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A mechanical model of Markov processes

By

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Abstract

We consider the motion of several massive particles (molecules) in an ideal gas of identical point particles (atoms) in d-dimensional Euclidean space $\mathbf{R}^d$, moving according to Newton mechanical laws, with certain interactions. It is widely believed that, in many cases, the motion of the molecules converges to a Markov process when the mass m of atoms converges to 0, heuristically be virtue of the central limit theorem for "independent identically distributed" atoms. However, since not only the molecules but also the atoms are effected by the interactions, the states (positions and velocities) of the atoms at each time are indeed not independent to each other, nor to the history of the system.

In this study, we consider the above mentioned problem for "plural molecules in an ideal gas under Newton laws" without the independence assumption (which, as explained, actually does not hold). We prove the existence of the solution of the corresponding infinite system of ordinary differential equations, and study its limit when m converges to 0. Details of the proofs can be found in [6].

§ 1. Preliminary

It is, in general, a very interesting and important question to derive the phenomena of statistical mechanics directly and rigorously from classical mechanics.

The simplest example would be the derivative of the Brownian motion. The Brownian motion was first observed, without knowing the reason, by Brown in 1827, as an irregular motion of a rather big particle put into water. This phenomenon was later explained by Einstein in the following way: since a big number of water atoms collide with the big particle randomly, the motion of the big particle could be considered as a result of a sum of many independent identically distributed random variables, so after taking limit, this will give us a Brownian motion. This is also the explanation which can be found in many physical textbooks.

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However, we have to notice that, even in the model of collisional interactions only, there exists the possibility of re-collision, so the states (*i.e.*, positions and velocities) of small particles at each time are not independent to each other, nor to the history of the system. This becomes more evident and significant drawback when considering the model of interactions caused by potentials. Actually, since the interactions between molecules and atoms at each time effect not only the molecules but also the atoms, the states of the atoms interacting with molecules at each moment could not satisfy the *i.i.d.* assumption all the time. By the same reason, the states of the atoms at any two moments could not be independent in general either. Indeed, the actual motion of the massive particles could not be a result of the sum of *i.i.d.* random variables, it is even not a Markov process. So to study this phenomenon more precisely, we need to construct some new model, which takes the mentioned re-interactions into account.

The mechanical model was first introduced and studied by Holley [5], for the case of only one massive particle and with the whole system in dimension $d = 1$, with the interactions given by collisions. This was later extended by, *e.g.*, Dürr-Goldstein-Lebowitz [2], [3], [4], Calderoni-Dürr-Kusuoka [1], and others, to the case of higher dimensional spaces. But all of these are for the model of one massive particle and collisional interactions.

The aim of this research is to extend the above problem to the case of plural massive particles. We consider the model of several massive particles (molecules) in an ideal gas of identical point particles (atoms) in d-dimensional Euclidean space $\mathbf{R}^d$, moving according to classical mechanical laws, with interactions given by potentials between molecules and atoms. Under certain assumptions (but do not include the independence assumption, which, as explained, actually does not hold), we show that the solution of the considered infinite system of ordinary differential equations exists almost surely, and study the limit behavior of the molecules as the mass of the atoms converges to 0.

We finally make the remark that, the model of potential-caused-interactions, although has the advantage that is less singular when compared with collisional interactions, has its own disadvantage that the total momentum of the whole system is not kept invariant.

§ 2. Introduction

Let us describe our problem in detail now. Let $N \geq 1$ and $d \geq 1$ be integers, and let $M_1, \cdots, M_N, m > 0$. Here N stands for the number of massive particles (molecules), d for the dimension of the space $\mathbf{R}^d$, in which the whole system is considered, $M_1, \cdots, M_N$ for the masses of each molecule, and m for the mass of the small particles (the environmental ideal gas atoms). We use $U_i \in C_0^\infty(\mathbf{R}^d)$, $i = 1, \cdots, N$, to denote the (cut off) potential functions, which, as the following equation shows, are assumed to be the potentials that depend only on the relative positions of the massive particles and the atoms. Also, let $X_{i,0}, V_{i,0} \in \mathbf{R}^d$, $i = 1, \cdots, N$, be given, which stand for the initial positions and the initial velocities of the massive particles.

Assume that the initial condition of the environment, *i.e.*, the positions and the velocities of the ideal gas atoms at time 0, is given by $\omega \in Conf(\mathbf{R}^d \times \mathbf{R}^d)$, with the distribution given later. Here $Conf(\mathbf{R}^d \times \mathbf{R}^d)$ stands for the set of all non-empty closed subsets of $\mathbf{R}^d \times \mathbf{R}^d$ which have no cluster points. Each ω is a subset of $\mathbf{R}^d \times \mathbf{R}^d$, and $(x, v) \in \omega$ means that there is a atom at position x with velocity v at time 0.

As claimed before, we assume that as long as the initial conditions $\omega \in Conf(\mathbf{R}^d \times \mathbf{R}^d)$ and $X_{i,0}, V_{i,0} \in \mathbf{R}^d$, $i = 1, \cdots, N$, are given, the whole system evolves according to Newton mechanical laws, with the forces given by potentials depending on the relative positions. Also, for the sake of simplicity, we assume that there is no direct interaction between massive particles or between small particles. Actually, adding the effect of interactions between massive particles causes totally no mathematical difficulty, but will make the formula more complicated only. We would rather say that one of the most interesting point of our results in this paper is that, even for the case with no direct interactions between massive particles, after taking limit $m \to 0$, we get a diffusion in which interactions between massive particles appear. (See the results, especially the definition of the generator L in Section 3).

We use $X_i^{(m)}(t, \omega), V_i^{(m)}(t, \omega) \in \mathbf{R}^d$ to denote the position and the velocity of the i-th massive particle at time t with initial environmental condition ω, and for each $(x, v) \in \omega$, we use $x_i^{(m)}(t, x, v, \omega), v_i^{(m)}(t, x, v, \omega) \in \mathbf{R}^d$ to denote the position and the velocity at time t of the small particle which had state (x, v) at time 0.

In conclusion, for each initial environmental condition ω, we assume that the motion of the system is described by the following infinite system of ordinary differential equations (ODE):

$$
(2.1)\qquad \begin{cases}
\dfrac{d}{dt}X_i^{(m)}(t,\omega) = V_i^{(m)}(t,\omega), \\
M_i \dfrac{d}{dt}V_i^{(m)}(t,\omega) = \\
\qquad\qquad -\displaystyle\int_{\mathbf{R}^d\times\mathbf{R}^d} \nabla U_i(X_i^{(m)}(t,\omega) - x^{(m)}(t,x,v,\omega))\mu_\omega(dx,dv), \\
(X_i^{(m)}(0,\omega), V_i^{(m)}(0,\omega)) = (X_{i,0}, V_{i,0}), \qquad i = 1,\cdots,N, \\
\\
\dfrac{d}{dt}x^{(m)}(t,x,v,\omega) = v^{(m)}(t,x,v,\omega), \\
m\dfrac{d}{dt}v^{(m)}(t,x,v,\omega) = -\displaystyle\sum_{i=1}^{N} \nabla U_i(x^{(m)}(t,x,v,\omega) - X_i^{(m)}(t,\omega)), \\
(x^{(m)}(0,x,v,\omega), v^{(m)}(0,x,v,\omega)) = (x,v), \qquad (x,v)\in\omega.
\end{cases}
$$

Here $\mu_\omega(\,\cdot\,)$ is defined as the counting measure determined by ω: $\mu_\omega(A) = \sharp(\omega\cap A)$ for any $A\in\mathcal{B}(\mathbf{R}^d\times\mathbf{R}^d)$.

We will omit the superscript (m) when there is no risk of confusion. Also, since we are only interested in the motion of the massive particles, from now on, whenever talking about the solution of (2.1), we always mean the value of $(\vec{X}^{(m)}(t,\omega), \vec{V}^{(m)}(t,\omega)) = ((X_1^{(m)}(t,\omega),\cdots,X_N^{(m)}(t,\omega)),(V_1^{(m)}(t,\omega),\cdots,V_N^{(m)}(t,\omega)))$.

Finally, let us give the distribution of the environmental initial condition ω. Let $\rho:\mathbf{R}\to[0,\infty)$ be a continuous function such that $\rho(s)\to 0$ rapidly as $s\to\infty$. Let λ_m be the non-atomic Radon measure on $\mathbf{R}^d\times\mathbf{R}^d$ given by

$$
\lambda_m(dx,dv) = m^{\frac{d-1}{2}}\rho\Big(\frac{m}{2}|v|^2 + \sum_{i=1}^{N} U_i(x - X_{i,0})\Big)dxdv,
$$

and let $P_m(d\omega)$ be the Poisson point process with the intensity measure λ_m. So P_m is a probability measure on $Conf(\mathbf{R}^d\times\mathbf{R}^d)$. We assume that the distribution of ω is given by such P_m.

We are mostly interested in the following two problems:

1. **Existence and uniqueness of the solution of (2.1)**.
2. **The limit behavior of the distribution of** $(\vec{X}^{(m)}(t,\omega),\vec{V}^{(m)}(t,\omega))$ **under** $P_m(d\omega)$ **as** $m\to 0$**.**

§ 3. Results

For the problem of existence and uniqueness, we have the following result.

Existence and Uniqueness. Assume

$$d \geq 2 \quad \text{and} \quad \int_{-\infty}^{\infty} (1+|s|)^d \rho(s) ds < \infty, \tag{3.1}$$

then there exists a unique solution to (2.1) for P_m-*a.s.* ω.

See [6] for the proof.

In order to answer the second question at the end of Section 2, the question of convergence, we need to modify our assumption (3.1) a little bit. Assume that the potential functions $U_i \in C_0^\infty(\mathbf{R}^d)$ are even, *i.e.*, $U_i(-x) = U_i(x)$ for any $x \in \mathbf{R}^d$, $i = 1, \cdots, N$. Let $R_i > 0$ be constants such that $U_i(x) = 0$ if $|x| \geq R_i$, $i = 1, \cdots, N$, and define constants $C_0 = \left(2\sum_{i=1}^N R_i \|\nabla U_i\|_\infty\right)^{1/2}$ and $e_0 = \frac{1}{2}(2C_0+1)^2 + \sum_{i=1}^N \|U_i\|_\infty$. Assume that $\rho : \mathbf{R} \to [0, \infty)$ is a measurable function satisfying the following.

1. $\rho(s) = 0$ if $s \leq e_0$,
2. for any $c > 0$, there exists a $\widetilde{\rho}_c : \mathbf{R} \to [0, \infty)$ such that

$$\sup_{|a| \leq c} \rho(s+a) \leq \widetilde{\rho}_c(s), \qquad \text{for any } s \in \mathbf{R},$$

and

$$\int_{\mathbf{R}^d} (1+|v|^3) \widetilde{\rho}_c(\frac{1}{2}|v|^2) dv < \infty.$$

The first condition above, combined with the expression of the intensity measure λ_m of P_m, implies that only those atoms moving fast enough are taken into consideration in our dynamics. This is a natural and acceptable assumption since, as the masses of atoms are small enough, the effects of slow atoms are negligible.

Also, assume that the initial position $(X_{1,0}, \cdots, X_{N,0})$ of the massive particles satisfies $|X_{i,0} - X_{j,0}| > R_i + R_j$ for any $i \neq j$. *i.e.*, we assume that the massive particles are far enough from each other at the beginning.

It is easy to check that under our present setting (instead of (3.1)), we still have the desired existence and uniqueness of the solution of our ODE, *i.e.*, there exists a unique solution to (2.1) for P_m-*a.s.* ω. Moreover, we have the convergence results as follows.

To describe the limit process as $m \to 0$, let us first define some notations. For any $\vec{X} = (X_1, \cdots, X_N) \in \mathbf{R}^{dN}$, let us consider the following ODE:

$$\begin{cases} \dfrac{d}{dt}\widetilde{x}(t,x,v;\vec{X}) = \widetilde{v}(t,x,v;\vec{X}), \\ \dfrac{d}{dt}\widetilde{v}(t,x,v;\vec{X}) = -\displaystyle\sum_{i=1}^N \nabla U_i(\widetilde{x}(t,x,v;\vec{X}) - X_i), \\ (\widetilde{x}(0,x,v;\vec{X}), \widetilde{v}(0,x,v;\vec{X})) = (x,v). \end{cases} \tag{3.2}$$

Notice that after a proper scaling change of time, this is nothing but the second half equations in (2.1) with the position of massive particles $\vec{X}^{(m)}(t,\omega)$ substituted by the given $\vec{X}$. So the solution of this ODE, after scaling change of time, gives us an approximation of the atoms' motion by keeping the massive particles fixed.

We also introduce the so-called ray representation Ψ. Let

$$E = \{(x,v) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus \{0\});\ x \cdot v = 0\},$$
$$E_v = \{x \in \mathbf{R}^d; x \cdot v = 0\}, \qquad v \in \mathbf{R}^d \setminus \{0\},$$

and let $\nu(dx,dv) = |v|\widetilde{\nu}(dx;v)dv$ be a measure on E, where $\widetilde{\nu}(dx;v)$ is the Lebesgue measure on E_v. Define

$$\Psi : \mathbf{R} \times E \to \mathbf{R}^d \times (\mathbf{R}^d \setminus \{0\}), (s,(x,v)) \mapsto (x - sv, v),$$

in other words, we decompose the position of each atom into two parts: one parallel to its velocity and the other perpendicular to its velocity.

Let

$$\psi^0(t,x,v;\vec{X}) = \lim_{s\to\infty} \widetilde{x}(t+s, \Psi(s,x,v); \vec{X}),$$

which is well-defined for any $t \in \mathbf{R}$ and $(x,v) \in E$. Here $(\widetilde{x}, \widetilde{v})$ stands for the solution of (3.2).

Now we are ready to give the quadratic term of the diffusion generator of the limit process: Let

$$a_{ik;jl}(\vec{X}) = \frac{1}{M_i M_j} \int_E \Big(\int_{-\infty}^{\infty} \nabla_k U_i(\psi^0(t,x,v;\vec{X}) - X_i) dt \Big) \times \Big(\int_{-\infty}^{\infty} \nabla_l U_j(\psi^0(t,x,v;\vec{X}) - X_j) dt \Big) \rho(\frac{1}{2}|v|^2) \nu(dx,dv).$$

We next give the definition of the drift term of the limit process. For any $(x,v) \in E$, $\vec{X}, \vec{V} \in \mathbf{R}^{dN}$ and $a \in \mathbf{R}$, let $z(t;x,v,\vec{X},\vec{V},a) \in \mathbf{R}^d$ denote the solution of

$$\begin{cases} \dfrac{d^2}{dt^2} z(t) = -\displaystyle\sum_{i=1}^N \nabla^2 U_i(\psi^0(t,x,v,\vec{X}) - X_i)(z(t) - (t+a)V_i), \\ \displaystyle\lim_{t\to-\infty} z(t) = \lim_{t\to-\infty} \frac{d}{dt} z(t) = 0. \end{cases}$$

Then $z(t;x,v,\vec{X},\vec{V},a)$ is a linear function of $\vec{V}$. Let $b_{ik;jl} : \mathbf{R}^{dN} \to \mathbf{R}$ be the C^∞-functions determined by the following:

$$-\frac{1}{2} \int_E \Big(\int_{-\infty}^{\infty} \nabla^2 U_i(\psi^0(t,x,v,\vec{X}) - X_i) z(t,x,v,\vec{X},\vec{V},-t) dt \Big) \times \rho(\frac{1}{2}|v|^2) \nu(dx,dv)$$
$$= \sum_{\ell=1}^d \sum_{j=1}^N b_{i\cdot;j\ell}(\vec{X}) V_j^\ell,$$

or equivalent, in the form of element expression,

$$-\frac{1}{2}\int_E\Big(\int_{-\infty}^{\infty}\sum_{p=1}^{d}\nabla_k\nabla_p U_i(\psi^0(t,x,v,\vec{X})-X_i)z_p(t,x,v,\vec{X},\vec{V},-t)dt\Big)$$
$$\times\rho(\frac{1}{2}|v|^2)\nu(dx,dv)$$
$$=\sum_{\ell=1}^{d}\sum_{j=1}^{N}b_{ik;j\ell}(\vec{X})V_j^\ell,\qquad k=1,\cdots,d,$$

where z_p means the p-th element of the vector z for $p=1,\cdots,d$.

Now we are in a position to give the definition of the limiting diffusion generator L on $\mathbf{R}^{2dN}$:

$$L=\sum_{i,j=1}^{N}\sum_{k,l=1}^{d}a_{ik,jl}(\vec{X})\frac{\partial^2}{\partial V_i^k\partial V_j^l}+\sum_{i,j=1}^{N}\sum_{k,l=1}^{d}b_{ik,jl}(\vec{X})V_j^\ell\frac{\partial}{\partial V_i^k}+\sum_{i=1}^{N}\sum_{k=1}^{d}V_i^k\frac{\partial}{\partial X_i^k}.$$

Our convergence results, formulated in three different situations, are the following.

Convergence Result 1. Assume $N=1$. Then $\{(X_1^{(m)}(t),V_1^{(m)}(t)),t\ge 0\}$ under P_m converges weakly to the diffusion process in $C([0,\infty);\mathbf{R}^{2d})$ with generator L as $m\to 0$.

Convergence Result 2. Assume $N\ge 2$. Let

$$\sigma_0(\omega)=\inf\big\{t>0;\min_{i\neq j}\{|X_i(t)-X_j(t)|-(R_i+R_j)\}\le 0\big\},$$

the first time that the positions of massive particles in some pair are too close. Then $\{(\vec{X}^{(m)}(t\wedge\sigma_0),\vec{V}^{(m)}(t\wedge\sigma_0)),t\ge 0\}$ under P_m converges weakly to the diffusion with generator L stopped at σ_0 in $C([0,\infty);\mathbf{R}^{2dN})$ as $m\to 0$.

Convergence Result 3. Let $N=2$ and $d\ge 3$. Assume that there exist functions h_1,h_2 such that

$$U_i(x)=h_i(|x|),\qquad i=1,2,$$

and there exists a constant $\varepsilon_0>0$ such that

$$(-1)^{i-1}h_i(s)>0,\quad(-1)^{i-1}h_i''(s)>0,\qquad s\in(R_i-\varepsilon_0,R_i),i=1,2.$$

Then we have that $\{(\vec{X}^{(m)}(t),\vec{V}^{(m)}(t)),t\ge 0\}$ under P_m converges weakly to a Markov process as $m\to 0$.

The description of the limit Markov process in Convergence Result 3, indeed a reflecting diffusion process, will be given in Section 4. The first half of the conditions in Result 3 requires that, the potential functions for the two massive particles depend only on the distances with the atoms. Also, the second half of the assumptions above implies that: at least when near to R_i, one massive particle has repulsive forces with the atoms, and the other massive particle has attractive forces with the atoms.

§ 4. Ideas of the Proof and Basic Lemma

We emphasize again that, as mentioned in Sections 1 and 2, in our present problem, the involved forces at any fixed time are not independent to the history. Therefore, since both the massive particles and the small "environmental" particles are moving, the system is very complicated. This is also one of the difficulties in this problem. One of our main ideas for the proof of convergence results is that, although all of the involved particles are moving all the time, since the velocities of the massive particles are very slow compared with the small particles, when considering the scattering of the small particles, we can use the approximation that the massive particles are not moving, with the caused error small enough. With the help of this approximation, the ODE for the motion of the small particles could be approximated by the one in which the massive particles are "fixed" (recall the definition of $\widetilde{x}(\,\cdot\,)$ is Section 3).

For any $n > 0$, let $\sigma_n = \inf\{t \geq 0; \max_{i=1,\cdots,N} |V_i(t)| \geq n\}$ be the first time that the velocity of some massive particles is greater than n. This is introduced only for convenience, since we will let $n \to \infty$, which implies $\sigma_n \to \infty$, at the end.

Our basic lemma is the following, which gives us a decomposition of $M_i V_i(t \wedge \sigma_n)$ into a martingale part, a "smooth part", a negligible part, and the term $-m^{-1/2} \int_0^{t\wedge\sigma} \nabla_i \widetilde{U}(\vec{X}(s))ds$ corresponding to a "new potential" function $\widetilde{U}$, in which the small "environmental" particles do not appear.

Lemma 4.1. *For any $n > 0$ and $i = 1, \cdots, N$, there exist an $\mathbf{R}^d$-valued $(\mathcal{F}_t)_t$-martingale $H_i(t)$, an $\mathbf{R}^d$-valued $(\mathcal{F}_t)_t$-adapted C^1-class (in t) process $P_i^{*1}(t)$ and an $\mathbf{R}^d$-valued $(\mathcal{F}_t)_t$-adapted process $\eta_i(t)$ such that the following four conditions are satisfied:*

1.
$$M_i(V_i(t \wedge \sigma_n) - V_i(0)) = H_i(t) + P_i^{*1}(t) + \eta_i(t) - m^{-1/2} \int_0^{t\wedge\sigma_n} \nabla_i \widetilde{U}(\vec{X}(s))ds$$
for $i = 1, \cdots, N$,

2. *there exists a constant C independent of m such that*
$$\left| d\langle H_i^k, H_j^\ell \rangle_t \right| \leq Cdt, \qquad P_m\text{-a.s.}$$
and the jumps of H_i satisfy $|\Delta H_i(t)| \leq Cm^{1/2}$ for any $k, \ell = 1, \cdots, d$, $i, j = 1, \cdots, N$ and $m \in (0, 1]$,

3.

$$\sup_{m\in(0,1]}\sup_{t\in[0,T]} E^{P_m}\Big[|\frac{d}{dt}P_i^{*1}(t)|^2\Big] < \infty$$

for $i = 1, \cdots, N$,

4.

$$E^{P_m}\big[\sup_{t\in[0,T]}|\eta_i(t)|\big] \to 0, \qquad \text{as } m \to 0$$

for $i = 1, \cdots, N$.

In particular, the distribution of $\{H_i(t) + P_i^{*1}(t) + \eta_i(t)\}_{t\in[0,T]}$ *under* P_m *is tight in* $\wp(D([0,T];\mathbf{R}^d))$ *as* $m \to 0$, *with limits as distributions of continuous processes.*

See [6] for the detailed expressions of $H_i(t)$, $P_i^{*1}(t)$ and $\eta_i(t)$.

Here $\wp(D([0,T];\mathbf{R}^d))$ means the space of probabilities on the complete metric space $(D([0,T];\mathbf{R}^d), d^0)$, where $D([0,T];\mathbf{R}^d)$ is the usual Skorohod space:

$$D([0,T];\mathbf{R}^d) = \Big\{w : [0,T] \to \mathbf{R}^d; \quad w(t) = w(t+) := \lim_{s\downarrow t} w(s), t \in [0,T), \\ \text{and } w(t-) := \lim_{s\uparrow t} w(s) \text{ exists}, t \in (0,T]\Big\},$$

with the metric d^0 given by

$$d^0(w, \widetilde{w}) = \inf_{\lambda\in\Lambda}\Big\{\|\lambda\|^0 \vee \|w - \widetilde{w}\circ\lambda\|_\infty\Big\}$$

for any $w, \widetilde{w} \in D([0,T];\mathbf{R}^d)$, where

$$\Lambda = \Big\{\lambda : [0,T] \to [0,T];\ \text{continuous, non-decreasing}, \lambda(0) = 0, \lambda(T) = T\Big\},$$

$\|w\|_\infty = \sup_{0\le t\le T}|w(t)|$, and

$$\|\lambda\|^0 = \sup_{0\le s<t\le T}\Big|\log\frac{\lambda(t) - \lambda(s)}{t - s}\Big|$$

for any $\lambda \in \Lambda$.

The new potential function $\widetilde{U}$ is by definition

$$\widetilde{U}(X_1, \cdots, X_N) = \int_{\mathbf{R}^{2d}}\Big\{\widetilde{\rho}\left(\frac{1}{2}|v|^2 + \sum_{i=1}^N U_i(x - X_i)\right) - \widetilde{\rho}(\frac{1}{2}|v|^2)\Big\}dxdv,$$

with $\widetilde{\rho}(t) = -\int_t^\infty \rho(s)ds, t \in \mathbf{R}$.

It is easy to be checked that the integral $\int_0^{t\wedge\sigma_n}\nabla_i\widetilde{U}(\vec{X}(s))ds$ in the last term of the decomposition in Lemma 4.1 keeps 0 until the positions of any of the two massive particles become too close, *i.e.*,

$$\nabla_i\widetilde{U}(\vec{X}) = 0, \qquad \text{if } |X_j - X_k| > R_j + R_k \text{ for any } j \neq k.$$

This combined with Lemma 4.1 gives us the tightness of the distributions of $\{V_i(t \wedge \sigma_n)\}_{t\in[0,T]}$ under P_m in $\wp(D([0,T];\mathbf{R}^d))$ as $m \to 0$, under the situations described in Convergence Results 1 and 2. After converting the problem into martingale problem, a more detailed discussion for each related terms gives us the desired convergences in Results 1 and 2.

In the special case of two massive particles described in Result 3, since we have the coefficient $-m^{-1/2}$, which converges to $-\infty$ as $m \to 0$, the last term $-m^{-1/2}\int_0^{t\wedge\sigma_n} \nabla_i \widetilde{U}(\vec{X}(s))ds$ gives us the reflecting force when the two massive particles are too close, more precisely, when $|X_1(t) - X_2(t)| \geq R_1 + R_2$. Therefore, our limit Markov process in Convergence Result 3 is the reflecting diffusion process with generator L, reflecting whenever the distance of the two massive particles is equal to $R_1 + R_2$.

Finally, we want to remark that, for any fixed $m > 0$, although $V_i(t)$ is continuous with respect to t (since it is described by the ODE (2.1)), our martingale part $H_i(t)$ in the decomposition of $V_i(t)$ in Lemma 4.1 needs not be continuous. The only thing we can say is that its jumps are dominated by some constant times $m^{1/2}$, (see Lemma 4.1). This is also one of our ideas: we only intend to use the martingale theory to the part for which it is applicable, without caring whether it is continuous or not. For the remaining term, instead of trying to deal with it in detail, we show that the whole term is negligible as $m \to 0$ from the beginning.

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