<table>
<thead>
<tr>
<th>Title</th>
<th>Variance of the linear statistics of the Ginibre random point field (Proceedings of RIMS Workshop on Stochastic Analysis and Applications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>OSADA, Hirofumi; SHIRAI, Tomoyuki</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2008), B6: 193-200</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2008-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/174211">http://hdl.handle.net/2433/174211</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Variance of the linear statistics of the Ginibre random point field

By
Hirofumi Osada, Tomoyuki Shirai

Abstract
We investigate the variance of linear statistics of the Ginibre random point field. We generalize the result obtained by the second author to higher order moments and also to functions with rotational and radial perturbations. Our result is motivated by the construction of a solution of the infinite dimensional stochastic differential equation related to the Ginibre random point field.

§1. Introduction
The Ginibre random point field (GRPF) $\mu_{\text{Gin}}$ is a probability measure on the configuration space over the complex plane $\mathbb{C}$. The GRPF $\mu_{\text{Gin}}$ is specified by the correlation functions $\{\rho^n\}_{n\in\mathbb{N}}$ with respect to the standard complex Gaussian measure $g(dz) = \frac{1}{\pi} \exp\{-|z|^2\}dz$ given by

\begin{equation}
\rho^n(z_1, \ldots, z_n) = \det[K(z_i, z_j)]_{i,j=1,\ldots,n},
\end{equation}

where $K: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is the exponential kernel defined by

\begin{equation}
K(z_1, z_2) = \exp\{z_1 \overline{z}_2\}.
\end{equation}

The GRPF $\mu_{\text{Gin}}$ is one of the typical examples of Fermion (determinantal) random point field ([7], [5]). It is well known that $\mu_{\text{Gin}}$ is rotation and translation invariant although the kernel $K$ is not translation invariant. Moreover, $\mu_{\text{Gin}}$ is the thermodynamic limit of the distribution $\mu_{\text{Gin}}^N$ of the spectrum of the random matrices called Ginibre ensemble ([5], [1]). The finite volume measure $\mu_{\text{Gin}}^N$ is given by

\begin{equation}
\mu_{\text{Gin}}^N(d\zeta) = \text{const.} e^{-\sum_{i=1}^{N}|z_i|^2} \prod_{i,j=1,\ldots,N, i<j} |z_i - z_j|^2 dz_1 \cdots dz_N.
\end{equation}

2000 Mathematics Subject Classification(s):
*Faculty of Mathematics, Kyushu University 33, Fukuoka 812-8581, JAPAN.
© 2008 Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.
Here $\zeta = \sum_{i=1}^{N} \delta_{z_i}$, and we naturally regard the right hand side as a probability measure on the configuration space.

In view of (1.1.3) the GRPF is the stationary distribution of particles interacting via the two dimensional Coulomb potential. Because of the strong and long range nature of the interaction the static property of the GRPF is quite different from that of Gibbs measures. Indeed, the following small fluctuation result of the variance of the linear statistics of the GRPF is known (cf. [6]):

(1.1.4) \[ \text{Var}(\langle \zeta, 1_{D_r} \rangle) \sim \frac{r}{\sqrt{\pi}} \text{ as } r \to \infty, \]

where $\text{Var}$ is the variation with respect to $\mu_{\text{Gin}}$, $\zeta = \sum_i \delta_{z_i}$ denotes an element of the set of the configurations, and $\{z_i\}$ is a sequence in $\mathbb{C}$ with no accumulation points in $\mathbb{C}$ and $D_r = \{ z \in \mathbb{C} ; |z| \leq r \}$. Here and after we set $\langle \zeta, g \rangle = \int_{\mathbb{C}} gd\zeta = \sum_i g(z_i)$. So $\langle \zeta, 1_{D_r} \rangle$ denotes the cardinality of the particles in the disk $D_r$ by definition.

Note that, if we replace $\mu_{\text{Gin}}$ by the Poisson random point field whose intensity is the Lebesgue measure, then the exponent of $r$ on the right hand side of (1.1.4) is 2 instead of 1.

The purpose of the paper is to generalize the above result to a wider class of functions. As for higher order moments and rotations we obtain the following.

**Theorem 1.1.** Let $I_{\nu}(x) = \sum_{k=0}^{\infty} \frac{1}{k!(k+\nu+1)} \left( \frac{x}{2} \right)^{2k+\nu}$ be the modified Bessel function of the first kind, where $\nu \geq 0$. Let $m$ be a nonnegative integer. Then

\[ \text{Var}(\langle \zeta, 1_{D_r} z^m \rangle) = \frac{r^{2(m+1)}e^{-2r^2}}{m+1} \sum_{k=-m-1}^{m} I_{|k|}(2r) \sim \frac{r^{2m+1}}{\sqrt{\pi}} \text{ as } r \to \infty. \]

Here $f \sim g$ as $r \to \infty$ means $\lim_{r \to \infty} f(r)/g(r) = 1$.

**Theorem 1.2.** Let $m$ be a nonnegative integer and $q \in \mathbb{Z}$. Then

\[ \text{Var}(\langle \zeta, 1_{D_r} z^m e^{\sqrt{-1}q \arg z} \rangle) \sim \frac{r^{2m+1}}{\sqrt{\pi}} \text{ as } r \to \infty. \]

**Theorem 1.3.** Let $f : \mathbb{C} \to \mathbb{C}$ be a measurable function such that

(1.1.5) $\sup_{z \in \mathbb{C}} |f(z)| < \infty$,

(1.1.6) $\sup_{|z|=r} |f(z) - \beta| = O(r^{-\alpha})$ as $r \to \infty$,

where $\alpha > 0$ and $\beta \in \mathbb{C}$ are constants. Let $m$ be a nonnegative integer and $q \in \mathbb{Z}$ as before. Then we have

(1.1.7) $\text{Var}(\langle \zeta, 1_{D_r} z^m e^{\sqrt{-1}q \arg z} f(z) \rangle) = O(r^{\max\{2m+1,2m+2-2\alpha\}})$ as $r \to \infty$. 
Here \( f(r) = O(g(r)) \) as \( r \to \infty \) means \( \limsup_{r \to \infty} |f(r)|/|g(r)| < \infty \).

The first author solved the following infinitely dimensional stochastic differential equation (SDE):

\[
(1.1.8) \quad dZ_t^i = dB_t^i - Z_t^i dt + \sum_{j \in \mathbb{Z}} \frac{Z_t^i - Z_t^j}{|Z_t^i - Z_t^j|^2} dt \quad (i \in \mathbb{Z}).
\]

The solution \( (Z_t^i)_{i \in \mathbb{Z}} \in \mathbb{C}^\mathbb{Z} \) is closely related to \( \mu_{\text{Gin}} \). Indeed, the associated unlabeled dynamics \( \mathbb{Z}_t = \sum_{i \in \mathbb{Z}} \delta_{Z_t^i} \) is a \( \mu_{\text{Gin}} \)-reversible diffusion. One of the difficulty to solve the SDE (1.1.8) is the control of the interaction term \( \sum_{j \in \mathbb{Z}} (Z_t^i - Z_t^j)/|Z_t^i - Z_t^j|^2 dt \). Since \( \mathbb{Z}_t = \sum_{i \in \mathbb{Z}} \delta_{Z_t^i} \) is \( \mu_{\text{Gin}} \)-reversible and \( \mu_{\text{Gin}} \) is translation invariant, the summation \( \sum_{j \in \mathbb{Z}} (Z_t^i - Z_t^j)/|Z_t^i - Z_t^j|^2 \) does not converge absolutely; one can expect only a conditional convergence. To prove the conditional convergence and other properties to solve the SDE (1.1.8) we need an estimate of the form

\[
(1.1.9) \quad \operatorname{Var}(\langle \zeta, 1_{D_r} \frac{z[|z|]}{|z|^2} \rangle) = O(r^{2-c}) \quad \text{for some } c > 0,
\]

where \([t]\) denote the maximal integer smaller than \( t \). The estimate (1.1.9) with \( c = 1 \) is immediate from Theorem 1.3 by taking \( m = 0, q = \beta = 1, \alpha = 1 \) and \( f(z) = [z]|z| \).

\section{Variance of higher order moments}

In this section we prove Theorem 1.1. We first recall the following.

\begin{lemma}
Let \( g \) be a bounded measurable function with compact support. Then
\[
(2.2.1) \quad \operatorname{Var}(\langle \zeta, g \rangle) = \int_{\mathbb{C}} |g(z)|^2 K(z, z) g(dz)
- \int_{\mathbb{C}^2} g(w)\overline{g(z)} |K(w, z)|^2 g(dw) g(dz).
\]
\end{lemma}

\begin{proof}
This lemma is well known. So we omit a proof. \qed
\end{proof}

\begin{remark}
It may be interesting to note that the variance \( \operatorname{Var}(\langle \zeta, g \rangle) \) can be written as follows.
\[
\operatorname{Var}(\langle \zeta, g \rangle) = \frac{1}{2} \int_{\mathbb{C}^2} |g(w) - g(z)|^2 |K(w, z)|^2 g(dw) g(dz).
\]
This follows from Lemma 2.1 and the formula \( K(z, z) = \int_{\mathbb{C}} |K(z, w)|^2 g(dw) \). Also, it follows from this formula that
\[
(2.2.2) \quad \operatorname{Var}(\langle \zeta, g \rangle) \leq 2 \int_{\mathbb{C}} |g(z)|^2 K(z, z) g(dz).
\]
\end{remark}
Lemma 2.2. For $m = 0, 1, 2, \ldots$, we have

\begin{equation}
\int_{D_r} |z^m|^2 K(z, z) g(dz) = \frac{r^{2(m+1)}}{m+1}.
\end{equation}

\textbf{Proof.} A direct calculation shows

\begin{equation}
\int_{D_r} |z^m|^2 K(z, z) g(dz) = \int_{D_r} |z^m|^2 \frac{dz}{\pi} = \frac{r^{2(m+1)}}{m+1}.
\end{equation}

\square

Lemma 2.3. Let $Y_{r^2}$ be the Poisson random variable with mean $r^2$. Let $S_m(n) = n!/(n-m)!$, where $m, n \in \mathbb{Z}$ such that $0 \leq m \leq n$. Then

\begin{equation}
\int_{D_r^2} w^m z^m |K(w, z)|^2 g(dw) g(dz) = \sum_{k=m}^{\infty} S_m(k) P(Y_{r^2} \geq k+1)^2.
\end{equation}

\textbf{Proof.} Since $K(w, z) = e^{wz} = \sum_{k=0}^{\infty} \frac{w^k z^k}{k!}$, we have

\begin{equation}
\int_{D_r^2} w^m z^m |K(w, z)|^2 g(dw) g(dz) = \sum_{k,l=0}^{\infty} \frac{1}{k!l!} \left| \int_{D_r} w^{k+m} \overline{z}^l g(dw) \right|^2
\end{equation}

\begin{align*}
= & \sum_{k=0}^{\infty} \frac{1}{k!(k+m)!} \left( \int_{D_r} |w|^{2(k+m)} g(dw) \right)^2 \\
= & \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \left( \int_{r^2} s^{k+m} e^{-s} \frac{ds}{(k+m)!} \right)^2 \\
= & \sum_{k=0}^{\infty} S_m(k+m) P(Y_{r^2} \geq k+m+1)^2.
\end{align*}

Hence Lemma 2.3 follows from (2.2.6) with the change of variable $k+m \mapsto k$ in the summation immediately. \square

Lemma 2.4. Let $p_k := P(Y_{r^2} = k) = e^{-r^2} \frac{r^{2k}}{k!}$. Then we have

\begin{equation}
\sum_{k=m}^{\infty} S_m(k) P(Y_{r^2} \geq k+1)^2 = \frac{r^{2m}}{m+1} \sum_{i,j=0}^{\infty} p_{i\land j} \cdot p_{(i\lor j)+m+1}.
\end{equation}
Proof. Since $P(Y_{r^{2}} \geq k+1)^{2} = \sum_{i,j=k+1}^{\infty} p_{i} p_{j}$, we have by using Fubini’s theorem

\begin{equation}
(2.2.8) \quad \sum_{k=m}^{\infty} S_{m}(k) P(Y_{r^{2}} \geq k+1)^{2} = \sum_{i,j=m+1}^{\infty} S_{m}(i) p_{i} p_{j}
\end{equation}

\begin{align*}
&= \frac{1}{m+1} \sum_{i,j=m+1}^{\infty} S_{m+1}(i \wedge j) p_{i} p_{j} = \frac{1}{m+1} \sum_{i,j=m+1}^{\infty} S_{m+1}(i \wedge j) p_{i \wedge j} p_{i \vee j} \\
&= \frac{r^{2(m+1)}}{m+1} \sum_{i,j=m+1}^{\infty} p_{(i \wedge j) - (m+1)} \cdot p_{i \vee j} \\
&= \frac{r^{2(m+1)}}{m+1} \sum_{i,j=0}^{\infty} p_{(i \wedge j) + m+1}.
\end{align*}

Here we used $S_{m}(k) = \frac{1}{m+1}(S_{m+1}(k+1) - S_{m+1}(k))$ for the second line, where we set $S_{m+1}(m) = 0$. Moreover, we used $S_{m+1}(k)p_{k} = r^{2(m+1)}p_{k-(m+1)}$ for the third line. \square

**Proof of Theorem 1.1.** First we note that

\begin{equation}
(2.2.9) \quad \sum_{n=0}^{\infty} p_{n} p_{n+k} = \sum_{n=0}^{\infty} \frac{r^{2n+2(n+k)}}{n!(n+k)!} e^{-2r^{2}} = e^{-2r^{2}} I_{k}(2r^{2})
\end{equation}

for $k \geq 0$, and

\begin{equation}
(2.2.10) \quad \sum_{i,j=0}^{\infty} f(i \wedge j) g(i \vee j) = \sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}} f(n) g(n + |k|)
\end{equation}

for any functions $f, g : \mathbb{N} \rightarrow \mathbb{C}$. Then, by using (2.2.9) and (2.2.10) together with the lemmas in this section, we get

\begin{equation}
(2.2.11) \quad \text{Var}(\langle \zeta, 1_{D_{r}} z^{m} \rangle) = \frac{r^{2(m+1)}}{m+1} \left\{ \sum_{i,j=0}^{\infty} p_{i \wedge j} p_{i \vee j} - \sum_{i,j=0}^{\infty} p_{i \wedge j} p_{(i \vee j) + m+1} \right\}
\end{equation}

\begin{align*}
&= \frac{r^{2(m+1)}}{m+1} \sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}} p_{n}(p_{n+|k|} - p_{n+|k|+m+1}) \\
&= \frac{r^{2(m+1)}}{m+1} \sum_{n=0}^{\infty} \sum_{k=-m-1}^{m} p_{n} p_{n+|k|} \\
&= \frac{r^{2(m+1)}e^{-2r^{2}}}{m+1} \sum_{k=-m-1}^{m} I_{|k|}(2r^{2}) \\
&\sim \frac{r^{2m+1}}{\sqrt{\pi}}.
\end{align*}
The last asymptotics immediately follows from

\[(2.2.12) \quad I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}} \left\{ 1 - \frac{4\nu^2 - 1}{8x} + O(x^{-2}) \right\} \quad (x \to \infty)\]

for fixed \(\nu\) (cf. [2], p.123).

\[\square\]

§ 3. Proof of Theorems 1.2 and 1.3.

In this section we prove Theorems 1.2 and 1.3. We begin with a calculation related to the kernel \(K(w, z)\).

**Lemma 3.1.** Let \(q \in \mathbb{Z}\) and \(F_{m,q}(z) = z^m e^{\sqrt{-1}q \arg z}\). Then

\[\int_{D_r} |F_{m,q}(w)|^2 K(w, w) g(dw) = \int_{D_r} |F_{m,0}(w)|^2 K(w, w) g(dw)\]

\[\int_{D_r} F_{m,q}(w) \overline{F_{m,q}(z)} |K(w, z)|^2 g(dw) g(dz) = \int_{[0,r]^2} (st)^m I_{|m+q|}(2st) e^{-s^2 - t^2} d(s^2) d(t^2).\]

**Proof.** Since \(|F_{m,q}(w)| = |F_{m,0}(w)|\), (3.3.1) is clear. Let \(|w| = s, |z| = t, \varphi = \arg z\) and \(\psi = \arg w\). By a direct calculation we have

\[\int_{D_r^2} F_{m,q}(w) \overline{F_{m,q}(z)} |K(w, z)|^2 g(dw) g(dz) = \sum_{k, \ell = 0}^{\infty} \int_{[0,r]^2 \times [0,2\pi]^2} (st)^m e^{\sqrt{-1}(m+q)(\varphi - \psi)} \cdot \frac{s^k t^k e^{\sqrt{-1}k(\varphi - \psi)}}{k!} \cdot \frac{s^\ell t^\ell e^{-\sqrt{-1}\ell(\varphi - \psi)}}{\ell!} \cdot \frac{ste^{-s^2 - t^2}}{\pi^2} ds dt d\varphi d\psi \]

which implies (3.3.2).

\[\square\]

**Lemma 3.2.**

\[\Var(\langle \zeta, 1_{D_r} F_{m,q} \rangle) - \Var(\langle \zeta, 1_{D_r} F_{m,0} \rangle)\]

\[= \int_{[0,r]^2} (st)^m \{I_m(2st) - I_{|m+q|}(2st)\} \cdot e^{-s^2 - t^2} d(s^2) d(t^2).\]
Proof. Lemma 3.2 is immediate from Lemma 2.1 and Lemma 3.1. \qed

Lemma 3.3. It holds that

\[(3.3.4) \quad \int_{[0,r]^2} (st)^m \{I_m(2st) - I_{|m+q|}(2st)\} e^{-s^2-t^2} d(s^2) d(t^2) = O((\int_1^r t^{2m-1} dt) \cdot r^{1/2}) \text{ as } r \to \infty.\]

Proof. By (2.2.12) we see there exists a constant \(C = C_{m,q} > 0\) such that

\[(3.3.5) \quad |I_m(t) - I_{|m+q|}(t)| e^{-t} \leq C(1+t)^{-3/2} \text{ for any } t \geq 0.\]

Therefore,

\[
\begin{align*}
\int_{[0,r]^2} (st)^m |I_m(2st) - I_{|m+q|}(2st)| e^{-s^2-t^2} d(s^2) d(t^2) &= O\left(\int_{[0,r]^2} (st)^{m+1} (1+2st)^{-3/2} e^{-(s-t)^2} d(sdt)\right) \\
&= O\left(\left(\int_{[0,r]^2} (st)^{2m+2} (1+2st)^{-3} d(sdt)\right)^{1/2}\right) \cdot O\left(\left(\int_{[0,r]^2} e^{-2(s-t)^2} d(sdt)\right)^{1/2}\right) \\
&= O\left(\int_1^r t^{2m-1} dt\right) \cdot O(r^{1/2}),
\end{align*}
\]

which implies (3.3.4) \qed

Proof of Theorem 1.2. Theorem 1.2 follows from Theorem 1.1, Lemma 3.2 and Lemma 3.3 immediately. \qed

Proof of Theorem 1.3. Let \(F_{m,q}\) be as in Lemma 3.1. Then it is easy to see

\[(3.3.6) \quad \text{Var}(\langle \zeta, 1_{D_r} z^m e^{\sqrt{-1} q \arg z} f(z) \rangle) \]

\[
= \text{Var}(\langle \zeta, 1_{D_r} \{\beta F_{m,q} + (f(z) - \beta) F_{m,q}\} \rangle) \\
\leq 2\beta^2 \text{Var}(\langle \zeta, 1_{D_r} F_{m,q} \rangle) + 2\text{Var}(\langle \zeta, 1_{D_r} (f(z) - \beta) F_{m,q} \rangle) \\
\leq 2\beta^2 \text{Var}(\langle \zeta, 1_{D_r} F_{m,q} \rangle) + 4 \int_{D_r} |(f(z) - \beta) F_{m,q}|^2 K(z,z) g(dz) \\
= 2\beta^2 \text{Var}(\langle \zeta, 1_{D_r} F_{m,q} \rangle) + 4 \int_{D_r} |(f(z) - \beta) F_{m,q}|^2 \frac{1}{\pi} dz.
\]

Here we used Lemma 2.1 and (2.2.2) for the fourth line. Combining (3.3.6) with Theorem 1.2 and (1.1.6) completes the proof of Theorem 1.3. \qed
References


