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On a Liouville type theorem for harmonic maps to convex spaces via Markov chains

By

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Abstract

We give a Liouville type theorem for harmonic maps from spaces admitting no constant bounded harmonic functions to convex spaces in terms of conservative Markov chains and barycenters. No differentiable structures for the domain and the target are assumed.

§ 1. Introduction

The purpose of this note is to give a Liouville type theorem for harmonic map to metric space having a convex property in terms of discrete time Markov chains. Liouville type theorems for harmonic maps between complete smooth Riemannian manifolds have been done by many authors including geometers and probabilists. Eells-Sampson [13] proved that any (bounded) harmonic map from a compact Riemannian manifold with positive Ricci curvature into a complete manifold with non-positive curvature is a constant map. Schoen-Yau [58] also proved that any harmonic map with finite energy from a complete smooth Riemannian manifold with non-negative Ricci curvature into a complete manifold with non-positive curvature is a constant map. Cheng [3] showed that any harmonic map with sublinear growth from a complete Riemannian manifold with non-negative Ricci curvature into an Hadamard manifold is a constant map. Hildebrandt-Jost-Widman [22] (see also [23],[24]) proved a Liouville type theorem for harmonic maps

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into regular geodesic (open) balls in a complete C^3 Riemannian manifold from a simple or compact C^1 Riemannian manifold. Choi [6] showed a Liouville type theorem for harmonic maps from a complete smooth Riemannian manifold with non-negative Ricci curvature into regular geodesic (open) balls in a complete smooth Riemannian manifold. Kendall (cf. [40],[25],[41]) showed that any bounded harmonic maps from complete smooth Riemannian manifold having the Liouville property for harmonic functions into regular geodesic (open) ball is a constant map using a stochastic tool. He called his result "ultimate Liouville property". By Yau's gradient estimate for harmonic functions (see [66],[5]), any complete smooth Riemannian manifold with non-negative Ricci curvature admits both the elliptic Harnack inequality and the *strong Liouville property*, that is, any positive (or lower bounded) harmonic function is always constant. (The strong Liouville property follows from the elliptic Harnack inequality, cf. 5.4.5 in [56].) So Kendall's result generalizes [6]. Tam [65] proved that a harmonic map u from a Riemannian manifold M quasi-isometric to a complete smooth Riemannian manifold with non-negative Ricci curvature into an Hadamard manifold N is a constant map if u satisfies $d_N(u(x), u(y)) = o(d_M^\gamma(x, y))$, $x, y \in M$, $\gamma \in]0, 1]$.

Sung-Tam-Wang [64] proved that if any bounded harmonic function on a complete smooth Riemannian manifold which is asymptotically constant on each non-parabolic end, then the same assertion holds for bounded harmonic maps to regular geodesic balls extending the Kendall's result. Stafford [59] also gave a probabilistic proof of the result of [3]. Jin [29] proved another type of Liouville type theorem on harmonic maps from Euclidean space to general Riemannian manifolds in terms of the asymptotic behavior of the C^2-map. Colding-Minicozzi II [8],[9] gave related results to [3] on harmonic functions. Rigoli-Setti [55] also gave a Liouville type theorem for harmonic maps from a certain Riemannian manifold with pole. Cheng-Tam-Wang [4], extending the earlier result [58], proved that any harmonic maps with finite energy into Hadamard manifolds also have the Liouville property provided any bounded harmonic function is always constant, and its stochastic proof is done by Atsuji [1] in the framework of L-harmonic maps.

The spaces in the above results are assumed to have differentiable structures in the strict sense. Our main interest is the theory of harmonic maps between singular geometric objects like Alexandrov spaces. In such spaces we can not expect differentiable structures in the classical sense. There are various papers on harmonic maps between singular spaces from geometric or analytic point of view (see [20],[45],[46],[14],[16],[17],[18],[19], [30],[31],[32],[34],[39],[36],[37],[38],[52],[53],[27]). In particular, Gromov-Schoen [20] established the existence of energy minimizing locally Lipschitz maps from Riemannian manifolds into Bruhat-Tits buildings and gave a Corlette's version of Margulis's super-rigidity theorem. After [20], Korevaar-Schoen [45] constructed harmonic maps from domains in Riemannian manifolds into CAT(0)-spaces as a boundary value problem. The same strategy can be applied between Riemannian polyhedra (see [14]). Very recently, Izeki-Nayatani [27] constructed combinatorial harmonic maps from complexes to CAT(0)-spaces and applied it to prove a fixed point theorem. So the theory of harmonic maps between singular objects is nowadays significant in view of geometric or analytic point.

However, there have not been systematic approach to Liouville type theorem for harmonic maps between singular objects. To begin with, we start to prove a Liouville theorem for harmonic maps between such spaces in the framework of discrete time Markov processes according to the argument by Kendall [40]. More concretely, the source space X is only assumed to have conservative Markov chains or Markovian kernel $P(x, dy)$, which is necessary to define the notion of harmonicity, so-called the *P-harmonicity*. The second author has studied P-harmonic maps taking values into CAT(0)-spaces (see [60],[61]). We shall treat more general target spaces than CAT(0)-spaces including a CAT(1)-space with diameter strictly less than $\pi/2$ (e.g. Example 2). The target Y which we consider is assumed to have an analogy of barycenter defined as in [12],[21], which is an extended notion of the usual barycenter in CAT(0)-space discussed in [33],[45],[60],[61],[62],[63],[7]. In this framework we can establish our Liouville type theorem, that is, if the conservative Markov chain has a Liouville property for harmonic functions, then the same property holds for harmonic maps under the condition that the target space is a proper metric space or a separable Banach space (Theorem 3.1).

§ 2. Framework

Throughout this note, (Y, d) denotes a complete separable metric space.

Definition 2.1 ((Admissible Function))**.** A non-negative finite function Φ on $Y \times Y$ is said to be *admissible* if Φ vanishes only on the diagonal, there exists an upper semi continuous function $\psi : [0, \infty[\to [0, \infty[$ such that $\psi > 0$ on $]0, \infty[$, $\psi(0) = 0$ and $\Phi(x, y) \le \psi(d(x, y))$ for $x, y \in Y$, and for each $y \in Y$, $x \mapsto \Phi(x, y)$ is upper semi continuous and for each $x \in Y$, $y \mapsto \Phi(x, y)$ is continuous. The triple (Y, d, Φ) is called *admissible* if Φ is an admissible function on $Y \times Y$.

Let $\mathcal{P}(Y)$ be the family of probability measures on $(Y, \mathcal{B}(Y))$. For an admissible function Φ, we set

$$\mathcal{P}^{\Phi}(Y) := \left\{ \mu \in \mathcal{P}(Y) \;\middle|\; \int_Y \Phi(w, z)\mu(dw) < \infty \text{ for all } z \in Y \right\}.$$

When $\Phi(x, y) = d^p(x, y)$, $p \ge 1$, we write $\mathcal{P}^p(Y)$ instead of $\mathcal{P}^{\Phi}(Y)$.

Definition 2.2 ((Φ-Barycenter)). Let (Y, d, Φ) be an admissible space. For any $\mu \in \mathcal{P}^{\Phi}(Y)$, a point $b(\mu) \in Y$ is said to be Φ-*barycenter* of μ if for each $z \in Y$,

$$\Phi(b(\mu), z) \leq \int_Y \Phi(w, z)\mu(dw)(< \infty). \tag{2.1}$$

Remark. Given $\mu \in \mathcal{P}^{\Phi}(Y)$, the Φ-barycenter of μ is not necessarily unique in general. See Example 6 below for the non-uniqueness of Φ-barycenter. Denote by $B^{\Phi}(\mu)$ the family of Φ-barycenters of μ.

There are many examples of admissible spaces (Y, d, Φ) admitting Φ-barycenters in our sense. Next is a list of examples.

Example 1 ((CAT(0)-Space)). A complete separable metric space (Y, d) is called the *CAT(0)-space* (*Hadamard space, or global NPC space*) if for any pair of points $\gamma_0, \gamma_1 \in Y$ and any $t \in [0, 1]$ there exists a point $\gamma_t \in Y$ such that for any $z \in Y$

$$d^2(z, \gamma_t) \leq (1-t)d^2(z, \gamma_0) + td^2(z, \gamma_1) - t(1-t)d^2(\gamma_0, \gamma_1). \tag{2.2}$$

By definition, $\gamma := (\gamma_t)_{t \in [0,1]}$ is the minimal geodesic joining γ_0 and γ_1. Any CAT(0)-space is simply connected. Hadamard manifolds, Euclidean Bruhat-Tits buildings (e.g. metric tree), spiders, booklets and Hilbert spaces are typical examples of CAT(0)-spaces (cf. [63]). Let (Y, d) be a CAT(0)-space. Then the distance function $d : Y \times Y \to [0, \infty[$ is convex (Corollary 2.5 in [63]) and Jensen's inequality (Theorem 6.3 in [63]) can be applied to the convex function $Y \ni w \mapsto d(w, z)$ for each $z \in Y$. In this case, Φ can be taken to be the distance d and the Φ-barycenter is given by the usual barycenter for $\mu \in \mathcal{P}^1(Y)$ over (Y, d) discussed in [63]. Then (Y, d, Φ) is an admissible space admitting Φ-barycenters.

Example 2 ((k-Convex Space; cf. Ohta [50],[51])). A complete separable metric space (Y, d) is called the k-*convex space* if (Y, d) is a geodesic space and for any three points $x, y, z \in Y$, any geodesic $\gamma := (\gamma_t)_{t \in [0,1]}$ in Y with $\gamma_0 = x$, $\gamma_1 = y$, and all $t \in [0, 1]$,

$$d^2(z, \gamma_t) \leq (1-t)d^2(z, x) + td^2(z, y) - \frac{k}{2}t(1-t)d^2(x, y). \tag{2.3}$$

By definition, putting $z = \gamma_t$, we see $k \in]0, 2]$. The inequality (2.3) yields the (strict) convexity of $Y \ni x \mapsto d^2(z, x)$ for a fixed $z \in Y$. Any closed convex subset of a k-convex space is again k-convex. Every CAT(0)-space is a 2-convex space. It is proved by Ohta [50] that any CAT(1)-space (in particular any spherical building ([2])) Y with $\mathrm{diam}(Y) \leq \frac{\pi}{2} - \varepsilon$, $\varepsilon \in]0, \frac{\pi}{2}[$ is a $\{(\pi - 2\varepsilon)\tan\varepsilon\}$-convex space, and any Banach space L^p with $p \in]1, 2]$ over a measurable space is a $2(p-1)$-convex space (Propositions 3.1

and 3.4 in [50]). He proved that any two points in a k-convex space can be connected by a unique minimal geodesic (Lemma 2.2 in [50]) and contractible (Lemma 2.3 and Corollary 2.4 in [50]). In this case, Φ can be taken to be the square of distance d^2 and the Φ-barycenter is given by the pure barycenter $\bar{b}(\mu)$ for $\mu \in \mathcal{P}^2(Y)$ over (Y, d) discussed in Section 4. Then (Y, d, Φ) is an admissible space admitting Φ-barycenters.

Example 3 ((Regular Geodesic Ball in Riemannian Manifold))**.** Let (Y, d) be the regular geodesic (closed) ball with center o and radius $r > 0$ in an m-dimensional complete smooth Riemannian manifold (M, g) $(m \geq 2)$, where d is the Riemannian distance from g. That is, $Y := \{x \in M \mid d(o, x) \leq r\}$ does not intersect the cut-locus of the center $o \in Y$ and the upper bound $\kappa(= \kappa(r))$ of sectional curvatures in Y satisfies $0 \leq \kappa < (\frac{\pi}{2r})^2$ (see [24],[41]). Then, for $\kappa > 0$, Φ given by

$$\Phi(w, z) := \frac{1}{\kappa} \cdot \frac{1 - \cos(\sqrt{\kappa} d(w, z))}{\cos(\sqrt{\kappa} d(w, o))}$$

determines a non-negative bounded convex function $w \mapsto \Phi(w, z)$ for each $z \in Y$ and $\Phi(w, z) \leq \frac{d^2(w,z)}{2 \cos \sqrt{\kappa} r}$ for $w, z \in Y$ (such form of Φ is originally indicated by Jäger-Kaul [28]). The barycenter $b(\mu)$ for $\mu \in \mathcal{P}(Y)$ defined as a local minimizer of $Y \ni z \mapsto \int_Y d^2(z, w)\mu(dw)$ ([41]) is called the *Karcher mean* of μ and it is uniquely determined in Y (see Theorem 7.3 in [41]). The following Jensen's inequality holds (see Lemma 7.2 in [41]): Let $b(\mu)$ be a Karcher mean of a probability measure μ on Y with compact support contained in the interior of Y. Then for any bounded convex function φ,

$$\varphi(b(\mu)) \leq \int_Y \varphi(w)\mu(dw). \tag{2.4}$$

When κ is independent of $r > 0$, (2.4) remains true for any $\mu \in \mathcal{P}(Y)$ and bounded convex function φ (we can construct a regular geodesic ball U containing Y). Hence $\Phi(b(\mu), z) \leq \int_Y \Phi(w, z)\mu(dw)$ for each $z \in Y$ if κ is independent of $r > 0$. Then (Y, d, Φ) is an admissible space admitting Φ-barycenters under this assumption.

Example 4 ((Banach Space))**.** Let (Y, d) be a separable Banach space, where d is given through the norm on Y. In this case, Φ can be taken to be the distance d and Φ-barycenter can be given by the barycenter $b(\mu)$ for $\mu \in \mathcal{P}^1(Y)$ defined as the Bochner integral $\int_Y x\mu(dx)$. Then (Y, d, Φ) is an admissible space admitting Φ-barycenters.

Let $(X, \mathcal{X})$ be a measurable space. Hereafter, we fix a conservative Markov chain $\mathbf{M}= (\Omega, X_n, \theta_n, \mathcal{F}_n, \mathcal{F}_\infty, \mathbb{P}_x)_{x \in X}$. Here $\Omega := X^{\mathbb{N} \cup \{0\}}$ is the family of sequences $\omega = \{\omega(n)\}_{n \in \mathbb{N} \cup \{0\}}$, $X_n(\omega) := \omega(n)$, $n \in \mathbb{N} \cup \{0\}$, θ_n is the shift operator defined by $X_{m+n}(\omega) = X_m(\theta_n \omega)$, $\mathcal{F}_\infty := \sigma\{X_n \mid n \in \mathbb{N} \cup \{0\}\}$, $\mathcal{F}_n := \sigma\{X_k \mid k \leq n\}$. Denote by $P(x, dy) := \mathbb{P}_x(X_1 \in dy)$ the transition kernel of $\mathbf{M}$, which is a kernel on $(X, \mathcal{X})$

and set $Pf(x) := \int_X f(y)P(x,dy) = \mathbb{E}_x[f(X_1)]$ if the integration has a meaning for an $\mathcal{X}$-measurable function f on X.

Definition 2.3 ((P-Harmonic Map))**.** Let (Y,d,Φ) be an admissible space and consider an $\mathcal{X}/\mathcal{B}(Y)$-measurable map $u : X \to Y$ satisfying $u_*P(x,\cdot) \in \mathcal{P}^\Phi(Y)$. We set

$$Pu(x) := b(u_*P(x,\cdot)).$$

Here $u_*P(x,\cdot)$ is a Borel measure defined by $u_*P(x,\cdot)(A) := P(x,u^{-1}(A))$, $A \in \mathcal{B}(Y)$ and we choose a Φ-barycenter $b(u_*P(x,\cdot)) \in B^\Phi(u_*P(x,\cdot))$ of $u_*P(x,\cdot)$. An $\mathcal{X}$-measurable function $f : X \to \mathbb{R}$ is said to be *P-subharmonic* if $f \le Pf$ on X and an $\mathcal{X}/\mathcal{B}(Y)$-measurable map $u : X \to Y$ is called *P-harmonic* if $u = Pu$ on X.

Remark.

(1) For an $\mathcal{X}/\mathcal{B}(Y)$-measurable map $u : X \to Y$, Pu depends on the choice of (Y,d,Φ) and the choice of Φ-barycenters. We do not require the $\mathcal{X}/\mathcal{B}(Y)$-measurability of $Pu : X \to Y$ (sufficient conditions for this measurability are given in [61] if Y is a CAT(0)-space). Note that P-harmonicity of an $\mathcal{X}/\mathcal{B}(Y)$-measurable $u : X \to Y$ implies the same measurability of $Pu : X \to Y$.

(2) If P is m-symmetric, i.e. $\int_X Pfg\,dm = \int_X fPg\,dm$ for any non-negative $\mathcal{X}$-measurable functions f,g, then one can extend P to $L^2(X;m)$. We call a map $u : X \to Y$ (weakly) P-harmonic if $Pu = u$ m-a.e. on X. In this setting, under that Y is a CAT(0)-space, $u : X \to Y$ is (weakly) P-harmonic if and only if it is a *minimizer of the energy*

$$E[v] := \frac{1}{2}\int_X d_Y(v(x),v(y))^2 P(x,dy)m(dx)$$

among all map $v : X \to Y$ (see [60]). This means that our P-harmonicity is closely related to the classical harmonic maps between Riemannian manifolds. The harmonic maps between singular geometric objects can be defined as a local minimizer of energy functional (see [45],[14]).

Lemma 2.1. *Let (Y,d,Φ) be an admissible space. If an $\mathcal{X}/\mathcal{B}(Y)$-measurable map $u : X \to Y$ satisfying $u_*P(x,\cdot) \in \mathcal{P}^\Phi(Y)$ is P-harmonic, then $x \mapsto \Phi(u(x),y_0)$ is P-subharmonic for each point $y_0 \in Y$. Moreover, if the range of u is bounded, then $\Phi(u,y_0)$ is also a bounded function.*

Proof. By definition of the Φ-barycenter of $u_*P(x,\cdot)$, we see

$$\Phi(u(x),y_0) = \Phi(Pu(x),y_0) \le P\Phi(u(\cdot),y_0) < \infty.$$

The last assertion is clear from $\Phi(u, y_0) \le \psi(d(u, y_0))$ and the upper-semi-continuity of ψ. $\square$

§ 3. Liouville Theorem

Theorem 3.1 ((Liouville Property)). *Let (Y, d, Φ) be an admissible space. We assume one of the following:*

(1) *(Y, d) is proper i.e. every closed bounded subset of Y is compact.*

(2) *(Y, d) is a separable Banach space with $\Phi = d$ and $b(\mu) = \int_Y x\mu(dx)$ for $\mu \in \mathcal{P}^1(Y)$.*

Suppose that any bounded P-harmonic function on X is always constant. Then the same property holds for any bounded P-harmonic map.

Remark.

(1) If $\mathbf{M}$ is irreducible and recurrent, i.e., every bounded P-subharmonic function is constant, then one can prove a Liouville type theorem for P-harmonic map from X to Y without assuming the properness of Y in (1).

(2) As in Theorem 5 of [40], one can prove a Liouville type theorem for Y-valued discrete time martingales over a filtered probability space in the framework of admissible space (Y, d, Φ) with Φ-barycenter (cf. [62],[7] for (discrete or continuous time) martingales taking values in CAT(0)-space). Precisely, if Y-valued martingale (Y_n) has a non-random limit Y_∞ as $n \to \infty$, then $Y_n = Y_\infty = Y_0$ a.s. Its proof is simpler than the proof of Theorem 3.1.

(3) In our setting, the conservativeness of $\mathbf{M}$ is needed for some technical reasons. On account of the conservativeness, that is, $P(x, X) = 1$, one can apply the Jensen's inequality for (pure) barycenter of the probability measure u_*P for any harmonic map u when Y is a CAT(0)-space or a k-convex space.

Example 5 ((Simple Random Walk on $\mathbb{Z}$)). Let $\mathbf{M}$= $(\Omega, X_n, \mathbb{P}_x)_{x\in\mathbb{Z}}$ be a simple random walk on $\mathbb{Z}$, i.e., $P(x, dy) := \mathbb{P}_x(X_1 \in dy) = p_x\delta_{x+1}(dy) + q_x\delta_{x-1}(dy)$ for $p_x, q_x \in]0, 1[$ with $p_x + q_x = 1$. Any P-harmonic function f is given by

$$f(x) = \begin{cases} f(0) + (f(1) - f(0)) \sum_{\ell=1}^{x} \prod_{k=1}^{\ell-1} \frac{q_x}{p_x} & \text{if } x > 0, \\ f(0) - (f(1) - f(0)) \sum_{\ell=x+1}^{0} \prod_{k=\ell}^{0} \frac{p_x}{q_x} & \text{if } x < 0, \\ f(0) + (f(1) - f(0))(x - 1) & \text{if } p_x = \frac{1}{2} \text{ for all } x \in \mathbb{Z}. \end{cases}$$

If for given $\varepsilon > 0$, $q_x > (1 + \varepsilon)p_x$ holds except for finite positive integers x, or $p_x > (1 + \varepsilon)q_x$ holds except for finite negative integers x, then any bounded P-harmonic function f satisfies $f(x) \equiv f(0)$. More strongly, if $p_x = \frac{1}{2}$ for all $x \in \mathbb{Z}$, then any non-negative P-harmonic function f satisfies $f(x) \equiv f(0)$, namely, $\mathbf{M}$ possesses the *strong Liouville property.*

A symmetric simple random walk on an infinite weighted graph treated in [47] satisfying elliptic or parabolic Harnack inequality possesses a strong Liouville property (cf. 5.4.5 in [56]). Other examples on (strong) Liouville property of Markov chains can be found in [49],[26],[15] (see also section 5 in [57]).

Proof of Theorem 3.1. Suppose that $u : X \to Y$ is a bounded P-harmonic map, that is, $u = Pu$ and $\Phi(u, y)$ is a bounded P-subharmonic function for any point $y \in Y$ by Lemma 2.1. We may assume that the image of u is contained in a closed ball $\bar{B}_R(o)$ for some $o \in Y$. We set $M_y := \sup_{x\in X} \Phi(u(x), y) < \infty$. Then $h_y(x) := M_y - \Phi(u(x), y)$ is a P-superharmonic function. According to the Riesz decomposition (Theorem 1.4 in [54]), we have the following:

$$h_y = h_{\infty,y} + Gg_y,$$

where $h_{\infty,y}(x) := \lim_{n\to\infty} P^n h_y(x)$ is the P-harmonic part of h_y and Gg_y with $g_y := (I - P)h_y \geq 0$ is the potential part. Here G is the potential kernel of P defined by $G := \sum_{n=0}^{\infty} P^n$. By assumption, $h_{\infty,y}$ is a constant c_y. Consequently, by putting $d_y := M_y - c_y$, we have $d_y - \Phi(u(x), y) = Gg_y(x)$ for all $x \in X$, in particular, $\Phi(u(x), y) \leq d_y$ holds for all $x \in X$.

Let $\{y_j\} \subset Y$ be a countable dense subcollection of Y. We denote g_{y_i} by g_i. For each $z \in X$, $r > 0$ and $N \in \mathbb{N}$, we claim that there exists $n \in \mathbb{N}$ such that

$$P^n\left(z, \bigcap_{i=1}^{N} \left\{x \in X \;\middle|\; Gg_i(x) \leq r\right\}\right) > 0. \tag{3.1}$$

If the left hand side of (3.1) vanishes for all $n \in \mathbb{N}$ with some $z \in X$, $r > 0$ and $N \in \mathbb{N}$, then

$$G(g_1 + g_2 + \cdots + g_N) > r \quad P^n(z, \cdot)\text{-a.s. on } X \text{ for all } n \in \mathbb{N}.$$

Hence, for such $z \in X$, $P^n G(g_1 + g_2 + \cdots + g_N)(z) > r$ for all $n \in \mathbb{N}$, which contradicts $\lim_{n\to\infty} P^n Gg_y(z) = 0$ for any $y \in Y$. Here we use $P^n 1 = 1$ on X.

From (3.1), for each $z \in X$, $r > 0$ and $N \in \mathbb{N}$,

$$\bigcap_{i=1}^{N} \left\{x \in X \;\middle|\; Gg_i(x) \leq r\right\} \neq \emptyset,$$

in particular,

$$\bigcap_{i=1}^{N} \left\{y \in \bar{B}_R(o) \;\middle|\; d_{y_i} - \Phi(y, y_i) \leq r\right\} \neq \emptyset.$$

If (Y,d) is proper, $\bar{B}_R(o)$ is compact. In particular, any decreasing sequence of non-empty closed subsets of $\bar{B}_R(o)$ has a non-empty intersection. Thus, for each $z \in X$,

$$\bigcap_{n=1}^{\infty}\bigcap_{i=1}^{\infty}\left\{y \in \bar{B}_R(o) \,\middle|\, d_{y_i} - \Phi(y,y_i) \le 1/n\right\} \ne \emptyset.$$

That is, there exists $y_0 \in \bar{B}_R(o)$ which may depend on z such that

$$d_{y_i} - \Phi(y_0,y_i) \le 1/n \text{ for all } i,n \in \mathbb{N},$$

which yields $d_{y_i} \le \Phi(y_0,y_i)$ for each $i \in \mathbb{N}$. Therefore, $\Phi(u(x),y_i) \le \Phi(y_0,y_i)$ for all $x \in X$ and $i \in \mathbb{N}$, hence $u(x) \equiv y_0$.

Next we prove the assertion for the case that Y is a separable Banach space. Recall that $\Phi(z,w)$ is given by the distance $d(z,w) = \|z-w\|_Y$ and Φ-barycenter is the barycenter defined by Bochner integrals. Let $u : X \to Y$ be a bounded P-harmonic map. Then for each $\ell \in Y^*$, we see $x \mapsto {}_{Y^*}\langle \ell, u(x)\rangle_Y$ is a P-harmonic function by use of a property of Bochner integrals. By assumption, ${}_{Y^*}\langle \ell, u(x)\rangle_Y \equiv c_\ell$. Hence for $x,y \in X$

$$\|u(x)-u(y)\|_Y = \sup_{\ell \in Y^*, \|\ell\|_{Y^*}\le 1} {}_{Y^*}\langle \ell, u(x)-u(y)\rangle_Y = 0. \tag{3.2}$$

Therefore, $u \equiv x_0$ for some $x_0 \in Y$.

□

§ 4. Appendix: k-convex space

In this section, we clarify that k-convex space can be an admissible space.

Lemma 4.1. *Let (Y,d) be a k-convex space. Then every open or closed ball B in Y is always convex. In particular, any closed ball B is again a k-convex space.*

Proof. Noting the convexity of $x \mapsto d^2(z,x)$, the proof is easy. □

We can define the notion of *barycenter* of probability measures on a k-convex space (cf. Section 5.3 in [51]):

Definition 4.1 ((Barycenter)). For $\mu \in \mathcal{P}^2(Y)$, we set

$$\mathrm{Var}(\mu) := \inf_{z\in Y}\int_Y d^2(z,y)\mu(dy)(<\infty)$$

and call it the *variance* of μ. If $\mu \in \mathcal{P}^2(Y)$ for $z \mapsto \int_Y d^2(z,x)\mu(dx)$ admits a unique minimizer, i.e., there exists a unique $b(\mu) \in Y$ such that

$$\mathrm{Var}(\mu) = \int_Y d^2(b(\mu),y)\mu(dy), \tag{4.1}$$

then $b(\mu)$ is called the *barycenter of* $\mu \in \mathcal{P}^2(Y)$. For $\mu \in \mathcal{P}^1(Y)$ and $w \in Y$, we consider a function F_w defined by

$$
\begin{aligned}
F_w(z) :&= \int_Y (d^2(z,x) - d^2(w,x))\mu(dx) \\
&\le d(z,w) \int_Y (d(z,x) + d(w,x))\mu(dx) < \infty. \qquad (4.2)
\end{aligned}
$$

If $Y \ni z \mapsto F_w(z)$ admits a unique minimizer $b(\mu)$, then it is also called the *barycenter for* $\mu \in \mathcal{P}^1(Y)$. The barycenter for $\mu \in \mathcal{P}^2(Y)$ is automatically the barycenter for $\mu \in \mathcal{P}^1(Y)$.

For any closed set F, we denote by $C(F)$ the closed convex hull of F, which is the smallest closed convex set containing F, that is, $C(F) := \bigcap \mathcal{C}_F$, $\mathcal{C}_F := \{A \mid A \text{ is closed convex and } F \subset A\}$.

Definition 4.2 ((Pure Barycenter))**.** Take $\mu \in \mathcal{P}^1(Y)$ and $w \in Y$. If $C(\mathrm{supp}[\mu]) \ni z \mapsto F_w(z)$ admits a unique minimizer $\bar{b}(\mu) \in C(\mathrm{supp}[\mu])$, then it is called the *pure barycenter of* $\mu \in \mathcal{P}^1(Y)$.

The following lemma can be similarly proved as in the case of CAT(0)-space.

Lemma 4.2. *Let (Y,d) be a k-convex space. Then every $\mu \in \mathcal{P}^1(Y)$ admits both the barycenter $b(\mu)$ and the pure barycenter $\bar{b}(\mu)$.*

Remark. Let (Y,d) be a k-convex space.

(1) The pure barycenter $\bar{b}(\mu)$ of $\mu \in \mathcal{P}^1(Y)$ is nothing but the barycenter of $\mu \in \mathcal{P}^1(C(\mathrm{supp}[\mu]))$ over the k-convex space $(C(\mathrm{supp}[\mu]), d)$. If $b(\mu) \in C(\mathrm{supp}[\mu])$, in particular, if μ has full support, then $\bar{b}(\mu) = b(\mu)$.

(2) Let K be a closed convex subset of (Y,d). The inequality (2.3) yields that there exists a unique minimizer of $K \ni y \to d^2(y,x)$ denoted by $\pi_K x$, called the *foot-point* or *projected point* to K from x and we call the map $\pi_K : Y \to K$ the *convex projection* to K. For $\mu \in \mathcal{P}^1(Y)$, we set $K = C(\mathrm{supp}[\mu])$. Assume that π_K is contractive, that is, $d(\pi_K x, \pi_K y) \le d(x,y)$ for all $x, y \in Y$. Then we have $\bar{b}(\mu) = b(\mu)$. Indeed, for $w, z \in Y$ we see

$$
\begin{aligned}
\int_Y (d^2(\bar{b}(\mu), y) - d^2(w,y))\mu(dy) &\le \int_K (d^2(\pi_K z, \pi_K y) - d^2(w,y))\mu(dy) \\
&\le \int_Y (d^2(z,y) - d^2(w,y))\mu(dy).
\end{aligned}
$$

From Proposition 2.6 in [63], if (Y,d) is a CAT(0)-space, then π_K is always contractive for any closed convex K, hence $\bar{b}(\mu) = b(\mu)$ for $\mu \in \mathcal{P}^1(Y)$.

(3) Let (Y,d) be a k-convex space. For $x_0, x_1 \in Y$ and $t \in [0,1]$, $\bar{b}((1-t)\delta_{x_0} + t\delta_{x_1}) = b((1-t)\delta_{x_0} + t\delta_{x_1}) = \gamma_t$, where γ_t is the point at t of the minimal geodesic γ joining $x_0 = \gamma_0$ and $x_1 = \gamma_1$. Indeed, $C(\mathrm{supp}[\mu])$ is the minimal geodesic joining x_0 and x_1 and we see for $\mu := (1-t)\delta_{x_0} + t\delta_{x_1}$ and any $z \in Y$,

$$\begin{aligned}\int_Y d^2(\gamma_t, y)\mu(dy) &= (1-t)d^2(\gamma_t,\gamma_0) + td^2(\gamma_t,\gamma_1)\\ &= (1-t)t^2 d(x_0,x_1) + t(1-t)^2 d^2(x_0,x_1)\\ &= t(1-t)d^2(x_0,x_1) \le (1-t)d^2(z,x_0) + td^2(z,x_1)\\ &= \int_Y d^2(z,y)\mu(dy).\end{aligned}$$

(4) Let (Y,d) be a 2-dimensional ℓ^p space $(\mathbb{R}^2, \|\cdot\|_p)$, $p \in]1,2[$. Then (Y,d) is a $2(p-1)$-convex space by [50]. In this case, $b(\mu) \neq \int_Y x\mu(dx)$ and $\bar{b}(\mu) \neq \int_Y x\mu(dx)$ for some $\mu \in \mathcal{P}^1(Y)$, where $\int_Y x\mu(dx)$ is the Bochner integral (see Example 6 below).

Proposition 4.1 ((Jensen's Inequality)). *Suppose that (Y,d) is a k-convex space. Let φ be a lower semi continuous convex function on Y and take $\mu \in \mathcal{P}^1(Y)$. Then we have*

$$\varphi(\bar{b}(\mu)) \le \int_Y \varphi d\mu \tag{4.3}$$

provided the right hand side is well-defined.

Proof. We may assume $Y = C(\mathrm{supp}[\mu])$ replacing Y with $C(\mathrm{supp}[\mu])$. Under this assumption, pure barycenter can be treated as the barycenter. Then the proof is quite same as in the proof for the case of CAT(0)-spaces (see Theorem 6.2 in [63]). □

By Proposition 4.1, we can see that every k-convex space (Y,d) is an admissible space admitting Φ-barycenter by taking $\Phi = d^2$ with the notion of pure barycenters. Finally, we show that Φ-barycenter of a probability measure on $(Y, \mathcal{B}(Y))$ is not unique over a k-convex space (Y,d).

Example 6. Let us consider a 2-dimensional ℓ^p space $(\mathbb{R}^2, d_{\ell^p})$, $p \in]1,2[$ with $d_{\ell^p}(x,y) := \|x-y\|_p = (|x_1-x_2|^p + |y_1-y_2|^p)^{1/p}$, $x = (x_1,x_2), y = (y_1,y_2)$. Take three points $0 = (0,0)$, $e_1 := (1,0)$, $e_2 = (0,1)$ in $\mathbb{R}^2$ and a probability measure $\mu := \frac{1}{3}\delta_{e_1} + \frac{1}{3}\delta_0 + \frac{1}{3}\delta_{e_2}$, and let $\triangle 0e_1e_2$ be the closed convex set surrounded by the geodesic triangle among $0, e_1, e_2$. Note that any line segment between two points in $\mathbb{R}^2$ is the minimal geodesic in $(\mathbb{R}^2, d_{\ell^p})$ joining them. We show $b(\mu) \neq \int_{\mathbb{R}^2} x\mu(dx)$ and $\bar{b}(\mu) \neq \int_{\mathbb{R}^2} x\mu(dx)$. For $(x_1,x_2) \in \mathbb{R}^2$, set

$$f_p(x_1,x_2) := (|1-x_1|^p + |x_2|^p)^{\frac{2}{p}} + (|x_1|^p + |x_2|^p)^{\frac{2}{p}} + (|x_1|^p + |1-x_2|^p)^{\frac{2}{p}}.$$

Then $b(\mu) \in \mathbb{R}^2$ (resp. $\bar{b}(\mu) \in \triangle 0e_1e_2$) is the unique minimizer of $\mathbb{R}^2 \ni (x_1, x_2) \mapsto f_p(x_1, x_2)$ (resp. $\triangle 0e_1e_2 \ni (x_1, x_2) \mapsto f_p(x_1, x_2)$). On the other hand, we see $\int_{\mathbb{R}^2} x\mu(dx) = (1/3, 1/3)$. Since

$$f_p\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{2(2^p+1)^{2/p} + 2^{2/p}}{9} \to \frac{22}{9} > 2 = f_p(0,0) \geq f_p(\bar{b}(\mu)) \geq f_p(b(\mu))$$

as $p \to 1$, we have $b(\mu) \neq (1/3, 1/3)$ and $\bar{b}(\mu) \neq (1/3, 1/3)$ for some $p \in]1, 2[$. We consider $\Phi(x, y) := \|x - y\|_p^2$ for $x, y \in \mathbb{R}^2$. Then $(\mathbb{R}^2, d_{\ell^p}, \Phi)$ is an admissible space. But we have two kinds of different Φ-barycenters for such μ and p. One is $\bar{b}(\mu)$, another is $\int_{\mathbb{R}^2} x\mu(dx)$.

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