RIMS Kôkyûroku Bessatsu **B6**
(2008), 85–95

Heat kernel estimates on the incipient infinite cluster for critical branching processes

By

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Abstract

We obtain heat kernel estimates for the simple random walk on the family tree of the critical branching process with finite variance, conditioned on non-extinction. We show that the spectral dimension of the random walk is 4/3.

§ 1. Introduction

There has been a lot of work by mathematical physicists on the behaviour of random walk on percolation clusters (see [6] and the references therein). Through numerical computations, it was observed that random walk on a supercritical percolation cluster on $\mathbb{Z}^d$ behaves in a diffusive fashion whereas at the criticality, it behaves anomalously.

On the other hand, mathematically rigorous results appear quite recently, even in the supercritical case, for the quenched estimates (i.e. almost sure estimates with respect to the randomness of the media). In [3], Barlow obtained both sides Gaussian-type quenched heat kernel estimates. Using these estimates, the quenched invariance principle was established ([7, 11, 12]).

Critical percolation clusters are believed to be finite in all dimensions, and it is rigorously proved when $d = 2$ or $d \geq 19$. To avoid finite-size issues associated with random walk on a finite cluster, it is convenient to consider random walk on the incipient infinite cluster (IIC), which can be understood as a critical percolation cluster conditioned to be infinite. Note that the existence of the IIC is in general a highly non-trivial problem; so far it has been constructed only when $d = 2$, and $d > 6$ in the spread-out case. (For trees, it is not difficult to construct the IIC, as mentioned below.)

On the IICs for $\mathbb{Z}^2$ and trees, Kesten [9, 10] proved that the random walk is subdiffusive, i.e., spread more slowly than the random walk on the Euclidean lattice. In

2000 Mathematics Subject Classification(s): Primary 60K37; Secondary 60J80, 60J35.
Research partially supported by the Grant-in-Aid for Scientific Research (B) 18340027.
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[4], heat kernel estimates for the random walk are given on the spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ for $d > 6$. As a consequence, it is proved that the random walk is subdiffusive. In [5], detailed sub-Gaussian heat kernel estimates are established for the simple random walk on the IICs of family trees of critical branching processes whose offspring distributions are binomial.

In this note, we estimate the heat kernel of the simple random walk on the IICs for more general family trees. We only assume that the offspring distribution has finite variance, so it does not need to be bounded.

In the following of this section, we give some notation, explain the framework we work on, and state our main results.

Let $\Gamma = (G, E)$ be an infinite graph, with the vertex set G and the edge set E. We assume that Γ is connected. We write $x \sim y$ if $\{x, y\} \in E$, and assume that (G, E) is locally finite, i.e. $\mu_y < \infty$ for each $y \in G$, where μ_y is the number of bonds that contain y. For $A \subset G$, set $\mu(A) = \sum_{x \in A} \mu_x$. Let $d(x, y)$ be the length of the shortest path connecting x and y, and denote

$$B(x, r) := \{y \in G : d(x, y) \leq r\}, \quad V(x, r) := \mu(B(x, r)).$$

Let $X = (X_n, n \in \mathbb{Z}_+, P^x, x \in G)$ be the discrete-time simple random walk on Γ. Then X has transition probabilities

$$P^x(X_1 = y) = \frac{1}{\mu_x}, \quad y \sim x.$$

We define the discrete-time heat kernel (or the transition density) of X by

$$p_n(x, y) = P^x(X_n = y)\frac{1}{\mu_y};$$

we have $p_n(x, y) = p_n(y, x)$. Let $0 \in G$ be fixed. For each $R \geq 0$, let

$$\tau_R = \min\{n \geq 0 : d(0, X_n) \geq R\}.$$

The *spectral dimension of* G, denoted $d_s(G)$, is defined by

$$d_s(G) = -2 \lim_{n \to \infty} \frac{\log p_{2n}(x, x)}{\log n}, \tag{1.1}$$

if the limit exists. Here $x \in G$; it is easy to see that the limit is independent of the choice of x. Note that $d_s(\mathbb{Z}^d) = d$.

Next, let $\{\mathcal{G}(\omega) : \omega \in \Omega\}$ be the realization of the IIC, where ω expresses the randomness of the media. For each $\omega \in \Omega$ we can define the simple random walk $X = (X_n, n \in \mathbb{Z}_+, P_\omega^x, x \in \mathcal{G}(\omega))$. Let $p_n^\omega(x, y)$ be the discrete-time heat kernel of X.

Alexander-Orbach [1] conjectured that, if $\mathcal{G}(\omega)$ is the IIC for the critical percolation on $\mathbb{Z}^d$, $d \geq 2$, then $d_s(\mathcal{G}(\omega)) = 4/3$. It is now thought that this is unlikely to be true for small d. This conjecture is true for the simple random walk on the IIC of the critical branching process with binomial offspring distributions ([5]), and for the random walk on the IIC of the spread-out oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ for $d > 6$ ([4]). We will show that the conjecture is true for the IIC for the critical branching process whose offspring distribution has finite variance.

We now introduce the random family tree and give the assertion completely. This graph is a family tree with randomness of a number of children for each vertex. We assume that $\{p_j\}_{j\geq 0}$ is a non-negative sequence with $\sum_{j=0}^{\infty} p_j = 1$.

First, put a vertex which we call a root. This is said to be in the zeroth generation. We denote it by 0. The root gives birth to j_0 vertices (children) with probability p_{j_0}. We denote them by $(0, l_1)$, $1 \leq l_1 \leq j_0$. They are said to be in the first generation. Second, each vertex $(0, l_1)$ in the first generation gives birth to $j_{(0,l_1)}$ vertices (children) with probability $p_{j_{(0,l_1)}}$, which we denote by $(0, l_1, l_2)$, $1 \leq l_2 \leq j_{(0,l_1)}$. They are said to be in the second generation. In general, each vertex $(0, l_1, l_2, \ldots, l_n)$ in the n-th generation gives birth to $j_{(0,l_1,l_2,\ldots,l_n)}$ vertices (children) with probability $p_{j_{(0,l_1,l_2,\ldots,l_n)}}$ independently, which we denote by $(0, l_1, l_2, \ldots, l_n, l_{n+1})$, $1 \leq l_{n+1} \leq j_{(0,l_1,\ldots,l_n)}$. These children are said to be in the $(n+1)$-th generation. The number of children for each parent in each generation obeys the law $\{p_j\}$ and is independent of each other. Finally, we connect the parent and their children with edges. We denote this random graph by $\mathcal{G}'$ and the law by P.

Let $\{Z_n\}_{n\geq 0}$ be random variables representing a number of vertices in the n-th generation. $\{Z_n\}_{n\geq 0}$ is called a Bienaymé-Galton-Watson branching process, and the law $\{p_j\}$ is called an offspring distribution. In particular, $P[Z_1 = j] = p_j$. When $E[Z_1] < 1$, the number of vertices of $\mathcal{G}'$ is finite. When $E[Z_1] > 1$, $\mathcal{G}'$ is an infinite graph with positive probability. The case where $E[Z_1] = 1$ is critical. Here we treat the critical case, i.e., we assume $E[Z_1] = \sum_{j=0}^{\infty} jp_j = 1$.

In this case $\mathcal{G}'$ is a finite graph P-a.s. So we modify $\mathcal{G}'$ to have infinite vertices.

Let A be a family tree. We denote the subgraph of A restricted to (resp. up to) the n-th generation by A_n (resp. $A_{\leq n}$). We have

Lemma 1.1. ([10, Lemma1.14]) *Let A be a tree up to the k-th generation. Then*

$$\lim_{n\to\infty} P[\mathcal{G}'_{\leq k} = A | Z_n \neq 0] = |A_k| P[\mathcal{G}'_{\leq k} = A]$$

and writing $\mathbb{P}_0[A] = |A_k| P[\mathcal{G}'_{\leq k} = A]$, $\mathbb{P}_0$ has a unique extension to a probability measure $\mathbb{P}$ on the set of infinite family trees.

Let $\mathcal{G}$ be a family tree chosen with the distribution $\mathbb{P}$: we call this the incipient infinite cluster (IIC).

We assume

$$\sigma^2 := Var[Z_1] = E[(Z_1 - 1)^2] = \sum_{j=0}^{\infty} (j-1)^2 p_j < \infty.$$

In order to obtain the assertions, the following estimates for the volume growth and the effective resistance are essential.

Proposition 1.1. (1) *There exist $q_0, c_1 > 0$ such that for each $R > 1$,*

$$\mathbb{P}[R_{\text{eff}}(0, B(0,R)^c) \geq \lambda^{-1} R] \geq 1 - \frac{c_1}{\lambda^{q_0}}. \tag{1.2}$$

(2) $\mathbb{E}[V(0,R)] \leq c_2 R^2$.
(3) $\mathbb{E}[1/V(0,R)] \leq c_3 R^{-2}$.

The definition of $R_{\text{eff}}(A, B)$ in (1.2) will be given in (3.3).

We prove this proposition in the following sections.

Once this proposition is proved, then using Proposition 1.6 and Theorem 1.7 in [4], we can obtain the following results.

Theorem 1.1. (1) *There exist $\alpha_1, \alpha_2 < \infty$, and a subset Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that the following statements hold.*
(a) *For each $\omega \in \Omega_0$ and $x \in \mathcal{G}(\omega)$ there exists $N_x(\omega) < \infty$ such that*

$$(\log n)^{-\alpha_1} n^{-2/3} \leq p^{\omega}_{2n}(x,x) \leq (\log n)^{\alpha_1} n^{-2/3}, \quad n \geq N_x(\omega).$$

In particular, $d_s(\mathcal{G}) = \frac{4}{3}$, $\mathbb{P}$-a.s., and the random walk is recurrent.
(b) *For each $\omega \in \Omega_0$ and $x \in \mathcal{G}(\omega)$ there exists $R_x(\omega) < \infty$ such that*

$$(\log R)^{-\alpha_2} R^3 \leq E^x_\omega \tau_R \leq (\log R)^{\alpha_2} R^3, \quad R \geq R_x(\omega).$$

Hence

$$\lim_{R\to\infty} \frac{\log E^x_\omega \tau_R}{\log R} = 3.$$

(2) *The following estimates hold:*

$$\begin{aligned} c_1 R^3 \leq \mathbb{E}(E^0_\omega \tau_R) \leq c_2 R^3 \ \textit{for all } R \geq 1, \\ c_3 n^{-2/3} \leq \mathbb{E}(p^\omega_{2n}(0,0)) \leq c_4 n^{-2/3} \ \textit{for all } n \geq 1, \\ c_5 n^{1/3} \leq \mathbb{E}(E^0_\omega d(0, X_n)) \ \textit{for all } n \geq 1. \end{aligned}$$

(1) gives quenched estimates, i.e., estimates for a.e. $\omega \in \Omega$, whereas (2) is the annealed estimates (i.e., taking the average over the randomness of the media, which is denoted by $\mathbb{E}$).

Remark. 1) We can not take α_1 to be 0 in general (see [5, Lemma5.1]).
2) We can deduce more estimates for the heat kernel etc. from Proposition 1.1. See Proposition 1.5 and Theorem 1.7, 1.8 in [4] (or Proposition 1.2.3 and Theorem 1.2.4 in [8]).
3) In [5], further off-diagonal heat kernel estimates are obtained, whereas we only obtain on-diagonal estimates. This is because we do not know how to maintain good uniform control of the laws $\mathbb{P}_x$ in our setting, where $\mathbb{P}_x$ is the law of the IIC conditioned that the vertex x is in the IIC.

Example 1. We have the Poisson distribution with parameter 1

$$p_j = \frac{1}{j!}e^{-1}, \quad j \in \mathbb{Z}_+$$

as an example of the offspring distribution. In this case, the expectation and the variance are 1, so it has finite variance. But it is not bounded, therefore is not treated in [5].

In Section 2, we will give some estimates for the branching process and in Section 3, we will prove Proposition 1.1.

This note is based on the Master Thesis by the first named author ([8]).

§ 2. Bienaymé-Galton-Watson branching process

As we mentioned above, we assume

$$\sigma^2 := Var[Z_1] = E[(Z_1 - 1)^2] < \infty.$$

We will use this assumption for the proof essentially.

In the following of this section, we estimate the volume of $\mathcal{G}'_{\leq n}$.

Let f be the generating function of the offspring distribution, so that

$$f(s) = E[s^{Z_1}] = \sum_{k=0}^{\infty} p_k s^k.$$

From [2, p.19 (2)] we have

$$P[Z_n > 0] \sim \frac{2}{n\sigma^2}. \tag{2.1}$$

Let

$$Y_n = \sum_{k=0}^{n} Z_k, \quad g_n(s) = E[s^{Y_n}], \quad f_n(s) = E[s^{Z_n}].$$

Then conditioning on Z_1 we obtain that $f_{n+1}(s) = f(f_n(s))$, $g_{n+1}(s) = sf(g_n(s))$. And

(2.2) $\qquad f_n(1-) = 1,\ f_n'(1-) = E[Z_n] = 1,\ f_n''(1-) = E[Z_n(Z_n-1)] < \infty,$

(2.3) $\qquad g_n(1-) = 1,\ g_n'(1-) = E[Y_n] = n+1.$

Lemma 2.1. *There exists $c > 0$ such that for any $n > 0$*

$$E[Y_n^2] \le cn^3.$$

Proof. Let $Z_i^{(l)}, l = 1, 2, \cdots$ be independent copies of Z_i. Then,

$$\begin{aligned} Var[Z_{n+1}] = E[(Z_{n+1}-1)^2] &= \sum_{y=0}^{\infty} E[(Z_{n+1}-1)^2 | Z_n = y] P[Z_n = y] \\ &= \sum_{y=0}^{\infty} E[(\sum_{l=1}^{y} Z_1^{(l)} - 1)^2] P[Z_n = y] \\ &= \sum_{y=0}^{\infty} E[\{\sum_{l=1}^{y} (Z_1^{(l)} - 1)\}^2 + (y-1)^2] P[Z_n = y] \\ &= \sum_{y=0}^{\infty} \{y\sigma^2 + (y-1)^2\} P[Z_n = y] = \sigma^2 E[Z_n] + E[(Z_n - 1)^2] \\ &= \sigma^2 + Var[Z_n] = (n+1)\sigma^2. \end{aligned}$$

Let $i < j$, then

$$E[Z_i Z_j] = E[E[Z_i Z_j | Z_i]] = \sum_{y=0}^{\infty} E[\sum_{l=0}^{y} y Z_{j-i}^{(l)}] P[Z_i = y] = E[Z_i^2],$$

since $E[\sum_{l=0}^{y} y Z_{j-i}^{(l)}] = y^2$. Using the two equalities above, we have

$$\begin{aligned} E[Y_n^2] = E[(\sum_{i=0}^{n} Z_i)^2] &= \sum_{i=0}^{n} E[Z_i^2] + 2\sum_{i<j} E[Z_i Z_j] = \sum_{i=0}^{n} E[Z_i^2] + 2\sum_{i<j} E[Z_i^2] \\ &= \sum_{i=0}^{n} (i\sigma^2 + 1) + 2\sum_{i=0}^{n} (n-i)(i\sigma^2 + 1) \le cn^3. \end{aligned}$$

□

The next lemma is Lemma 2.3(a) in [5]. Since the proof is the same, we omit it.

Lemma 2.2. *There exist $c_0 > 0, p_0 > 0$ such that*

$$P[Y_n > c_0 n^2] \ge \frac{p_0}{n}.$$

We will need to consider the following modified branching process. Let $\tilde{Z} = (\tilde{Z}_n, n \geq 0)$ be a branching process with $\tilde{Z}_0 = 1$ and the same offspring distribution as Z, except that at the first generation we have

$$P[\tilde{Z}_1 = j] = (j+1)p_{j+1}.$$

For the generating function of $\tilde{Z}_1$, we have

$$E[s^{\tilde{Z}_1}] = \sum_{k=0}^{\infty}(k+1)p_{k+1}s^k = f'(s),$$

and $\sum_{k=0}^{\infty}(k+1)p_{k+1} = f'(1-) = 1$ by (2.2). So $\{(j+1)p_{j+1}\}$ is a probability.

The generating function of $\tilde{Z}_n$ is expressed by f as

$$\begin{aligned} E[s^{\tilde{Z}_n}] &= E[E[s^{\tilde{Z}_n}|\tilde{Z}_1]] = \sum_{y=0}^{\infty} E[s^{\sum_{l=1}^{y} Z_{n-1}^{(l)}}]P[\tilde{Z}_1 = y] \\ &= E[E[s^{Z_{n-1}}]^{\tilde{Z}_1}] = E[f_{n-1}(s)^{\tilde{Z}_1}] = f'(f_{n-1}(s)), \end{aligned}$$

where $\{Z_{n-1}^{(l)}\}_l$ are independent copies of Z_{n-1}. The expectation of $\tilde{Z}_n$ is finite since we assume the offspring distribution has finite variance:

$$E[\tilde{Z}_n] = \frac{d}{ds}f'(f_{n-1}(s))|_{s=1-} = f''(f_{n-1}(1-))f'_{n-1}(1-) = f''(1-) < \infty. \tag{2.4}$$

Let ζ be a random variable. We write $\zeta[n]$ for a random variable with the distribution of $\sum_{i=1}^{n}\zeta_i$, where ζ_i are i.i.d. with $\zeta_i \overset{(\mathrm{d})}{=} \zeta$. Using (2.4) and the Chebyshev inequality, we can easily obtain the following lemma.

Lemma 2.3. *There exists $c > 0$ such that for any $n, \lambda > 0$*

$$P[\tilde{Z}_n[n] \geq \lambda n] \leq \frac{c}{\lambda}.$$

Let $\tilde{Y}_n = \sum_{k=0}^{n}\tilde{Z}_k$. We then have the following, which corresponds to Lemma 2.3(b), 2.5(b) in [5]. Again, we omit the proof since it is the same as that of [5], given the estimates above.

Lemma 2.4. *There exists $c_1 > 0$ such that for any λ, n with $0 < \lambda < \frac{c_0}{4}, n > \sqrt{\frac{c_0}{4\lambda}}$,*

$$P[\tilde{Y}_n[n] < \lambda n^2] \leq \mathrm{e}^{-\frac{c_1}{\sqrt{\lambda}}},$$

where c_0 is the constant in Lemma 2.2.

§ 3. Proof of Proposition 1.1

In this section, we estimate the graph $\mathcal{G}(\omega)$ and give the proof of Proposition 1.1.

We remark that $\mathbb{P}$-a.s. $\mathcal{G}$ has exactly one infinite descending path from 0, which we call the backbone, and denote by B. We denote the vertex on the backbone in the n-th generation by b_n.

By [10, Corollary 2.13], we see that a parent on the backbone has l children, excluding the one on the backbone, with probability $P[\tilde{Z}_1 = l] = (l+1)p_{l+1}$. On the other hand, we can easily see that a parent off the backbone has l children with probability $P[Z_1 = l] = p_l$.

For each $x, y \in \mathcal{G}$, let $\gamma(x,y)$ be the unique geodesic path connecting x and y. We write $D(x)$ for the set of descendants of x. Note that $x \in D(x)$. We set

$$D(x;z) := \{y \in D(x) : \gamma(x,y) \cap \gamma(x,z) = \{x\}\}.$$

We also set

$$D_r(x;z) := \{y \in D(x;z) : d(x,y) = r\}, \quad D_{\leq r}(x;z) := \cup_{i=0}^r D_i(x;z).$$

We estimate the volume of balls with a center at the origin.

If the root has no bond, we define $\mu_0 = 1$ for convenience. Note that as $\mathcal{G}$ is a tree, we have

$$|B(x,r)| \leq V(x,r) \leq 2|B(x,r+1)|. \tag{3.1}$$

Lemma 3.1. *There exists $c > 0$ such that for any $\lambda > 0$,*

$$\mathbb{E}[V(0,n)] \leq cn^2.$$

Proof. By (3.1), it is enough to bound $|B(0,n)|$. Recall that b_n is the vertex on the backbone in the n-th generation.

$$\begin{aligned}
h(s) :&= \mathbb{E}[s^{|B(0,n)|}] = \mathbb{E}[s^{|\cup_{k=0}^n D_{\leq k}(b_{n-k};b_{n-k-1})|}] = E[s^{\sum_{k=0}^n \tilde{Y}_k^{(k)}}] = \Pi_{k=0}^n E[s^{\tilde{Y}_k}] \\
&= s\Pi_{k=1}^n E[E[s^{\tilde{Y}_k}|\tilde{Z}_1]] = s\Pi_{k=1}^n \sum_{y=1}^\infty E[s^{1+\sum_{l=1}^y Y_{k-1}^{(l)}}]P[\tilde{Z}_1 = y] \\
&= s\Pi_{k=1}^n \sum_{y=1}^\infty sE[s^{Y_{k-1}}]^y P[\tilde{Z}_1 = y] = s\Pi_{k=1}^n (sE[g_{k-1}(s)^{\tilde{Z}_1}]) \\
&= s^{n+1}\Pi_{k=1}^n f'(g_{k-1}(s)) = s^{n+1}\Pi_{k=0}^{n-1} f'(g_k(s)),
\end{aligned}$$

where $\{Y_\cdot^{(k)}\}_k$ are independent copies of $Y_\cdot$. Using this, we have

$$h'(s) = (n+1)s^n \Pi_{k=0}^{n-1} f'(g_k(s)) + s^{n+1} \sum_{l=0}^{n-1} f''(g_l(s))g_l'(s)\Pi_{m=0,m\neq l}^{n-1} f'(g_m(s)).$$

So using (2.2), (2.3),

$$\mathbb{E}[|B(0,n)|] = h'(1-) = n+1+\sum_{l=0}^{n-1} f''(1-)(l+1) \leq cn^2. \tag{3.2}$$

□

Using Lemma 2.4, we can prove the following in the same way as Proposition 2.7 of [5].

Proposition 3.1. *There exists* $c_1 > 0$ *such that for any* λ, n *with* $0 < \lambda < \frac{c_0}{4}, n > 3\sqrt{\frac{c_0}{4\lambda}}$,

$$\mathbb{P}[V(0,r) < \lambda r^2] \leq \mathrm{e}^{-\frac{c_1}{\sqrt{\lambda}}},$$

where c_0 *is the constant in Lemma 2.2.*

Proposition 3.2. *There exists* $c > 0$ *such that for any* $r > 0$,

$$\mathbb{E}[\frac{1}{V(0,r)}] \leq \frac{c}{r^2}.$$

Proof. First, note that $V(0,r) \geq 1$. From Proposition 3.1, we have

$$\mathbb{P}[\frac{1}{V(0,r)} > \frac{\lambda}{r^2}] \leq \mathrm{e}^{-c_1\sqrt{\lambda}}$$

for $4/c_0 \leq \lambda \leq 4n^2/9c_0$. So

$$\begin{aligned}
&\mathbb{E}[\frac{1}{V(0,n)}] \\
\leq &\frac{4\cdot 3^2}{9c_0n^2}\mathbb{P}[\frac{1}{V(0,n)} \leq \frac{4\cdot 3^2}{9c_0n^2}] \\
&+\sum_{k=3}^{n-1}\frac{4\cdot(k+1)^2}{9c_0n^2}\mathbb{P}[\frac{4\cdot k^2}{9c_0n^2} < \frac{1}{V(0,n)} \leq \frac{4\cdot(k+1)^2}{9c_0n^2}] + 1\cdot\mathbb{P}[\frac{4\cdot n^2}{9c_0n^2} < \frac{1}{V(0,n)}] \\
\leq &\frac{4\cdot 3^2}{9c_0n^2} + \sum_{k=3}^{n-1}\frac{4\cdot(k+1)^2}{9c_0n^2}\mathbb{P}[\frac{4\cdot k^2}{9c_0n^2} < \frac{1}{V(0,n)}] + \mathbb{P}[\frac{4\cdot n^2}{9c_0n^2} < \frac{1}{V(0,n)}] \\
\leq &\frac{4\cdot 3^2}{9c_0n^2} + \sum_{k=3}^{n-1}\frac{4\cdot(k+1)^2}{9c_0n^2}\mathrm{e}^{-ck} + \mathrm{e}^{-cn} \leq \frac{c}{n^2}.
\end{aligned}$$

□

In the following of this section, we estimate the effective resistance for $\mathcal{G}$. For the purpose, we first estimate the connectivity in the ball.

Definition 3.1. Let $x \in \mathcal{G}$, $r \geq 1$. Let $M(x,r)$ be the smallest number m such that there exists $A = \{z_1, \dots, z_m\} \subset \mathcal{G}$ with $d(x, z_i) \in [r/4, 3r/4]$ for each i, such that any path γ from x to $B(x,r)^c$ must pass through the set A.

Since $\mathcal{G}$ is a tree, the best choice of such a set A will in fact be the points at a distance $\lceil r/4 \rceil$ from x.

Using the previous estimates such as (2.1) and Lemma 2.3, we can prove the following similarly (in fact, more easily) to the proof of [5, Proposition 2.10].

Proposition 3.3. *There exists $c > 0$ such that for each $m \geq 2, r \geq 4$,*

$$\mathbb{P}[M(0,r) \geq m] \leq \frac{c}{m}.$$

Finally, we estimate the effective resistance. To define the effective resistance, we define a quadratic form $\mathcal{E}$ by

$$\mathcal{E}(f,f) = \frac{1}{2} \sum_{x,y \in \mathcal{G}, x \sim y} (f(x) - f(y))^2.$$

Let A, B be disjoint subsets of $\mathcal{G}$. The effective resistance between A and B is defined by

$$R_{\mathrm{eff}}(A,B)^{-1} := \inf\{\mathcal{E}(f,f) : f|_A = 1, f|_B = 0\}. \tag{3.3}$$

Proposition 3.4. *There exists $c > 0$ such that for any $r \geq 4, \lambda \leq 1/4$,*

$$\mathbb{P}[R_{\mathrm{eff}}(0, B(0,r)^c) < \lambda r] \leq c\lambda.$$

Proof. Let

$$A = \bigcup_{z \in \gamma(0, b_{\lceil r/4 \rceil}) \setminus \{b_{\lceil r/4 \rceil}\}} D_{\lceil r/4 \rceil}(z, b_{\lceil r/4 \rceil}), \quad A^* = \{z \in A : D_{\lceil r/4 \rceil}(z) \neq \emptyset\}.$$

Then any path from 0 to $B(0,r)^c$ must pass through $A^* \cup \{b_{\lceil r/4 \rceil}\}$, so $M(0,r) \leq |A^*| + 1$. Let A^{**} be the set of ancestors at level $\lceil r/4 \rceil$ of $A^* \cup \{b_{\lceil r/4 \rceil}\}$, and we define a function f on $\mathcal{G}$ as follows: if $z \in \gamma(0,x)$ for some $x \in A^{**}$, $f(z) = |z|/\lceil r/4 \rceil$, otherwise $f(z) = f(a(z,1))$, where $|z|$ is the level of the vertex and $a(z,1)$ is the parent of z. Since $\mathcal{G}$ is a tree, we see that $M(0,r) = |A^{**}|$. Then

$$\frac{1}{R_{\mathrm{eff}}(0, B(0,r)^c)} \leq \mathcal{E}(f,f) \leq \frac{1}{2}(\frac{1}{\lceil r/4 \rceil})^2 \cdot \lceil r/4 \rceil \cdot M(0,r) = \frac{2M(0,r)}{r},$$

so $R_{\mathrm{eff}}(0, B(0,r)^c) \geq \frac{r}{2M(0,r)}$. Using Proposition 3.3, we deduce that

$$\mathbb{P}[R_{\mathrm{eff}}(0, B(0,r)^c) < \lambda r] \leq \mathbb{P}[\frac{r}{2M(0,r)} < \lambda r] \leq \mathbb{P}[M(0,r) > \frac{1}{2\lambda}] \leq c\lambda.$$

□

By Lemma 3.1, Proposition 3.2 and Proposition 3.4, the proof of Proposition 1.1 is completed.

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