RIMS Kôkyûroku Bessatsu **B6**
(2008), 193–200

Variance of the linear statistics of the Ginibre random point field

By

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Abstract

We investigate the variance of linear statistics of the Ginibre random point field. We generalize the result obtained by the second author to higher order moments and also to functions with rotational and radial perturbations. Our result is motivated by the construction of a solution of the infinite dimensional stochastic differential equation related to the Ginibre random point field.

§ 1. Introduction

The Ginibre random point field (GRPF) μ_{Gin} is a probability measure on the configuration space over the complex plane $\mathbb{C}$. The GRPF μ_{Gin} is specified by the correlation functions $\{\rho^n\}_{n\in\mathbb{N}}$ with respect to the standard complex Gaussian measure $\mathsf{g}(dz) = \frac{1}{\pi}\exp\{-|z|^2\}dz$ given by

$$\rho^n(z_1, \dots, z_n) = \det[K(z_i, z_j)]_{i,j=1,\dots,n}, \tag{1.1.1}$$

where $K:\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ is the exponential kernel defined by

$$K(z_1, z_2) = \exp\{z_1\bar{z}_2\}. \tag{1.1.2}$$

The GRPF μ_{Gin} is one of the typical examples of Fermion (determinantal) random point field ([7], [5]). It is well known that μ_{Gin} is rotation and translation invariant although the kernel K is not translation invariant. Moreover, μ_{Gin} is the thermodynamic limit of the distribution μ_{Gin}^N of the spectrum of the random matrices called Ginibre ensemble ([5], [1]). The finite volume measure μ_{Gin}^N is given by

$$\mu_{\text{Gin}}^N(d\zeta) = \text{const.}e^{-\sum_{i=1}^N |z_i|^2} \prod_{i,j=1,\dots,N,i<j} |z_i - z_j|^2 dz_1 \cdots dz_N. \tag{1.1.3}$$

2000 Mathematics Subject Classification(s):

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Here $\zeta = \sum_{i=1}^{N} \delta_{z_i}$ and we naturally regard the right hand side as a probability measure on the configuration space.

In view of (1.1.3) the GRPF is the stationary distribution of particles interacting via the two dimensional Coulomb potential. Because of the strong and long range nature of the interaction the static property of the GRPF is quite different from that of Gibbs measures. Indeed, the following small fluctuation result of the variance of the linear statistics of the GRPF is known (cf. [6]):

$$\mathrm{Var}(\langle \zeta, 1_{D_r} \rangle) \sim \frac{r}{\sqrt{\pi}} \quad \text{as } r \to \infty, \tag{1.1.4}$$

where Var is the variation with respect to μ_{Gin}, $\zeta = \sum_i \delta_{z_i}$ denotes an element of the set of the configurations, and $\{z_i\}$ is a sequence in $\mathbb{C}$ with no accumulation points in $\mathbb{C}$ and $D_r = \{z \in \mathbb{C}; |z| \le r\}$. Here and after we set $\langle \zeta, g \rangle = \int_{\mathbb{C}} g d\zeta = \sum_i g(z_i)$. So $\langle \zeta, 1_{D_r} \rangle$ denotes the cardinality of the particles in the disk D_r by definition.

Note that, if we replace μ_{Gin} by the Poisson random point field whose intensity is the Lebesgue measure, then the exponent of r on the right hand side of (1.1.4) is 2 instead of 1.

The purpose of the paper is to generalize the above result to a wider class of functions. As for higher order moments and rotations we obtain the following.

Theorem 1.1. *Let $I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\nu+1)} (\frac{x}{2})^{2k+\nu}$ be the modified Bessel function of the first kind, where $\nu \ge 0$. Let m be a nonnegative integer. Then*

$$\begin{aligned} \mathrm{Var}(\langle \zeta, 1_{D_r} z^m \rangle) &= \frac{r^{2(m+1)} e^{-2r^2}}{m+1} \sum_{k=-m-1}^{m} I_{|k|}(2r^2) \\ &\sim \frac{r^{2m+1}}{\sqrt{\pi}} \quad \text{as } r \to \infty. \end{aligned}$$

Here $f \sim g$ as $r \to \infty$ means $\lim_{r \to \infty} f(r)/g(r) = 1$.

Theorem 1.2. *Let m be a nonnegative integer and $q \in \mathbb{Z}$. Then*

$$\mathrm{Var}(\langle \zeta, 1_{D_r} z^m e^{\sqrt{-1} q \arg z} \rangle) \sim \frac{r^{2m+1}}{\sqrt{\pi}} \quad \text{as } r \to \infty.$$

Theorem 1.3. *Let $f : \mathbb{C} \to \mathbb{C}$ be a measurable function such that*

$$\sup_{z \in \mathbb{C}} |f(z)| < \infty, \tag{1.1.5}$$

$$\sup_{|z|=r} |f(z) - \beta| = O(r^{-\alpha}) \quad \text{as } r \to \infty, \tag{1.1.6}$$

where $\alpha > 0$ and $\beta \in \mathbb{C}$ are constants. Let m be a nonnegative integer and $q \in \mathbb{Z}$ as before. Then we have

$$\mathrm{Var}(\langle \zeta, 1_{D_r} z^m e^{\sqrt{-1} q \arg z} f(z) \rangle) = O(r^{\max\{2m+1, 2m+2-2\alpha\}}) \text{ as } r \to \infty. \tag{1.1.7}$$

Here $f(r) = O(g(r))$ *as* $r \to \infty$ *means* $\limsup_{r\to\infty} |f(r)|/|g(r)| < \infty$.

The first author solved the following infinitely dimensional stochastic differential equation (SDE):

$$dZ_t^i = dB_t^i - Z_t^i dt + \sum_{j\in\mathbb{Z}} \frac{Z_t^i - Z_t^j}{|Z_t^i - Z_t^j|^2} dt \qquad (i \in \mathbb{Z}). \tag{1.1.8}$$

The solution $(Z_t^i)_{i\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is closely related to μ_{Gin}. Indeed, the associated unlabeled dynamics $\mathbb{Z}_t = \sum_{i\in\mathbb{Z}} \delta_{Z_t^i}$ is a μ_{Gin}-reversible diffusion. One of the difficulty to solve the SDE (1.1.8) is the control of the interaction term $\sum_{j\in\mathbb{Z}} (Z_t^i - Z_t^j)/|Z_t^i - Z_t^j|^2 dt$. Since $\mathbb{Z}_t = \sum_{i\in\mathbb{Z}} \delta_{Z_t^i}$ is μ_{Gin}-reversible and μ_{Gin} is translation invariant, the summation $\sum_{j\in\mathbb{Z}} (Z_t^i - Z_t^j)/|Z_t^i - Z_t^j|^2$ does not converge absolutely; one can expect only a conditional convergence. To prove the conditional convergence and other properties to solve the SDE (1.1.8) we need an estimate of the form

$$\mathrm{Var}(\langle \zeta, 1_{D_r} \frac{z[|z|]}{|z|^2} \rangle) = O(r^{2-c}) \quad \text{for some } c > 0, \tag{1.1.9}$$

where $[t]$ denote the maximal integer smaller than t. The estimate (1.1.9) with $c = 1$ is immediate from Theorem 1.3 by taking $m = 0$, $q = \beta = 1$, $\alpha = 1$ and $f(z) = [|z|]/|z|$.

§ 2. Variance of higher order moments

In this section we prove Theorem 1.1. We first recall the following.

Lemma 2.1. *Let* g *be a bounded measurable function with compact support. Then*

$$\begin{aligned}\mathrm{Var}(\langle \zeta, g \rangle) = &\int_{\mathbb{C}} |g(z)|^2 K(z,z) \mathsf{g}(dz) \\ &- \int_{\mathbb{C}^2} g(w)\overline{g(z)} |K(w,z)|^2 \mathsf{g}(dw)\mathsf{g}(dz).\end{aligned} \tag{2.2.1}$$

Proof. This lemma is well known. So we omit a proof. □

Remark. It may be interesting to note that the variance $\mathrm{Var}(\langle \zeta, g \rangle)$ can be written as follows.

$$\mathrm{Var}(\langle \zeta, g \rangle) = \frac{1}{2} \int_{\mathbb{C}^2} |g(w) - g(z)|^2 |K(w,z)|^2 \mathsf{g}(dw)\mathsf{g}(dz).$$

This follows from Lemma 2.1 and the formula $K(z,z) = \int_{\mathbb{C}} |K(z,w)|^2 \mathsf{g}(dw)$. Also, it follows from this formula that

$$\mathrm{Var}(\langle \zeta, g \rangle) \le 2 \int_{\mathbb{C}} |g(z)|^2 K(z,z) \mathsf{g}(dz). \tag{2.2.2}$$

Lemma 2.2. *For $m = 0, 1, 2, \ldots,$ we have*

$$\int_{D_r} |z^m|^2 K(z,z)\mathsf{g}(dz) = \frac{r^{2(m+1)}}{m+1}. \tag{2.2.3}$$

Proof. A direct calculation shows

$$\int_{D_r} |z^m|^2 K(z,z)\mathsf{g}(dz) = \int_{D_r} |z^m|^2 \frac{dz}{\pi} = \frac{r^{2(m+1)}}{m+1}. \tag{2.2.4}$$

□

Lemma 2.3. *Let Y_{r^2} be the Poisson random variable with mean r^2. Let $S_m(n) = n!/(n-m)!$, where $m, n \in \mathbb{Z}$ such that $0 \le m \le n$. Then*

$$\int_{D_r^2} w^m \overline{z}^m |K(w,z)|^2 \mathsf{g}(dw)\mathsf{g}(dz) = \sum_{k=m}^{\infty} S_m(k) P(Y_{r^2} \ge k+1)^2. \tag{2.2.5}$$

Proof. Since $K(w,z) = e^{w\overline{z}} = \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{k!}$, we have

$$\begin{aligned} \int_{D_r^2} w^m \overline{z}^m |K(w,z)|^2 \mathsf{g}(dw)\mathsf{g}(dz) &= \sum_{k,l=0}^{\infty} \frac{1}{k!l!} \left| \int_{D_r} w^{k+m} \overline{w}^l \mathsf{g}(dw) \right|^2 \\ &= \sum_{k=0}^{\infty} \frac{1}{k!(k+m)!} \left(\int_{D_r} |w|^{2(k+m)} \mathsf{g}(dw) \right)^2 \\ &= \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \left(\int_0^{r^2} \frac{s^{k+m} e^{-s}}{(k+m)!} ds \right)^2 \\ &= \sum_{k=0}^{\infty} S_m(k+m) P(Y_{r^2} \ge k+m+1)^2. \end{aligned} \tag{2.2.6}$$

Hence Lemma 2.3 follows from (2.2.6) with the change of variable $k+m \mapsto k$ in the summation immediately. □

Lemma 2.4. *Let $p_k := P(Y_{r^2} = k) = e^{-r^2} \frac{r^{2k}}{k!}$. Then we have*

$$\sum_{k=m}^{\infty} S_m(k) P(Y_{r^2} \ge k+1)^2 = \frac{r^{2m}}{m+1} \sum_{i,j=0}^{\infty} p_{i \wedge j} \cdot p_{(i \vee j)+m+1}. \tag{2.2.7}$$

Proof. Since $P(Y_{r^2} \geq k+1)^2 = \sum_{i,j=k+1}^{\infty} p_i p_j$, we have by using Fubini's theorem

$$\sum_{k=m}^{\infty} S_m(k) P(Y_{r^2} \geq k+1)^2 = \sum_{i,j=m+1}^{\infty} \sum_{k=m}^{i \wedge j - 1} S_m(k) p_i p_j \tag{2.2.8}$$

$$= \frac{1}{m+1} \sum_{i,j=m+1}^{\infty} S_{m+1}(i \wedge j) p_i p_j = \frac{1}{m+1} \sum_{i,j=m+1}^{\infty} S_{m+1}(i \wedge j) p_{i \wedge j} p_{i \vee j}$$

$$= \frac{r^{2(m+1)}}{m+1} \sum_{i,j=m+1}^{\infty} p_{(i \wedge j)-(m+1)} \cdot p_{i \vee j}$$

$$= \frac{r^{2(m+1)}}{m+1} \sum_{i,j=0}^{\infty} p_{i \wedge j} \cdot p_{(i \vee j)+m+1}.$$

Here we used $S_m(k) = \frac{1}{m+1}(S_{m+1}(k+1) - S_{m+1}(k))$ for the second line, where we set $S_{m+1}(m) = 0$. Moreover, we used $S_{m+1}(k) p_k = r^{2(m+1)} p_{k-(m+1)}$ for the third line. □

Proof of Theorem 1.1. First we note that

$$\sum_{n=0}^{\infty} p_n p_{n+k} = \sum_{n=0}^{\infty} \frac{r^{2n+2(n+k)}}{n!(n+k)!} e^{-2r^2} = e^{-2r^2} I_k(2r^2) \tag{2.2.9}$$

for $k \geq 0$, and

$$\sum_{i,j=0}^{\infty} f(i \wedge j) g(i \vee j) = \sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}} f(n) g(n+|k|) \tag{2.2.10}$$

for any functions $f, g : \mathbb{N} \to \mathbb{C}$. Then, by using (2.2.9) and (2.2.10) together with the lemmas in this section, we get

$$\mathrm{Var}(\langle \zeta, 1_{D_r} z^m \rangle) = \frac{r^{2(m+1)}}{m+1} \{ \sum_{i,j=0}^{\infty} p_{i \wedge j} p_{i \vee j} - \sum_{i,j=0}^{\infty} p_{i \wedge j} p_{(i \vee j)+m+1} \} \tag{2.2.11}$$

$$= \frac{r^{2(m+1)}}{m+1} \sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}} p_n (p_{n+|k|} - p_{n+|k|+m+1})$$

$$= \frac{r^{2(m+1)}}{m+1} \sum_{n=0}^{\infty} \sum_{k=-m-1}^{m} p_n p_{n+|k|}$$

$$= \frac{r^{2(m+1)} e^{-2r^2}}{m+1} \sum_{k=-m-1}^{m} I_{|k|}(2r^2)$$

$$\sim \frac{r^{2m+1}}{\sqrt{\pi}}.$$

The last asymptotics immediately follows from

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}}\left\{1 - \frac{4\nu^2 - 1}{8x} + O(x^{-2})\right\} \quad (x \to \infty) \tag{2.2.12}$$

for fixed ν (cf. [2], p.123). □

§ 3. Proof of Theorems 1.2 and 1.3.

In this section we prove Theorems 1.2 and 1.3. We begin with a calculation related to the kernel $K(w,z)$.

Lemma 3.1. *Let $q \in \mathbb{Z}$ and $F_{m,q}(z) = z^m e^{\sqrt{-1}q \arg z}$. Then*

$$\int_{D_r} |F_{m,q}(w)|^2 K(w,w)\mathsf{g}(dw) = \int_{D_r} |F_{m,0}(w)|^2 K(w,w)\mathsf{g}(dw) \tag{3.3.1}$$

$$\int_{D_r^2} F_{m,q}(w)\overline{F_{m,q}(z)}|K(w,z)|^2 \mathsf{g}(dw)\mathsf{g}(dz) = \int_{[0,r]^2} (st)^m I_{|m+q|}(2st)e^{-s^2-t^2} d(s^2)d(t^2). \tag{3.3.2}$$

Proof. Since $|F_{m,q}(w)| = |F_{m,0}(w)|$, (3.3.1) is clear. Let $|w| = s$, $|z| = t$, $\varphi = \arg z$ and $\psi = \arg w$. By a direct calculation we have

$$\begin{aligned}
&\int_{D_r^2} F_{m,q}(w)\overline{F_{m,q}(z)}|K(w,z)|^2 \mathsf{g}(dw)\mathsf{g}(dz) \\
&= \int_{D_r^2} F_{m,q}(w)\overline{F_{m,q}(z)}\{\sum_{k=0}^{\infty} \frac{1}{k!} w^k \overline{z}^k\}\{\sum_{\ell=0}^{\infty} \frac{1}{\ell!} \overline{w}^\ell z^\ell\}\mathsf{g}(dw)\mathsf{g}(dz) \\
&= \sum_{k,\ell=0}^{\infty} \int_{[0,r]^2 \times [0,2\pi)^2} (st)^m e^{\sqrt{-1}(m+q)(\varphi-\psi)} \\
&\qquad \cdot \frac{s^k t^k e^{\sqrt{-1}k(\varphi-\psi)}}{k!} \cdot \frac{s^\ell t^\ell e^{-\sqrt{-1}\ell(\varphi-\psi)}}{\ell!} \cdot \frac{ste^{-s^2-t^2}}{\pi^2} dsdtd\varphi d\psi \\
&= \sum_{k=0}^{\infty} \int_{[0,r]^2} (st)^m \frac{s^{|m+q|+2k}}{k!} \cdot \frac{t^{|m+q|+2k}}{(k+|m+q|)!} \cdot e^{-s^2-t^2} d(s^2)d(t^2),
\end{aligned}$$

which implies (3.3.2). □

Lemma 3.2.

$$\begin{aligned}
&\mathrm{Var}(\langle \zeta, 1_{D_r} F_{m,q}\rangle) - \mathrm{Var}(\langle \zeta, 1_{D_r} F_{m,0}\rangle) \\
&= \int_{[0,r]^2} (st)^m \{I_m(2st) - I_{|m+q|}(2st)\} \cdot e^{-s^2-t^2} d(s^2)d(t^2).
\end{aligned} \tag{3.3.3}$$

Proof. Lemma 3.2 is immediate from Lemma 2.1 and Lemma 3.1. □

Lemma 3.3. *It holds that*

$$\int_{[0,r]^2}(st)^m\{I_m(2st)-I_{|m+q|}(2st)\}e^{-s^2-t^2}d(s^2)d(t^2) = O((\int_1^r t^{2m-1}dt)\cdot r^{1/2}) \quad \textit{as } r\to\infty. \tag{3.3.4}$$

Proof. By (2.2.12) we see there exists a constant $C=C_{m,q}>0$ such that

$$|I_m(t)-I_{|m+q|}(t)|e^{-t}\le C(1+t)^{-3/2} \quad \text{for any } t\ge 0. \tag{3.3.5}$$

Therefore,

$$\begin{aligned}
&\int_{[0,r]^2}(st)^m\left|I_m(2st)-I_{|m+q|}(2st)\right|e^{-s^2-t^2}d(s^2)d(t^2)\\
=&O(\int_{[0,r]^2}(st)^{m+1}(1+2st)^{-3/2}e^{-(s-t)^2}dsdt)\\
=&O(\left(\int_{[0,r]^2}(st)^{2m+2}(1+2st)^{-3}dsdt\right)^{1/2})\cdot O(\left(\int_{[0,r]^2}e^{-2(s-t)^2}dsdt\right)^{1/2})\\
=&O(\int_1^r t^{2m-1}dt)\cdot O(r^{1/2}),
\end{aligned}$$

which implies (3.3.4) □

Proof of Theorem 1.2. Theorem 1.2 follows from Theorem 1.1, Lemma 3.2 and Lemma 3.3 immediately. □

Proof of Theorem 1.3. Let $F_{m,q}$ be as in Lemma 3.1. Then it is easy to see

$$\begin{aligned}
&\mathrm{Var}(\langle\zeta,1_{D_r}z^m e^{\sqrt{-1}q\arg z}f(z)\rangle)\\
&=\mathrm{Var}(\langle\zeta,1_{D_r}\{\beta F_{m,q}+(f(z)-\beta)F_{m,q}\}\rangle)\\
&\le 2\beta^2\mathrm{Var}(\langle\zeta,1_{D_r}F_{m,q}\rangle)+2\mathrm{Var}(\langle\zeta,1_{D_r}(f(z)-\beta)F_{m,q}\rangle)\\
&\le 2\beta^2\mathrm{Var}(\langle\zeta,1_{D_r}F_{m,q}\rangle)+4\int_{D_r}|(f(z)-\beta)F_{m,q}|^2K(z,z)\mathsf{g}(dz)\\
&= 2\beta^2\mathrm{Var}(\langle\zeta,1_{D_r}F_{m,q}\rangle)+4\int_{D_r}|(f(z)-\beta)F_{m,q}|^2\frac{1}{\pi}dz.
\end{aligned} \tag{3.3.6}$$

Here we used Lemma 2.1 and (2.2.2) for the fourth line. Combining (3.3.6) with Theorem 1.2 and (1.1.6) completes the proof of Theorem 1.3. □

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