{N}.$$

Then we show the convergence of $\overline{X}^{i,\varepsilon,\delta}$ to $\overline{X}^{i,\delta}$ as $\varepsilon \to 0$:

Proposition 3.2. *For each $\delta > 0$ there is a number $K(\delta)$ (which also depends on T and V) such that*

$$\sup_{0\leq s\leq T}\left|\overline{X}_s^{i,\varepsilon,\delta} - \overline{X}_s^{i,\delta}\right| \leq K(\delta)\varepsilon^2.$$

Then we use the following analytical result due to Bénilan and Crandall [1]:

Proposition 3.3. *For each $T \geq 0$:*

$$\sup_{0\leq t\leq T}\|\rho^\delta(t,\cdot) - \rho(t,\cdot)\|_{L^1(\mathbb{R}^d)} \to 0 \qquad (\delta \to 0).$$

Combining Propositions 3.1, 3.2 and 3.3 we obtain the following crucial result:

Proposition 3.4 ((Propagation of Chaos)). *Let m be a fixed natural number, and let $P_t^{N,m,\varepsilon,\delta}$ be the joint distribution of the random variables $X_t^{N,i,\varepsilon,\delta}$, $i = 1, \ldots, m$. When $N \to \infty$, $\varepsilon \to 0$ and $\delta \to 0$ in such a way that $N \gg 1/\varepsilon$ and $\varepsilon \ll \delta$, then $P_t^{N,m,\varepsilon,\delta}$ converges weakly to $P_t^{\otimes m}$.*

Proposition 3.4 obviously implies the second and the third statement of Theorem 2.1. It also implies the first statement thanks to the general fact (see [11], Chapter I.2, Proposition 2.2) that propagation of chaos is equivalent to weak convergence of the empirical measure to a deterministic measure.

Proof of Proposition 3.4. Let P_t^δ be the distribution of $\overline{X}_t^\delta$. Thanks to the conditions $N \gg 1/\varepsilon$ and $\varepsilon \ll \delta$, Propositions 3.1 and 3.2 imply that for $N \to \infty$, $\varepsilon \to 0$ and $\delta \to 0$ the measure $P_t^{N,m,\varepsilon,\delta} - P_t^{\delta\otimes m}$ converges weakly to 0. Proposition 3.3 implies the weak convergence of P_t^δ to P_t as $\delta \to 0$. It follows that for $N \to \infty$, $\varepsilon \to 0$ and $\delta \to 0$ the measure $P_t^{N,m,\varepsilon,\delta}$ converges weakly to $P_t^{\otimes m}$. □

§ 4. Sketch of the proof of Theorem 2.2

We now give an overview of the proof of Theorem 2.2. For details we refer to [9].

As intermediate objects between the system of discrete vortices (2.4) and the vorticity equation (2.3) we introduce processes $(\overline{X}^{i,\varepsilon}, \overline{a}^{i,\varepsilon})$ $(i = 1, \ldots, N,\ \varepsilon > 0)$ defined by the following stochastic differential equations:

$$\begin{aligned} \overline{X}_t^{i,\varepsilon} &= \xi^i + \int_0^t u^\varepsilon(s, \overline{X}_s^{i,\varepsilon})ds + \sqrt{2\nu}W_t^i \\ \overline{a}_t^{i,\varepsilon} &= \alpha^i + \int_0^t \nabla u^\varepsilon(s, \overline{X}_s^{i,\varepsilon})\overline{a}_s^{i,\varepsilon} ds. \end{aligned} \tag{4.1}$$

Here $u^\varepsilon(x) := \mathcal{K}^\varepsilon(w^\varepsilon)(x) := \int_{\mathbb{R}^3} K^\varepsilon(x-y) \times w^\varepsilon(y)dy$, where w^ε is the solution of the smoothed vorticity equation

$$\begin{aligned} \frac{\partial w^\varepsilon}{\partial t} + (\mathcal{K}^\varepsilon(w^\varepsilon)\cdot\nabla)w^\varepsilon &= (w^\varepsilon\cdot\nabla)\mathcal{K}^\varepsilon(w^\varepsilon) + \nu\Delta w^\varepsilon \\ w^\varepsilon(0,\cdot) &= w_0. \end{aligned} \tag{4.2}$$

By setting $K^0 := K$ we obtain (2.3) as a special case of (4.2) (namely for $\varepsilon = 0$). We use the following analytical result due to Fontbona ([2], Theorem 3.1 and Remark 6.3):

Lemma 4.1. *There is a $T^* > 0$ such that for each $\varepsilon \geq 0$ the smoothed vorticity equation (4.2) has a unique mild solution in the class $L^\infty([0,T^*], L^p(\mathbb{R}^3, \mathbb{R}^3))$.*

Because of the Lipschitz continuity and the boundedness of $u^\varepsilon = \mathcal{K}^\varepsilon(w^\varepsilon)$ and ∇u^ε, the stochastic differential equations (4.1) have a unique strong solution on $[0,T^*]$. Note that the i-th process $(\overline{X}^{i,\varepsilon}, \overline{a}^{i,\varepsilon})$ has the *same* initial value (ξ^i, α^i) and is driven by the *same* Brownian motion W^i as the i-th discrete vortex of the system (2.4). For different i these processes are independent copies of each other: the initial values are independent and identically distributed, and there is no interaction.

The following two properties are crucial:

Proposition 4.1. *The processes $(\overline{a}_t^{i,\varepsilon})_{0\leq t\leq T^*}$ are uniformly bounded. More precisely, there is a constant $R_0 = R_0(T^*, \|w_0\|_{L^p}, \|h\|_{L^\infty}, \nu) < \infty$ such that*

$$|\overline{a}_t^{i,\varepsilon}| \leq R_0$$

for all $t \in [0,T^*]$, all $N \in \mathbb{N}$, all $i \in \{1, \ldots, N\}$ and all $\varepsilon > 0$.

Proposition 4.2. *The functions w^ε and u^ε can be recovered from the process $(\overline{X}^{i,\varepsilon}, \overline{a}^{i,\varepsilon})$ in the following way:*

$$\begin{aligned} \int_{\mathbb{R}^3} f(x)w^\varepsilon(t,x)dx &= E\left[f(\overline{X}_t^{i,\varepsilon})\overline{a}_t^{i,\varepsilon}\right] \quad \textit{for all test functions } f \in H, \\ u^\varepsilon(t,x) &= E\left[K^\varepsilon(x - \overline{X}_t^{i,\varepsilon}) \times \overline{a}_t^{i,\varepsilon}\right]. \end{aligned}$$

Combining Propositions 4.1 and 4.2 one obtains that for each $R \geq R_0$ the process $(\overline{X}^{i,\varepsilon}, \overline{a}^{i,\varepsilon})$ satisfies the following nonlinear stochastic differential equation:

$$\begin{aligned} \overline{X}_t^{i,\varepsilon} &= \xi^i + \int_0^t u^\varepsilon(s, \overline{X}_s^{i,\varepsilon})ds + \sqrt{2\nu}W_t^i \\ \overline{a}_t^{i,\varepsilon} &= \alpha^i + \int_0^t \nabla u^\varepsilon(s, \overline{X}_s^{i,\varepsilon})\chi_R(\overline{a}_s^{i,\varepsilon})ds \\ u^\varepsilon(s,x) &= E\left[K^\varepsilon(x - \overline{X}_s^{i,\varepsilon}) \times \chi_R(\overline{a}_s^{i,\varepsilon})\right]. \end{aligned}$$

The proof of Theorem 2.2 now consists of the following parts: one first shows (for $N \to \infty$) pathwise convergence of $(X^{N,i,\varepsilon,R}, a^{N,i,\varepsilon,R})$ to $(\overline{X}^{i,\varepsilon}, \overline{a}^{i,\varepsilon})$:

Proposition 4.3. *There are constants $C_3, C_4 < \infty$ (only depending on φ) such that for each $N \in \mathbb{N}$, each $\varepsilon \in (0,1]$, each $R \geq R_0$, each $T \leq T^*$ and each $i \in \{1, \ldots, N\}$:*

$$\begin{aligned} &E\left[\sup_{0\leq t\leq T}\left|X_t^{N,i,\varepsilon,R} - \overline{X}_t^{i,\varepsilon}\right|^2 + \sup_{0\leq t\leq T}\left|a_t^{N,i,\varepsilon,R} - \overline{a}_t^{i,\varepsilon}\right|^2\right] \\ &\leq C_3\varepsilon^{12}R^{-4}T^{-2}\exp(C_4\varepsilon^{-10}R^4T^2)\frac{1}{N}. \end{aligned}$$

Then one shows (for $\varepsilon \to 0$) convergence of w^ε to w:

Proposition 4.4. *There is a constant $C_5 = C_5(T^*, \|w_0\|_{L^p}, \nu, \varphi) < \infty$ such that*

$$\|w(t,\cdot) - w^\varepsilon(t,\cdot)\|_{L^p} \leq C_5\varepsilon$$

for all $t \in [0, T^*]$.

Now Theorem 2.2 follows easily from Propositions 4.2, 4.3 and 4.4.

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