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A scaling limit for weakly pinned Gaussian random walks

By

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Abstract

The aim of this paper is threefold. First, we review the results of [1] on a scaling limit for weakly pinned Gaussian random walks. Secondly, two different proofs, one based on the FKG inequality and the other relying on the renewal theory, are given for an exponential decay estimate on certain probabilities related to the random walks. This plays a crucial role to establish the large deviation principle. Finally, randomly pinned random walks are introduced.

§ 1. Model and results

In this section, we introduce the d-dimensional Gaussian random walks with pinning at the origin 0. Then the results of [1] on a macroscopic scaling limit for such random walks are reviewed. See Section 6 of [3] for further related results. The paper [4] discusses the same problem in a critical situation for the pinned Wiener measures with weak self potentials.

§ 1.1. Weakly pinned Gaussian random walks

For $\epsilon \geq 0$, let $\phi = (\phi_i)_{i \in D_N}$ be the Markov chain on $\mathbb{R}^d$ with transition probability

$$P^\epsilon(x, dy) = \frac{1}{Z^\epsilon(x)} e^{-|x-y|^2/2}\big(\epsilon\, \delta_0(dy) + dy\big), \quad x, y \in \mathbb{R}^d,$$

where $D_N = \{0, 1, 2, \dots, N\}, N \in \mathbb{N}$ and $Z^\epsilon(x) = \epsilon\, e^{-|x|^2/2} + (2\pi)^{d/2}$ is the normalizing constant. If $\epsilon = 0$, the Markov chain ϕ is the d-dimensional Brownian motion viewed at integer times. The parameter ϵ represents the strength of the pinning at the origin

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0. In [1], the random walks weakly pinned at a general subspace M of $\mathbb{R}^d$ are studied, but in the present paper we take $M = \{0\}$ for simplicity.

Given $a \in \mathbb{R}^d$, the distribution on $(\mathbb{R}^d)^{D_N}$ of the Markov chain ϕ starting at aN is denoted by $\mu_N^{a,F,\epsilon}$, i.e.,

$$\mu_N^{a,F,\epsilon}(d\phi) = \frac{1}{Z_N^{a,F,\epsilon}} e^{-H_N(\phi)} \delta_{aN}(d\phi_0) \prod_{i=1}^{N} \big(\epsilon\, \delta_0(d\phi_i) + d\phi_i\big), \tag{1.1}$$

where $Z_N^{a,F,\epsilon}$ is the normalizing constant and

$$H_N(\phi) = \frac{1}{2} \sum_{i=0}^{N-1} |\phi_{i+1} - \phi_i|^2.$$

Given $b \in \mathbb{R}^d$, the distribution of the Markov chain conditioned to arrive at bN is denoted by $\mu_N^{a,b,\epsilon}$; in other words, $\mu_N^{a,b,\epsilon}$ is the probability measure given by the formula (1.1) with the last factor $(\epsilon\, \delta_0(d\phi_N) + d\phi_N)$ replaced by $\delta_{bN}(d\phi_N)$ and $Z_N^{a,F,\epsilon}$ by a new normalizing constant $Z_N^{a,b,\epsilon}$, respectively.

We also consider the Markov chains moving only on the upper half space $\mathbb{R}_+^d = \mathbb{R}^{d-1} \times [0,\infty)$ of $\mathbb{R}^d$. Namely, for $a, b \in \mathbb{R}_+^d$, we consider the conditional distributions $\mu_N^{a,F,\epsilon,+}$ and $\mu_N^{a,b,\epsilon,+}$ of $\mu_N^{a,F,\epsilon}$ and $\mu_N^{a,b,\epsilon}$, respectively, on the event $\{\phi; \phi_i \in \mathbb{R}_+^d$ for all $i \in D_N\}$ $\big(= (\mathbb{R}_+^d)^{D_N}\big)$. The corresponding normalizing constants are denoted by $Z_N^{a,F,\epsilon,+}$ and $Z_N^{a,b,\epsilon,+}$, respectively. These Markov chains are defined under the presence of a wall at $\partial\mathbb{R}_+^d$.

The following table shows the difference of these four measures in short:

	No Wall	Wall at $\partial\mathbb{R}_+^d$
Pinned at N	$\mu_N^{a,b,\epsilon}$	$\mu_N^{a,b,\epsilon,+}$
Free at N	$\mu_N^{a,F,\epsilon}$	$\mu_N^{a,F,\epsilon,+}$

Table 1. Distributions of Markov chains

§ 1.2. Results

1.2.1. Free energies and phase transitions

We define the free energies associated with the weakly pinned random walks by the thermodynamic limits under the absence or presence of a wall:

$$\xi^\epsilon = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z_N^{0,0,\epsilon}}{Z_N^{0,0}} \left(= \lim_{N\to\infty} \frac{1}{N} \log \frac{Z_N^{0,F,\epsilon}}{Z_N^{0,F}} \right),$$

$$\xi^{\epsilon,+} = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z_N^{0,0,\epsilon,+}}{Z_N^{0,0,+}} \left(= \lim_{N\to\infty} \frac{1}{N} \log \frac{Z_N^{0,F,\epsilon,+}}{Z_N^{0,F,+}} \right).$$

We have chosen $a = b = 0$ and the partition functions (normalizing constants) in the denominators are defined without pinning; for example, $Z_N^{0,0} = Z_N^{0,0,0}$ (i.e., $\epsilon = 0$). One can show that the limits exist and are independent of the conditions at $i = N$ (pinned or free), and $\xi^\epsilon > \xi^{\epsilon,+} \geq 0$ for all $\epsilon \geq 0$ unless $\xi^\epsilon = 0$. The random walks exhibit the phase transition (localization/delocalization transition or recurrence/transience transition) depending on the strength ϵ of the pinning as is summarized in the next table:

Pinning transition (Absence of wall)	$d \geq 3 : ^\exists\ \epsilon_c > 0$ s.t. $\epsilon > \epsilon_c \Rightarrow \xi^\epsilon > 0$ (positive recurrent regime) $0 \leq \epsilon \leq \epsilon_c \Rightarrow \xi^\epsilon = 0$ (null recur./transient regime) $d = 1, 2 : \epsilon_c = 0$
Wetting transition (Presence of wall)	$d \geq 1 : ^\exists\ \epsilon_c^+ > \epsilon_c$ s.t. $\epsilon > \epsilon_c^+ \Rightarrow \xi^{\epsilon,+} > 0$ (positive recurrent regime) $0 \leq \epsilon \leq \epsilon_c^+ \Rightarrow \xi^{\epsilon,+} = 0$ (null recur./transient regime)

Table 2. Phase transitions

Explicit formulae determining ξ^ϵ and ϵ_c are found in Section 2.2.1, while those for $\xi^{\epsilon,+}$ and ϵ_c^+ are in Section 2.2.2. Asymptotic behaviors of ξ^ϵ or $\xi^{\epsilon,+}$ as $\epsilon \downarrow \epsilon_c$ or $\epsilon \downarrow \epsilon_c^+$ can be studied and, especially, the critical exponents associated with the free energies can be computed explicitly, see [1].

1.2.2. Sample path large deviation principle

We are interested in a macroscopic behavior of the Markov chains. Let $h^N = \{h^N(t), t \in D\}$ be the macroscopic path of the Markov chain determined from the microscopic one ϕ under the scaling defined through a polygonal approximation of $\left(h^N(i/N) = \phi_i/N\right)_{i\in D_N}$:

$$h^N(t) = \frac{[Nt] - Nt + 1}{N}\phi_{[Nt]} + \frac{Nt - [Nt]}{N}\phi_{[Nt]+1}, \quad t \in D,$$

where $D = [0, 1]$. Then, as $N \to \infty$, the sample path large deviation principle holds for h^N under each of the four distributions $\mu_N^{a,b,\epsilon}$, $\mu_N^{a,b,\epsilon,+}$, $\mu_N^{a,F,\epsilon}$ and $\mu_N^{a,F,\epsilon,+}$ introduced in

Section 1.1 on the space $\mathcal{C} = C(D, \mathbb{R}^d)$ equipped with the uniform topology. The speed of the large deviation principle is always N and the unnormalized rate functionals are given by $\Sigma^{a,b,\epsilon}$, $\Sigma^{a,b,\epsilon,+}$, $\Sigma^{a,F,\epsilon}$ and $\Sigma^{a,F,\epsilon,+}$, respectively, of the form

$$\Sigma(h) = \frac{1}{2}\int_D |\dot{h}(t)|^2 dt - \xi\big|\{t \in D; h(t) = 0\}\big|, \tag{1.2}$$

for $h \in H^1(D, \mathbb{R}^d)$, where $\xi = \xi^\epsilon$ for $\Sigma = \Sigma^{a,b,\epsilon}$, $\Sigma^{a,F,\epsilon}$ and $\xi = \xi^{\epsilon,+}$ for $\Sigma = \Sigma^{a,b,\epsilon,+}$, $\Sigma^{a,F,\epsilon,+}$, $|\cdot|$ stands for the Lebesgue measure on D and $H^1(D, \mathbb{R}^d)$ is the usual Sobolev space.

The large deviation principle immediately implies the concentration for each of the distributions $\mu_N^\epsilon = \mu_N^{a,b,\epsilon}, \mu_N^{a,b,\epsilon,+}, \mu_N^{a,F,\epsilon}$ and $\mu_N^{a,F,\epsilon,+}$:

$$\lim_{N\to\infty} \mu_N^\epsilon\big(\mathrm{dist}_\infty(h^N, \mathcal{H}) \le \delta\big) = 1, \tag{1.3}$$

for every $\delta > 0$, where $\mathcal{H} = \{h^*; \text{minimizers of } \Sigma\}$ and dist_∞ denotes the distance on $\mathcal{C}$ under the uniform norm $\|\cdot\|_\infty$.

There are at most two possible minimizers of Σ:

Functionals	$\Sigma^{a,b,\epsilon}$	$\Sigma^{a,b,\epsilon,+}$	$\Sigma^{a,F,\epsilon}$	$\Sigma^{a,F,\epsilon,+}$
Possible minimizers	$\bar{h}, \hat{h}$	$\bar{h}, \hat{h}^+$	$\bar{h}^F, \hat{h}^F$	$\bar{h}^F, \hat{h}^{F,+}$

Table 3. Possible minimizers

Here, the functions $\bar{h}, \hat{h}, \bar{h}^F$ and $\hat{h}^F$ are defined as follows:

$$\bar{h}(t) = (1-t)a + tb, \quad \hat{h}(t) = \begin{cases} (t_1 - t)a/t_1, & t \in [0, t_1], \\ 0, & t \in [t_1, 1-t_2], \\ (t + t_2 - 1)b/t_2, & t \in [1-t_2, 1], \end{cases}$$

$$\bar{h}^F(t) = a, \qquad \hat{h}^F(t) = \begin{cases} (t_1 - t)a/t_1, & t \in [0, t_1], \\ 0, & t \in [t_1, 1], \end{cases}$$

for $t \in D$, where $t_1 = |a|/\sqrt{2\xi^\epsilon}$ and $t_2 = |b|/\sqrt{2\xi^\epsilon}$. The functions $\hat{h}^+$ and $\hat{h}^{F,+}$ are defined similarly to $\hat{h}$ and $\hat{h}^F$, respectively, with ξ^ϵ replaced by $\xi^{\epsilon,+}$. The functions $\bar{h}$ and $\bar{h}^F$ are linear, i.e., $\bar{h}$ linearly interpolates between a and b, while $\bar{h}^F$ stays at a. On the other hand, the functions $\hat{h}, \hat{h}^F, \hat{h}^+$ and $\hat{h}^{F,+}$ visits 0 to gain a reward from the second term of (1.2), paying a penalty for the first term.

In particular, if the minimizer of Σ is unique, we see from (1.3) that the law of large numbers holds for h^N under μ_N^ϵ and the limit is the unique minimizer.

1.2.3. Scaling limits in a critical situation

Of particular interest is the critical situation that the two possible minimizers are actually the minimizers of Σ simultaneously: $\Sigma^{a,b,\epsilon}(\bar{h}) = \Sigma^{a,b,\epsilon}(\hat{h})$ and others. The results, summarized in the next table, are superficially independent of the presence or absence of a wall, but depend on the conditions at $i = N$ and the dimension d of the space.

		Limits
Pinned at N $(\mu_N^{a,b,\epsilon}, \mu_N^{a,b,\epsilon,+})$	$d = 1$	$\hat{h}$ (or $\hat{h}^+$)
	$d = 2$	Coexistence
	$d \geq 3$	$\bar{h}$
Free at N $(\mu_N^{a,F,\epsilon}, \mu_N^{a,F,\epsilon,+})$	$d = 1$	Coexistence
	$d \geq 2$	$\bar{h}^F$

Table 4. Scaling limits at criticality

Here, "Coexistence" means that the distribution on $\mathcal{C}$ of h^N under μ_N^ϵ converges weakly to the superposition of positive masses at two minimizers as $N \to \infty$. See [1] for details.

The central limit theorem is also established in [1] for the first or last hitting times of the Markov chains ϕ at 0.

§ 2. Large deviation type estimates

The following estimates play a crucial role to establish the sample path large deviation principle for h^N under $\mu_N^{a,b,\epsilon}, \mu_N^{a,b,\epsilon,+}, \mu_N^{a,F,\epsilon}$ and $\mu_N^{a,F,\epsilon,+}$, cf. Section 4 of [1]. In fact, the proof of the large deviation principle can be reduced to the following rough large deviation estimates under the 0-boundary conditions (i.e., $a = b = 0$).

Proposition 2.1. *For every $\delta > 0$, there exists $C, c > 0$ independent of N such that*

$$\mu_N^\epsilon(\|h^N\|_\infty \geq \delta) \leq Ce^{-cN}$$

for $\mu_N^\epsilon = \mu_N^{0,0,\epsilon}, \mu_N^{0,0,\epsilon,+}, \mu_N^{0,F,\epsilon}$ and $\mu_N^{0,F,\epsilon,+}$.

We give two different proofs of this proposition. Section 2.1, based on a stochastic domination for Euclidean norms, is for two measures $\mu_N^{0,0,\epsilon}$ and $\mu_N^{0,F,\epsilon}$, i.e., under the absence of a wall; also based on a stochastic domination, [1] covered all four measures by viewing conditional distributions. The method of Section 2.2, based on a renewal theory and announced in Remark 4.1-(2) of [1], is applicable to all four measures.

§ 2.1. Proof based on a stochastic domination

Let $R : (\mathbb{R}^d)^{D_N} \to (\mathbb{R}_+)^{D_N}$ be a mapping defined by $R\phi = x$ with $x = (x_i = |\phi_i|)_{i \in D_N}$ for $\phi = (\phi_i)_{i \in D_N}$. The space $(\mathbb{R}_+)^{D_N}$ is equipped with a natural partial order $x \le y$ for $x = (x_i)_{i \in D_N}$ and $y = (y_i)_{i \in D_N}$ defined by $x_i \le y_i$ for every $i \in D_N$. For two probability measures ν_1 and ν_2 on $(\mathbb{R}_+)^{D_N}$, we say that ν_2 stochastically dominates ν_1 (written as $\nu_1 \le \nu_2$) if $E^{\nu_1}[F] \le E^{\nu_2}[F]$ holds for all bounded non-decreasing functions F on $(\mathbb{R}_+)^{D_N}$.

Lemma 2.1. *We have the stochastic dominations:*

$$\mu_N^{0,0,\epsilon} \circ R^{-1} < \mu_N^{0,0} \circ R^{-1},$$
$$\mu_N^{0,F,\epsilon} \circ R^{-1} < \mu_N^{0,F} \circ R^{-1}.$$

Proof. Since $x = (x_i)_{i \in D_N}$ is a d-dimensional Bessel process viewed at integer times under $\mu_N^{0,F} \circ R^{-1}$, it enjoys the FKG inequality; see Proposition 5.6 of [5]. However, the probability measure $\mu_N^{0,F,\epsilon} \circ R^{-1}$ is given by the weak limit of a sequence of certain probability measures $\{\nu_k\}_{k \in \mathbb{N}}$ having non-increasing densities with respect to $\mu_N^{0,F} \circ R^{-1}$. Therefore, the FKG inequality for $\mu_N^{0,F} \circ R^{-1}$ implies the stochastic domination $\nu_k < \mu_N^{0,F} \circ R^{-1}$ for every k, and this concludes $\mu_N^{0,F,\epsilon} \circ R^{-1} < \mu_N^{0,F} \circ R^{-1}$ by taking the limit $k \to \infty$.

On the other hand, the density of $\mu_N^{0,0} \circ R^{-1}$ satisfies the so-called Holley's condition (since we may only consider the density of the joint distributions of the d-dimensional Bessel processes at integer times under the condition $x_N = 0$) and thus the FKG inequality holds for $\mu_N^{0,0} \circ R^{-1}$. The rest of the proof for $\mu_N^{0,0,\epsilon} \circ R^{-1} < \mu_N^{0,0} \circ R^{-1}$ is similar. □

Proof of Proposition 2.1 for $\mu_N^{0,0,\epsilon}$ and $\mu_N^{0,F,\epsilon}$. The proof for $\mu_N^{0,0,\epsilon}$ is immediate:

$$\mu_N^{0,0,\epsilon}(\|h^N\|_\infty \ge \delta) \le \mu_N^{0,0}(\|h^N\|_\infty \ge \delta) \le Ce^{-cN},$$

where we have applied Lemma 2.1 for the first inequality and then the standard large deviation principle for $\mu_N^{0,0}$, under which ϕ is a Brownian bridge viewed at integer times. The proof for $\mu_N^{0,F,\epsilon}$ is similar. □

§ 2.2. Proof based on a renewal theory

We give the proof only for $\mu_N^{0,0,\epsilon}$ and $\mu_N^{0,0,\epsilon,+}$, the case where the Markov chains are pinned at N, since the free case can be discussed in a parallel way.

2.2.1. The pinned case without a wall

For $\delta > 0$ and $\epsilon \ge 0$, set

$$p_n^\epsilon(\delta) = \frac{Z_n^{0,0,\epsilon}}{Z_n^{0,0}} \mu_n^{0,0,\epsilon}(\|h^n\|_\infty \ge \delta).$$

Then, by expanding the measure as in the proof of Lemma 2.1 (or (3.3)) of [1], we have

$$\begin{aligned} p_n^\epsilon(\delta) =& \mu_n^{0,0}(\|h^n\|_\infty \ge \delta) \\ &+ \sum_{i=1}^{n-1} \epsilon \frac{Z_i^{0,0} Z_{n-i}^{0,0,\epsilon}}{Z_n^{0,0}} \mu_i^{0,0}(\|h^n\|_{\infty,[0,i/n]} \ge \delta) \mu_{n-i}^{0,0,\epsilon}(\|h^n\|_{\infty,[i/n,1]} \ge \delta) \\ =& p_n^0(\delta) + \sum_{i=1}^{n-1} \epsilon Z_{i,n-i}^{0,0} p_i^0(n\delta/i) p_{n-i}^\epsilon(n\delta/(n-i)), \end{aligned}$$

where

$$Z_{i,n-i}^{0,0} \equiv \frac{Z_i^{0,0} Z_{n-i}^{0,0}}{Z_n^{0,0}} = \frac{(2\pi n)^{d/2}}{(2\pi i)^{d/2}(2\pi(n-i))^{d/2}}.$$

The last line for $p_n^\epsilon(\delta)$ and the last identity for $Z_{i,n-i}^{0,0}$ follow from $\{\|h^n\|_{\infty,[0,i/n]} \ge \delta\} = \{\|h^i\|_\infty \ge n\delta/i\}$, $\{\|h^n\|_{\infty,[i/n,1]} \ge \delta\} = \{\|h^{n-i}\|_\infty \ge n\delta/(n-i)\}$ and $Z_n^{0,0} = (2\pi)^{dn/2}/(2\pi n)^{d/2}$, respectively. Since $p_i^0(n\delta/i) \le p_i^0(\delta)$ and $p_{n-i}^\epsilon(n\delta/(n-i)) \le p_{n-i}^\epsilon(\delta)$, the above calculation leads to the inequality:

$$u_n \le b_n + \sum_{i=0}^{n} a_i u_{n-i}, \quad n \ge 0,$$

for three sequences $\{u_n \equiv u_n(x;\epsilon), a_n \equiv a_n(x;\epsilon), b_n \equiv b_n(x)\}_{n=0}^\infty$ defined by

$$\begin{aligned} u_n &= \frac{x^n}{(2\pi n)^{d/2}} p_n^\epsilon(\delta), \\ b_n &= \frac{x^n}{(2\pi n)^{d/2}} p_n^0(\delta) \ \big(= u_n(x;0)\big), \\ a_n &= \frac{\epsilon x^n}{(2\pi n)^{d/2}} p_n^0(\delta) \ \big(= \epsilon b_n\big), \end{aligned}$$

for $n \ge 1$ and $u_0 = a_0 = b_0 = 0$, respectively, where $x \ge 0$.

Compared with the solution $\{v_n \equiv v_n(x;\epsilon)\}_{n=0}^\infty$ of the renewal equation:

$$v_n = b_n + \sum_{i=0}^{n} a_i v_{n-i}, \quad n \ge 0, \tag{2.1}$$

it is immediate to see that $0 \le u_n \le v_n$ by induction, since the coefficients a_i are non-negative.

To find the asymptotic behavior of $v_n(x;\epsilon)$ as $n \to \infty$, define an increasing function $f(x)$ for $x \geq 0$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{(2\pi n)^{d/2}} \mu_n^{0,0}(\|h^n\|_\infty \geq \delta) \left(\equiv \sum_{n=1}^{\infty} b_n(x) \right).$$

Then we have the following lemma.

Lemma 2.2. (1) *Let x_* be the radius of convergence of f. Then $x_* > 1$.*
(2) *For $\epsilon > 1/f(x_*-) \big(\in [0,\infty)\big)$, we have*

$$\lim_{n\to\infty} v_n(\bar{x}^\epsilon;\epsilon) = \frac{1}{\epsilon^2 \bar{x}^\epsilon f'(\bar{x}^\epsilon)} \ (>0),$$

where $\bar{x}^\epsilon \in (0, x_*)$ is the unique solution of $f(x) = 1/\epsilon$.
(3) *If $f(x_*-) < \infty$ and $0 < \epsilon \leq 1/f(x_*-)$, the sequence $\{v_n(x_* - \delta;\epsilon) > 0\}_{n\in\mathbb{N}}$ is bounded for every $\delta \in (0, x_*)$.*

Proof. The assertion (1) follows from the fact that $\mu_n^{0,0}(\|h^n\|_\infty \geq \delta)$ decays exponentially fast in n. For (2), since $\sum_{n=0}^\infty a_n(\bar{x}^\epsilon;\epsilon) = 1$ under the choice of $x = \bar{x}^\epsilon$, the renewal theory (Chapter XIII of [2]) shows that

$$v_n \sim \frac{\sum_{n=0}^\infty b_n}{\sum_{n=0}^\infty n a_n} = \frac{1}{\epsilon^2 \bar{x}^\epsilon f'(\bar{x}^\epsilon)},$$

as $n \to \infty$, where $\sim$ means that the ratio of both sides tends to 1. The assertion (3) follows again by applying the renewal theory to $v_n(x_* - \delta;\epsilon)$ noting that $\sum_{n=0}^\infty a_n(x_* - \delta;\epsilon) = \epsilon f(x_* - \delta) < 1$. □

Let g be the function defined by

$$g(x) = \sum_{n=1}^{\infty} \frac{x^n}{(2\pi n)^{d/2}},$$

for $x \geq 0$. It is shown in [1] that the free energy ξ^ϵ is given by $\xi^\epsilon = -\log x^\epsilon$, where $x^\epsilon \in (0,1]$ is the unique solution of $g(x) = 1/\epsilon$ for $\epsilon > \epsilon_c := 1/g(1-)$ and $x^\epsilon = 1$ for $0 < \epsilon \leq \epsilon_c$. Since $0 < f(x) < g(x)$ for $x > 0$ (as long as $f(x) < \infty$), choosing $\bar{x}^\epsilon \in (0, x_*)$ as in Lemma 2.2 for $\epsilon > 1/f(x_*-)$ and $\bar{x}^\epsilon \in (1, x_*)$ arbitrarily for $\epsilon \leq 1/f(x_*-)$ (if $f(x_*-) < \infty$), we have

$$0 < x^\epsilon < \bar{x}^\epsilon \tag{2.2}$$

for all $\epsilon > 0$. Under the choice of $x = \bar{x}^\epsilon$, we have

$$\mu_n^{0,0,\epsilon}(\|h^n\|_\infty \geq \delta) = (2\pi n)^{d/2} (\bar{x}^\epsilon)^{-n} u_n(\bar{x}^\epsilon;\epsilon) \left(\frac{Z_n^{0,0,\epsilon}}{Z_n^{0,0}} \right)^{-1},$$

and therefore

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log\mu_n^{0,0,\epsilon}(\|h^n\|_\infty\geq\delta)\\ &\leq-\log\bar{x}^\epsilon+\limsup_{n\to\infty}\frac{1}{n}\log v_n(\bar{x}^\epsilon;\epsilon)-\xi^\epsilon\\ &\leq\log x^\epsilon-\log\bar{x}^\epsilon<0,\end{aligned}$$

from Lemma 2.2 and (2.2). This completes the proof of Proposition 2.1 for $\mu_N^{0,0,\epsilon}$.

2.2.2. The pinned case with a wall

For $\delta>0$ and $\epsilon\geq 0$, set

$$p_n^{\epsilon,+}(\delta)=\frac{Z_n^{0,0,\epsilon,+}}{Z_n^{0,0,+}}\mu_n^{0,0,\epsilon,+}(\|h^n\|_\infty\geq\delta).$$

Then, an identity similar to that for $p_n^\epsilon(\delta)$ holds for $p_n^{\epsilon,+}(\delta)$ by replacing $Z_{i,n-i}^{0,0}$ with

$$Z_{i,n-i}^{0,0,+}\equiv\frac{Z_i^{0,0,+}Z_{n-i}^{0,0,+}}{Z_n^{0,0,+}}.$$

Since $Z_n^{0,0,+}=Z_n^{0,0}/n$, this proves

$$u_n^+\leq b_n^++\sum_{i=0}^n a_i^+u_{n-i}^+,\quad n\geq 0,$$

for three sequences $\{u_n^+\equiv u_n^+(x;\epsilon),a_n^+\equiv a_n^+(x;\epsilon),b_n^+\equiv b_n^+(x)\}_{n=0}^\infty$ defined by

$$\begin{aligned}u_n^+&=\frac{x^n}{n(2\pi n)^{d/2}}p_n^{\epsilon,+}(\delta),\\ b_n^+&=u_n^+(x;0),\\ a_n^+&=\epsilon b_n^+,\end{aligned}$$

for $n\geq 1$ and $u_0^+=a_0^+=b_0^+=0$, respectively, where $x\geq 0$. We define two increasing functions $f^+(x)$ and $g^+(x)$ for $x\geq 0$ by

$$\begin{aligned}f^+(x)&=\sum_{n=1}^\infty\frac{x^n}{n(2\pi n)^{d/2}}\mu_n^{0,0,+}(\|h^n\|_\infty\geq\delta)\left(\equiv\sum_{n=1}^\infty b_n^+(x)\right),\\ g^+(x)&=\sum_{n=1}^\infty\frac{x^n}{n(2\pi n)^{d/2}}.\end{aligned}$$

It is shown in [1] that the free energy $\xi^{\epsilon,+}$ is given by $\xi^{\epsilon,+}=-\log x^{\epsilon,+}$, where $x^{\epsilon,+}\in(0,1]$ is the unique solution of $g^+(x)=1/\epsilon$ for $\epsilon>\epsilon_c^+:=1/g^+(1)$ and $x^{\epsilon,+}=1$ for

$0 < \epsilon \le \epsilon_c^+$. Since $0 < f^+(x) < g^+(x)$ for $x > 0$ (as long as $f^+(x) < \infty$), the rest of the proof of Proposition 2.1 for $\mu_N^{0,0,\epsilon,+}$ is similar to Section 2.2.1.

2.2.3. Further application of the renewal theory

As another application of the renewal theory, we give the proof of the following proposition. This proposition is shown in [1] as a direct consequence of the precise asymptotics for the ratios of partition functions, see Remark 2.1-(2) of [1]. In a sense, we repeat the argument there from a slightly different view point.

Proposition 2.2. *If $\epsilon > \epsilon_c^+$, we have that*

$$\mu_N^{0,0,\epsilon}(\phi_i \in \mathbb{R}_+^d \text{ for all } i \in D_N) \sim Ce^{-N(\xi^\epsilon - \xi^{\epsilon,+})},$$

as $N \to \infty$, where $C = x^\epsilon g'(x^\epsilon)/g(x^{\epsilon,+})$.

Proof. As in Section 2.2.1, we have

$$q_n^\epsilon = q_n^0 + \sum_{i=1}^{n-1} \epsilon Z_{i,n-i}^{0,0} q_i^0 q_{n-i}^\epsilon,$$

for

$$q_n^\epsilon = \frac{Z_n^{0,0,\epsilon}}{Z_n^{0,0}} \mu_n^{0,0,\epsilon}(\phi_i \in \mathbb{R}_+^d \text{ for all } i \in D_n).$$

Therefore, three sequences $\{v_n \equiv v_n(x;\epsilon), a_n \equiv a_n(x;\epsilon), b_n \equiv b_n(x)\}_{n=0}^\infty$ defined by

$$\begin{aligned} v_n &= \frac{x^n}{(2\pi n)^{d/2}} q_n^\epsilon, \\ b_n &= v_n(x;0), \\ a_n &= \epsilon b_n, \end{aligned}$$

with $v_0 = a_0 = b_0 = 0$ fulfill the renewal equation (2.1), where $x \ge 0$. Since

$$q_n^0 = \mu_n^{0,0}(\phi_i \in \mathbb{R}_+^d \text{ for all } i \in D_n) = \frac{1}{n},$$

we see that

$$\sum_{n=1}^\infty a_n = \epsilon \sum_{n=1}^\infty \frac{x^n}{n(2\pi n)^{d/2}} = \epsilon g^+(x) = 1$$

under the choice of $x = x^{\epsilon,+} = e^{-\xi^{\epsilon,+}}$. Thus, the renewal theory implies that

$$v_n \sim \frac{1}{\epsilon^2 x^{\epsilon,+} (g^+)'(x^{\epsilon,+})} = \frac{1}{\epsilon^2 g(x^{\epsilon,+})},$$

as $n \to \infty$. The conclusion follows from

$$\frac{Z_n^{0,0,\epsilon}}{Z_n^{0,0}} \sim \frac{(2\pi n)^{d/2}}{\epsilon^2 x^\epsilon g'(x^\epsilon)} e^{n\xi^\epsilon},$$

which is shown in Proposition 2.2 of [1]. □

§ 3. Pinning model in a random environment

Finally in this section, we propose a model of Gaussian random walks perturbed by an attractive force toward a subspace of $\mathbb{R}^d$, which is chosen randomly at every time i. The model is stated for the free case, but the pinned case can be discussed similarly. Related random polymer models are studied in Giacomin [6], for instance, see case A in Section 1.1 of [6].

Let $\boldsymbol{\epsilon} = (\epsilon^\alpha)_{\alpha=1}^d \in (0,\infty)^d$ be given; $\boldsymbol{\epsilon}$ represents the direction-dependent strength of the pinning. Set

$$\epsilon(\mathfrak{a}) = \prod_{\alpha \in \mathfrak{a}} \epsilon^\alpha \quad \text{and} \quad \rho_{\mathfrak{a}}(dx) = \prod_{\alpha \in \mathfrak{a}} \delta_0(dx^\alpha) \prod_{\beta \notin \mathfrak{a}} dx^\beta,$$

for $\mathfrak{a} \in \bar{\Omega}$ and $x = (x^\alpha)_{\alpha=1}^d \in \mathbb{R}^d$, where $\bar{\Omega} = \{\mathfrak{a}; \text{non-empty subsets of } \{1, \dots, d\}\}$. The measure $\rho_{\mathfrak{a}}$ on $\mathbb{R}^d$ is concentrated on its subspace:

$$M_{\mathfrak{a}} = \{x \in \mathbb{R}^d; x^\alpha = 0 \text{ for } \alpha \in \mathfrak{a}\}.$$

Let p be an arbitrary probability measure on $\bar{\Omega}$ such that $p(\mathfrak{a}) > 0$ for each $\mathfrak{a} \in \bar{\Omega}$. We denote $\Omega = \bar{\Omega}^{D_N}$.

Definition 3.1. We define a (quenched) pinning model in an environment $\omega = (\mathfrak{a}_1, \dots, \mathfrak{a}_N) \in \Omega$ by

$$\mu_N^{a,F,\omega}(d\phi) = \frac{1}{Z_N^{a,F,\omega}} e^{-H_N(\phi)} \delta_{aN}(d\phi_0) \prod_{i=1}^N \left(\frac{\epsilon(\mathfrak{a}_i)}{p(\mathfrak{a}_i)} \rho_{\mathfrak{a}_i}(d\phi_i) + d\phi_i \right),$$

where $a \in \mathbb{R}^d$ and $Z_N^{a,F,\omega}$ is the normalizing constant. □

Under $\mu_N^{a,F,\omega}$, the pinning occurs randomly in i to the subspace $M_{\mathfrak{a}_i}$ with the strength $\epsilon(\mathfrak{a}_i)/p(\mathfrak{a}_i)$. Denoting $a = (a^\alpha)_{\alpha=1}^d \in \mathbb{R}^d$, let us consider the distribution $\mu_N^{a^\alpha,F,\epsilon^\alpha}$ on $\mathbb{R}^{D_N}$ of the one-dimensional Markov chains defined by (1.1) for each $1 \le \alpha \le d$. We then define the distribution on $(\mathbb{R}^d)^{D_N}$ by the product

$$\mu_N^{a,F,\boldsymbol{\epsilon}}(d\phi) = \prod_{\alpha=1}^d \mu_N^{a^\alpha,F,\epsilon^\alpha}(d\phi^\alpha).$$

Proposition 3.1. *We have that*

$$\mu_N^{a,F,\boldsymbol{\epsilon}}(d\phi) = \mathbb{E}[Z^\omega \mu_N^{a,F,\omega}(d\phi)],$$

where $\mathbb{E}$ *stands for the expectation on the probability space* $(\Omega, p^{\otimes N})$, $Z^\omega = Z_N^{a,F,\omega}/Z_N^{a,F,\boldsymbol{\epsilon}}$ *and* $Z_N^{a,F,\boldsymbol{\epsilon}} = \prod_{\alpha=1}^d Z_N^{a^\alpha,F,\epsilon^\alpha}$.

Proof. Rewrite $\mu_N^{a,F,\boldsymbol{\epsilon}}$ as

$$\mu_N^{a,F,\boldsymbol{\epsilon}}(d\phi) = \frac{1}{Z_N^{a,F,\boldsymbol{\epsilon}}} e^{-H_N(\phi)} \delta_{aN}(d\phi_0) \prod_{i=1}^N \prod_{\alpha=1}^d \big(\epsilon^\alpha \delta_0(d\phi_i^\alpha) + d\phi_i^\alpha\big),$$

and expand the measure as

$$\begin{aligned}\prod_{\alpha=1}^d \big(\epsilon^\alpha \delta_0(dx^\alpha) + dx^\alpha\big) &= \sum_{\mathfrak{a}' \subset \{1,\dots,d\}} \epsilon(\mathfrak{a}')\rho_{\mathfrak{a}'}(dx) \\ &= \sum_{\mathfrak{a}\neq\emptyset} \epsilon(\mathfrak{a})\rho_{\mathfrak{a}}(dx) + dx \\ &= \sum_{\mathfrak{a}\in\bar\Omega} \big(\epsilon(\mathfrak{a})\rho_{\mathfrak{a}}(dx) + p(\mathfrak{a})dx\big),\end{aligned}$$

for $x = (x^\alpha)_{\alpha=1}^d \in \mathbb{R}^d$, where $\epsilon(\emptyset) = 1$ and $\rho_\emptyset(dx) = dx$. Thus, we have

$$\begin{aligned}\mu_N^{a,F,\boldsymbol{\epsilon}}(d\phi) =& \frac{1}{Z_N^{a,F,\boldsymbol{\epsilon}}} e^{-H_N(\phi)} \delta_{aN}(d\phi_0) \prod_{i=1}^N \sum_{\mathfrak{a}_i\in\bar\Omega} \big(\epsilon(\mathfrak{a}_i)\rho_{\mathfrak{a}_i}(d\phi_i) + p(\mathfrak{a}_i)d\phi_i\big) \\ =& \frac{1}{Z_N^{a,F,\boldsymbol{\epsilon}}} \sum_{\omega=(\mathfrak{a}_1,\dots,\mathfrak{a}_N)\in\Omega} \left(\prod_{i=1}^N p(\mathfrak{a}_i)\right) e^{-H_N(\phi)} \\ & \times \delta_{aN}(d\phi_0) \prod_{i=1}^N \left(\frac{\epsilon(\mathfrak{a}_i)}{p(\mathfrak{a}_i)}\rho_{\mathfrak{a}_i}(d\phi_i) + d\phi_i\right) \\ =& \mathbb{E}\left[\frac{Z_N^{a,F,\omega}}{Z_N^{a,F,\boldsymbol{\epsilon}}} \mu_N^{a,F,\omega}(d\phi)\right].\end{aligned}$$

This concludes the proof. □

An asymptotic behavior of h^N under the annealed measure $\mu_N^{a,F,\boldsymbol{\epsilon}}$ is not difficulty to study. In fact, since it is a product measure, the large deviation principle holds for $h^N \in \mathcal{C}$ under $\mu_N^{a,F,\boldsymbol{\epsilon}}$ and the corresponding unnormalized rate functional is given by

$$\Sigma(h) = \frac{1}{2}\int_D |\dot h(t)|^2\, dt - \sum_{\alpha=1}^d \xi^{\epsilon^\alpha} \big|\{t \in D; h^\alpha(t) = 0\}\big|.$$

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