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Prüfer angle methods in spectral analysis of Krein–Feller–operators

By

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Abstract

Generalized second order differential operators of the form $\frac{d}{d\mu}\frac{d}{d\nu}$ are considered. They act on the space $L_2(K,\mu)$, where K is the (compact) support of ν and μ is an atomless measure which is in general singular with respect to the Lebesgue measure. In the particular case that μ is self-similar, one obtains Weyl asymptotics of the eigenvalues which can be refined by applying renewal theory. In some special cases, the method of Prüfer angles leads to exact renormalization formulas for the Neumann eigenvalues, allowing a better study of the spectral asymptotics in the lattice case.

§ 1. Introduction

In [3], a measure geometric Laplacian $\Delta^{\mu,\nu} = \frac{d}{d\mu}\frac{d}{d\nu}$ is introduced as the second derivative with respect to two atomless finite Borel measures ν and μ with compact supports supp $\mu \subseteq$ supp $\nu \subseteq \mathbb{R}$. These operators allow interpretations from two different points of view: On the one hand side, $\Delta^{\mu,\nu}$ is a generalization of the second order differential operator $\frac{d}{d\mu}\frac{d}{d\mu}$ given by the second (weak) derivative with respect to a measure μ as considered in [7]. This operator has an interpretation as Laplacian on certain compact (maybe fractal) subsets of the real line. So this model is one of the possibilities to complete the theory of analysis on fractals which was developed for higher dimensions by several approaches (we refer the interested reader to Kigami's monograph [12] and the references therein). On the other hand, $\Delta^{\mu,\nu}$ generalizes the notion of well–known Sturm–Liouville– (or, Krein–Feller–) operator of the form $\frac{d}{d\mu}\frac{d}{dx}$ introduced for example in [10]. In the case that μ is a Cantor type measure, spectral asymptotics of $\frac{d}{d\mu}\frac{d}{dx}$ are presented in [8], which is a special case of the results obtained

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in [4] dealing with $\Delta^{\mu,\nu} = \frac{d}{d\mu}\frac{d}{d\nu}$. This operator is the infinitesimal generator of a so called quasi– (or, gap–) diffusion. The theory of Dirichlet forms shows that also the more general operator $\Delta^{\mu,\nu}$ is the infinitesimal generator of a strong Markov process with almost sure continuous paths on supp μ (see [5]). Note that eigenvalues of the operator $\frac{d}{d\mu}\frac{d}{dx}$ have an interpretation as eigenfrequencies of a vibrating string with (singular) mass distribution μ (see [11]).

In the self–similar case, the eigenvalue counting function $N^{\mu,\nu}_{D/N}(x)$ – under Dirichlet, or Neumann boundary conditions – behaves asymptotically like x^{γ} where the spectral exponent γ is expressed in terms of μ and ν (see [4]). In [6], using renewal theory, a sufficient condition for the convergence of the term $N^{\mu,\nu}_{D/N}(x) \cdot x^{-\gamma}$, as $x \to \infty$, is given (see Theorem 2.1). If the self–similar measures μ and ν have a "large number of symmetries in common" (the so–called "lattice case") one can expect asymptotic oscillation of the term $N^{\mu,\nu}_{D/N}(x) \cdot x^{-\gamma}$. Very roughly, this phenomenon can be explained by a huge number of localized eigenfunction to the same eigenvalue (due to the interplay of self–similarity and symmetry) creating high jumps in the eigenvalue counting function. For the convenience of the reader, we will recall these results in Section 2.

Unfortunately, Theorem 2.1 provides only a necessary – but not sufficient – condition for such oscillations. This fact is illustrated at the beginning of Section 3 with the help of an example where we have convergence of $N^{\mu,\nu}_{D/N}(x) \cdot x^{-\gamma}$ in the lattice case as $x \to \infty$. In Section 3, we will provide an instrument which allows – in some special cases – to decide if convergence of the term $N^{\mu,\nu}_{D/N}(x) \cdot x^{-\gamma}$ in the lattice case occurs or not. This criterion will be provided in terms of an exact renormalization property of the Neumann eigenvalues (see Theorem 3.1). The proof of Theorem 3.1 uses the method of Prüfer angles (see Subsection 3.2), introduced for Atkinson eigenvalue problems (see [1] as a standard reference). Moreover, we will use a trick introduced by Volkmer in [14] which allows us to transform the eigenvalue problem associated with $-\Delta^{\mu,\nu}$ into an Atkinson eigenvalue problem (see Subsection 3.3). Note that the main result Theorem 3.1 (stated in Subsection 3.4) has been already obtained in [14] for the special case of the middle third Cantor set. Our results hold for a much wider class of fractal sets and fractals measures, but the idea of the proof remains the same. In Subsection 3.5, we finally discuss same examples and open problems.

§ 2. Preliminaries

§ 2.1. Definition and analytic properties of the operator

In this section, we firstly recall the definition and the main analytic properties of the operator $\frac{d}{d\mu}\frac{d}{d\nu}$. For details and proofs we refer to [3].

We are given two Borel probability measures μ and ν with compact supports $L :=$ supp μ and $K :=$ supp ν with $L \subseteq K \subseteq [a, b] \subset \mathbb{R}$. Without loss of generality we may assume that $a, b \in L$. The measure geometric Laplacian $\Delta^{\mu,\nu}$ is introduced as the second (week) derivative $\frac{d}{d\mu}\frac{d}{d\nu}$ with respect to the measures ν and μ as follows.
We introduce the linear space

$$\mathcal{D}_1^{\nu} := \{f : K \longrightarrow \mathbb{R} : \exists f' \in L_2(K, \nu) \ : \ f(x) = f(a) + \int_a^x f'(y) d\nu(y), x \in K\}.$$

As the above function f' is unique in $L_2(K, \nu)$, the (first) *ν–derivative of f*

$$\nabla^{\nu} f = \frac{df}{d\nu} := f', \qquad f \in \mathcal{D}_1^{\nu},$$

is well–defined. Iterating this procedure, the *$\mu - \nu$–Laplacian* is introduced on the subspace

$$\mathcal{D}_2^{\mu,\nu} := \{f \in \mathcal{D}_1^{\nu} : \exists f'' \in L_2(L, \mu) \ : \ \nabla^{\nu} f(x) = \nabla^{\nu} f(a) + \int_a^x f''(y) d\mu(y), x \in K\}$$

as the composition of the derivatives $\frac{d}{d\nu}$ and $\frac{d}{d\mu}$, i.e.

$$\Delta^{\mu,\nu} f = \nabla^{\mu}\Big(\nabla^{\nu} f\Big) = \frac{d}{d\mu}\Big(\frac{df}{d\nu}\Big) := \begin{cases} f'' & \text{on} \quad L \\ 0 & \text{on} \quad K \backslash L \end{cases}.$$

So $\Delta^{\mu,\nu}$ is a linear operator on the space $L_2(K, \mu)$. Note that in general it holds that supp $\mu \subsetneq K$. A standard example for this case is $K = [a, b] = [0, 1]$, ν is the Lebesgue measure on $[0, 1]$, and μ is the d–dimensional Hausdorff measure $\mathcal{H}^d$ supported on the middle third Cantor set, where $d = \frac{\log 2}{\log 3}$ is the Hausdorff dimension of the Cantor set. But one also can regard cases where μ, ν and the Lebesgue measure are pairwise mutually singular.
Denote $\Delta_D^{\mu,\nu}$ and $\Delta_N^{\mu,\nu}$ the restrictions of $\Delta^{\mu,\nu}$ to those $\mathcal{D}_2^{\mu,\nu}$–functions f which satisfy Dirichlet (i.e. $f(a) = f(b) = 0$) – or, Neumann (i.e. $\nabla^{\nu} f(a) = \nabla^{\nu} f(b) = 0$) resp.– boundary conditions. These operators are non–positive and self–adjoint, and the sequences of their eigenvalues have no accumulation point except $-\infty$ (hence, in particular, the eigenvalues have finite multiplicities). Therefore the eigenvalue counting functions of $-\Delta_{D/N}^{\mu,\nu}$ given by

$$N_{D/N}^{\mu,\nu}(x) := \# \Big\{\kappa_k \leq x : \kappa_k \text{ is eigenvalue of } -\Delta_{D/N}^{\mu,\nu}\Big\}$$

– counting according to multiplicities – are well-defined.

§ 2.2. Spectral asymptotics in the self–similar case

For the special case of self–similar measures – Hausdorff measures, or, more general, anisotropic self–similar measures supported on generalized Cantor sets, one can describe the asymptotic behaviour of $N^{\mu,\nu}_{D/N}(x)$ as x tends to infinity. This was done in [4] via proving some renormalization properties of the traces of the corresponding resolvents reflecting the self–similarity of the measures and applying Tauberian theory. In order to introduce the basic notations, we recall the main result here.

Let $[a,b] \subset \mathbb{R}$ be a closed interval and $S = \{S_1, \dots , S_M\}$, $M \geq 2$, a finite family of affine contractions, i.e.

$$S_i(x) = r_i x + b_i, \quad 0 < r_i < 1, \quad b_i \in \mathbb{R}, \quad i = 1, \dots , M,$$

such that every S_i maps $[a,b]$ into $[a,b]$. Moreover, we are given a M–dimensional vector of weights $\varrho = (\varrho_1, \dots , \varrho_M)$, i.e. $\varrho_1, \dots , \varrho_M$ are real numbers from the interval $(0,1)$ and $\sum_{i=1}^M \varrho_i = 1$. Then classical results of Hutchinson (see [9]) imply the existence of a unique nonempty compact set $L = L(S) \subseteq [a,b]$ which is self similar w.r.t. S, i.e. $L = \bigcup_{i=1}^M S_i L$. Moreover, there is a unique Borel probability measure $\mu = \mu(S,\varrho)$ which is self similar w.r.t. S and ϱ, i.e. $\mu(A) = \sum_{i=1}^M \varrho_i \mu(S_i^{-1}(A))$ for any Borel set A in $[a,b]$. Furthermore, it holds that supp $\mu = L$.
We denote d the unique positive solution of $\sum_{i=1}^M r_i^d = 1$, which is the so–called *similarity dimension of* S. Furthermore, we assume that for any $i,j \in \{1 \dots M\}, i \neq j$ the set $S_i([a,b]) \cap S_j([a,b])$ consists of at most one point (i.e. different images of $[a,b]$ are disjoint or just–touching). Note that this assumption is equivalent to the well–known open set condition. Then it holds that $d = \dim_H L$ and $0 < \mathcal{H}^d(L) < \infty$, where $\dim_H$ and $\mathcal{H}^d$ denote Hausdorff dimension and d–dimensional Hausdorff measure. Moreover, we have $\mathcal{H}^d(S_i(L) \cap S_j(L)) = 0$ for any $i \neq j$. If we set $\varrho_i = r_i^d$ (which is the "natural choice" of the weights) then it holds that $\mu(A) = (\mathcal{H}^d(L))^{-1}\mathcal{H}^d(A \cap L)$ for any Borel set A in $[a,b]$, i.e. the unique self similar measure μ is given by the normalized d–dimensional Hausdorff measure, restricted to L.

These are our requirements to the measure μ. For the measure ν, we assume that is satisfies a property which we call "S–homogeneity", i.e. we assume that there exist real numbers $\sigma_i > 0$, $i = 1, \dots , M$, such that

$$\nu\left(A \cap \left(\bigcup_{i=1}^M S_i K\right)\right) = \sum_{i=1}^M \sigma_i \nu\left(S_i^{-1}(A)\right) \tag{2.1}$$

for any Borel set $A \subset [a,b]$. By the other assumptions it follows immediately that $\sum_{i=1}^{M} \sigma_i \leq 1$.
Note that in the particular case that ν is the Lebesgue measure on $K = [0,1]$ and μ is an arbitrary self–similar measure with respect to a family of contractions S and a vector of weights ϱ, assumption (2.1) is always satisfied with $\sigma_i = r_i$, $i = 1, \ldots, M$.
Under these assumptions it holds that (see [4], Theorem 4.1)

$$N_{D/N}^{\mu,\nu}(x) \asymp x^{\gamma} \qquad \text{as} \quad x \to \infty,$$

i.e. there exist positive constants C_1, C_2 and x_0, such that

$$C_1 x^{\gamma} \leq N_{D/N}^{\mu,\nu}(x) \leq C_2 x^{\gamma} \qquad \text{for} \quad x \geq x_0, \tag{2.2}$$

where $\gamma \in (0,1)$ is given as the unique solution of $\sum_{i=1}^{M} (\varrho_i \sigma_i)^{\gamma} = 1$.

Remark 1. For the particular case that ν is the Lebesgue measure on $K = [0,1]$, and μ is the normalized d–dimensional Hausdorff measure on a self–similar subset $L \subseteq [0,1]$ satisfying the open set condition, the result (2.2) was obtained by Fujita (see [8]). In this case, it holds that $\gamma = \frac{d}{d+1}$.

Remark 2. The particular case that $\mu = \nu$, and hence $K = L$, has been treated in [7]. In this case, we have $\gamma = \frac{1}{2}$, i.e. we observe the same spectral asymptotic behaviour as in the Euclidean case (cf. also (2.5) below). Roughly spoken, the reason for this phenomenon is that the operator $\frac{d}{d\mu}\frac{d}{d\mu}$ is just the usual one–dimensional Laplacian composed with a spatial fractal transformation.

§ 2.3. Refinement of the spectral asymptotics

In view of formula (2.2) it is natural to ask whether the limit

$$\lim_{x \to \infty} \left(x^{-\gamma} \cdot N_{D/N}^{\mu,\nu}(x) \right) \tag{2.3}$$

exists or not. The following theorem provides a sufficient condition for the existence of the limit (2.3) which will be expressed in terms of the weights $(\varrho_i)_{i=1}^{M}$ of the self–similar measure μ and the "homogeneity numbers" $(\sigma_i)_{i=1}^{M}$ of the measure ν. For the proof and a detailed discussion we refer to the author's paper [6].

Theorem 2.1. *Under the assumptions made in Subsection 2.2 one can distinguish two cases for the asymptotical behaviour of the term $N_{D/N}^{\mu,\nu}(x) \cdot x^{-\gamma}$, $x \to \infty$, namely:*

<u>Non–lattice case:</u> *If the additive group $\sum_{i=1}^{M} \mathbb{Z} \log(\varrho_i \sigma_i)$ is a dense subset of $\mathbb{R}$, then it*

follows that the term $N^{\mu,\nu}_{D/N}(x)\cdot x^{-\gamma}$ *converges as* $x\to\infty$*, and the limit is given by*

$$\lim_{x\to\infty} N^{\mu,\nu}_{D/N}(x)\cdot x^{-\gamma} = \left(-\sum_{i=1}^{M}(\varrho_i\sigma_i)^\gamma\log(\varrho_i\sigma_i)\right)^{-1}\int_{-\infty}^{\infty} e^{-\gamma t}R(e^t)dt,$$

where R *is a non–negative, bounded, right continuous function defined by*

$$R(x) := N^{\mu,\nu}_{D}(x) - \sum_{i=1}^{M} N^{\mu,\nu}_{D}(\varrho_i\sigma_i x), \quad x\ge 0. \tag{2.4}$$

Lattice case: If $\sum_{i=1}^{M}\mathbb{Z}\log(\varrho_i\sigma_i)$ *lies in a discrete subgroup of* $\mathbb{R}$*, i.e.* $\sum_{i=1}^{M}\mathbb{Z}\log(\varrho_i\sigma_i) = T\mathbb{Z}$ *for some* $T>0$*, then it holds that*

$$N^{\mu,\nu}_{D/N}(x) = \left(G(\ln x) + o(1)\right)\cdot x^\gamma,$$

where $o(1)$ *denotes a term which vanishes as* $x\to\infty$*, and* G *is a positive,* T*–periodic function given by*

$$G(t) := T\cdot\left(-\sum_{i=1}^{M}(\varrho_i\sigma_i)^\gamma\log(\varrho_i\sigma_i)\right)^{-1}\sum_{j=-\infty}^{\infty} e^{-\gamma(t+jT)}R(e^{(t+jT)}),$$

where the function R *is defined in (2.4). Moreover,* G *is right–continuous, and there exist constants* $0<l<L<\infty$ *such that* $l\le G(t)\le L, t\in\mathbb{R}$.

Remark 3. For $M=2$ the lattice case occurs if and only if it holds that

$$\frac{\log(\varrho_1\sigma_1)}{\log(\varrho_2\sigma_2)}\in\mathbb{Q},$$

i.e. if and only if there exist non zero integers p and q such that $(\varrho_1\sigma_1)^q = (\varrho_2\sigma_2)^p$. For $M>2$ one can give similar criteria.

Let us compare our results with the classical result for the Euclidean space $\mathbb{R}^n$. We consider the Dirichlet eigenvalue problem on a bounded domain $\Omega\subset\mathbb{R}^n$ with smooth boundary $\partial\Omega$

$$\begin{cases} -\Delta_n u = \lambda u \text{ on } \Omega \\ u_{|\partial\Omega}\equiv 0, \end{cases}$$

where $\Delta_n = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is the Laplacian in $\mathbb{R}^n$. Define the eigenvalue counting function

$$N_n(x) := \#\left\{\lambda_k\le x \quad : \quad -\Delta_n u = \lambda_k u\right\}$$

(counting according to multiplicities), then for all $n \in \mathbb{N}$ it holds that

$$N_n(x) = (2\pi)^{-n} c_n |\Omega|_n x^{n/2} + o(x^{n/2}), \quad \text{as} \quad x \to \infty, \tag{2.5}$$

where $|\Omega|_n$ denotes the n–dimensional Lebesgue measure of Ω and c_n is the n–volume of the unit ball in $\mathbb{R}^n$ (see Weyl [15]). Hence, in the Euclidean case we always have convergence of the term $N_n(x) \cdot x^{-\frac{n}{2}}$ as $x \to \infty$.

§ 3. The lattice case - Exact renormalization of the Neumann–eigenvalues

§ 3.1. Statement of the problem - An example: The interval as a fractal

Now we want to illustrate an interesting problem related with the lattice case. To this end, we regard the following example.

We fix a number $r \in (0, 1)$, and we introduce the family $S = \{S_1, S_2\}$ of contractions acting on the interval $[0, 1]$ given by $S_1(x) = rx$ and $S_2(x) = (1 - r)x + r$. Obviously, the unique self–similar set L with respect to the family S is the interval $[0, 1]$, hence we interpret the unit interval as a self–similar set, as a "degenerated" fractal of Hausdorff dimension one. If we choose the corresponding vector of weights to be $\varrho = (r, 1-r)$, the corresponding measure μ is just the Lebesgue measure restricted to the unit interval. Now we investigate the particular case $\nu = \mu$ (hence $K = L$), i.e. the operator $\Delta^{\mu,\nu}$ is given by the usual second derivative, hence by the one–dimensional Laplacian on $[0, 1]$. On the one hand side, we know from (2.5) that the limit $N_D(x) \cdot x^{-1/2}$ exists for any choice of $r \in (0, 1)$. On the other hand side, Remark 3 tells us that we are in the non–lattice case if and only if

$$\frac{\log r}{\log(1 - r)} \in \mathbb{R} \setminus \mathbb{Q}.$$

Thus, convergence of the term $N_{D/N}(x) \cdot x^{-\gamma}$ as $x \to \infty$ may also emerge in the lattice case. The mathematical reason behind this fact is that the periodic function occurring in the renewal theorem might be a constant.

By contrast, Sabot constructed in [13] an example, where we have an oscillating function G. He regarded the interval as a self–similar set with respect to the same family $S = \{S_1, S_2\}$ as above but equipped with the measure which is self–similar with respect to the family S and the vector of weights $\tilde{\varrho} = (1 - r, r)$. Obviously, this leads to the lattice case in Theorem 2.1 for any $r \in (0, 1)$, because of $\varrho_1\sigma_1 = \varrho_2\sigma_2 = r(1 - r)$ (see Remark 3 again). If $r = \frac{1}{2}$, we obtain the Euclidean case and the arising periodic function is a constant. However, for $r \neq \frac{1}{2}$, the corresponding periodic function G in

Theorem 2.1 is not a constant. Note that Sabots's example satisfies the assumptions of Theorem 3.1 below.

In the rest of this section, we will provide an instrument which allows – in some special cases – to decide if the periodic function in the lattice case is a constant or not. To this end, we will use the method of Prüfer angles (see Subsection 3.2) as well as a trick introduced by Volkmer in [14] which allows us to transform the eigenvalue problem associated with $-\Delta^{\mu,\nu}$ into an Atkinson eigenvalue problem (see Subsection 3.3).

§ 3.2. The method of Prüfer angles

For the convenience of the reader, we briefly recall the definition of an Atkinson eigenvalue problem and the method of Prüfer angles. For details we refer to [1].

Let $[c,d] \subseteq \mathbb{R}$ be an interval and $B \subseteq [c,d]$ a Borel set. We assume that the Lebesgue measure of B is less than $d-c$, and that the sets $B \cap [c,e)$ and $B \cap (e,d]$ have positive Lebesgue measure for any $e \in (c,d)$.
Consider the Atkinson eigenvalue problem

$$\begin{aligned} u' &= (1-\mathbf{1}_B(x)) \\ v' &= -\lambda \mathbf{1}_B(x) u \end{aligned} \qquad \text{for a.e. } x \in [c,d], \tag{3.1}$$

($\mathbf{1}_B$ denotes the indicator function of the set B) with boundary conditions

$$\begin{aligned} \cos\alpha\ u(c) &= \sin\alpha\ v(c) \\ \cos\beta\ u(d) &= \sin\beta\ v(d), \end{aligned} \tag{3.2}$$

where $\alpha \in [0,\pi)$ and $\beta \in (0,\pi]$. Obviously, we obtain Dirichlet boundary conditions for $\alpha = 0$, $\beta = \pi$ and Neumann boundary conditions for $\alpha = \beta = \frac{\pi}{2}$. It is well–known (see [1], [2] and [14]), that the eigenvalues form an increasing sequence of real numbers

$$\lambda_0 < \lambda_1 < \lambda_2 < \ldots$$

which are all positive in the Dirichlet case, while the eigenvalues in the Neumann problem satisfy $0 = \lambda_0 < \lambda_n$, $n \geq 1$. Note that the number of eigenvalues is finite if and only if there is a finite union A of intervals such that $A \triangle B$ is a Lebesgue zero set (see Theorem 4.3 in [2]). If there are infinitely many eigenvalues, it holds that $\lambda_n \to \infty$ as $n \to \infty$.
Now we introduce the concept of Prüfer angles. For $\lambda \in \mathbb{R}$, let the pair $(u(x,\lambda), v(x,\lambda))$ denote the solution of (3.1) with *initial* values

$$u(c,\lambda) = \sin\alpha, \qquad v(c,\lambda) = \cos\alpha.$$

Define the Prüfer angle $\theta(x) = \theta(x, \lambda)$ by

$$\theta(x) := \arg(v(x, \lambda) + iu(x, \lambda)), \quad \theta(c) = \alpha.$$

It easily proves that the Prüfer angle satisfies the (ordinary) differential equation

$$\theta' = (1 - \mathbf{1}_B(x)) \cos^2 \theta + \lambda \mathbf{1}_B(x) \sin^2 \theta. \tag{3.3}$$

Obviously, λ is an eigenvalue of the problem (3.1, 3.2) if and only if the boundary conditions are also satisfied in the endpoint d. The function $\theta(d, \lambda)$ is increasing, and in [1] is shown the following:

Proposition 3.1. *The eigenvalues λ_n of the problem (3.1, 3.2) are the solutions of*

$$\theta(d, \lambda_n) = \beta + n\pi, \quad n = 0, 1, 2, \ldots .$$

§ 3.3. Transformation into an Atkinson–problem

For the rest of the paper, we assume that the measure ν is the Lebesgue measure, and that $[a, b] = [0, 1]$. The measure μ is assumed to be a atomless probability measure on $[0, 1]$ with support L satisfying

$$\mu([0, x)) > 0 \quad \text{and} \quad \mu((x, 1]) > 0 \qquad \text{for any} \quad x \in (0, 1).$$

Note that any self–similar measure introduced in Section 2.1 obeys this property. Now the eigenvalue problem for $-\Delta^{\mu,x} = -\frac{d}{d\mu}\frac{d}{dx}$ can be written as

$$\begin{aligned} U(t) - U(0) &= \textstyle\int_0^t V(z)dz \\ V(t) - V(0) &= -\lambda \textstyle\int_0^t U(z)d\mu(z) \end{aligned} \qquad t \in [0, 1]. \tag{3.4}$$

We consider this problem subject to the (mixed) boundary conditions

$$\begin{aligned} \cos\alpha\ U(0) &= \sin\alpha\ V(0) \\ \cos\beta\ U(1) &= \sin\beta\ V(1), \end{aligned} \tag{3.5}$$

where $\alpha \in [0, \pi)$ and $\beta \in (0, \pi]$. This is Krein's eigenvalue problem for a vibrating string with mass distribution μ (see [11]). A number λ is an eigenvalue of (3.4, 3.5), if there exists a nontrivial continuous solution (U, V) of bounded variation satisfying (3.4, 3.5). Then the function U is the eigenfunction to the eigenvalue λ, while the function V is its first (week) derivative. In [14] is described, how this problem can be transformed into an eigenvalue problem of the form (3.1). We sketch the construction here.

Define $h : [0, 1] \longrightarrow [0, 2]$ to be the distribution function of the measure which is the sum of the Lebesgue measure and the measure μ, i.e.

$$h(t) := t + \mu([0, t)), \qquad t \in [0, 1].$$

Then h is continuous, strictly increasing and surjective. Setting $B := h(L)$ it is easy to see that $B \cap [0, e)$ and $B \cap (e, 1]$ have positive Lebesgue measure for any $e \in (0, 1)$. Consider the solution $(u(x, \lambda), v(x, \lambda))$ of the system (3.1, 3.2) with $c = 0$, $d = 2$. It is easy to check that

$$U(t, \lambda) := u(h(t), \lambda), \quad V(t, \lambda) := v(h(t), \lambda), \qquad t \in [0, 1],$$

solve the system (3.4) with the initial values

$$U(0, \lambda) = \sin \alpha, \quad V(0, \lambda) = \cos \alpha.$$

Hence, the problems (3.1) and (3.4) have the same eigenvalues (subject to the boundary conditions (3.2) and (3.5) with the same angles α and β), and $\theta(h(t), \lambda)$ is the Prüfer angle for the problem (3.4), i.e.

$$\theta(h(t), \lambda) = \arg(V(t, \lambda) + iU(t, \lambda)).$$

Remark 4. Note that the (fractal) set $B \subseteq [0, 2]$ is in general no longer self–similar.

§ 3.4. Main result

Now we are going to apply this transformation to a special class of self–similar measures μ leading to the lattice case in the setting of Theorem 2.1.
We are given $M \geq 2$ linear contractions $S_i : [0, 1] \longrightarrow [0, 1]$ with ratios r_i, $i = 1, \ldots, M$, such that the sets $S_i([0, 1])$ are pairwise disjoint or just–touching. Moreover, we assume μ to be the unique probability measure which is self–similar with respect to the family S and a vector $\varrho = (\varrho_1, \ldots, \varrho_M) \in \mathbb{R}^M$.
In addition, we require that the products $\varrho_i r_i$ do not depend on the index i, i.e. we assume that there exists a number $R^{-1} \in (0, 1)$ such that

$$\varrho_i r_i \equiv: R^{-1}, \qquad i = 1, \ldots, M. \tag{3.6}$$

Note that – in the language of stochastics – this assumption means that the associated quasi–diffusion spends in average the same amount of time in any of the sets $S_i(L)$, $i = 1, \ldots, M$. From the statements of Subsection 2.2 (see (2.2)) it follows that the eigenvalue counting function of the operator $-\frac{d}{d\mu}\frac{d}{dx}$ behaves asymptotically like x^γ where the spectral exponent γ is given by $\gamma = \frac{\log M}{\log R}$. Obviously, this model fits into the lattice case of Theorem 2.1, because $\sum_{i=1}^M \mathbb{Z} \log(\varrho_i r_i) = (\log R)\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$. In order to find out if the term $N_{D/N}(x) \cdot x^{-\gamma}$ converges as $x \to \infty$ or not, we prove the following exact renormalization property of the Neumann eigenvalues. The proof deeply relies on Proposition 3.1, the differential equation (3.3) for the Prüfer angle, and the following lemma.

Lemma 3.1. *Denote $U(t,\lambda)$ and $V(t,\lambda)$ the functions introduced in Subsection 3.3. Then for every $i = 1,\dots,M$ it holds that*

$$U(t,R\lambda) = U(S_i^{-1}(t),\lambda)$$

and

$$V(t,R\lambda) = r_i^{-1}V(S_i^{-1}(t),\lambda) \tag{3.7}$$

for $S_i(0) \le t \le S_i(1)$.

Proof. Fixing i and substituting $t = S_i(s)$ in (3.4), the prove is an easy exercise. For a detailed proof we refer the reader to [6]. □

Now we are going to state our main result which generalizes a recent result of Volkmer (see [14]), who obtained Theorem 3.1 for the special case of the (classical) middle third Cantor set.

Theorem 3.1. *Under the above assumptions denote $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$ the eigenvalues of the operator $-\Delta_N^{\mu,x} = -\frac{d}{d\mu}\frac{d}{dx}$. Then it holds that*

$$\lambda_{Mn} = R\lambda_n, \quad n = 0,1,2,\dots .$$

Proof. Without loss of generality we may assume that the images $S_1([0,1]),\dots,S_M([0,1])$ are sorted from the left to the right, i.e.

$$S_1(0) = 0, \quad S_M(1) = 1 \quad \text{and} \quad S_i(1) \le S_{i+1}(0), \quad i = 1,\dots,M-1.$$

As we regard the eigenvalue problem subject to Neumann boundary conditions, it holds that

$$V(0,\lambda_n) = V(1,\lambda_n) = 0,$$

and the Prüfer angle satisfies

$$\theta(h(S_1(1)),R\lambda_n) = \theta(2,\lambda_n) = \frac{\pi}{2} + n\pi.$$

Next we claim that

$$\theta(x,R\lambda_n) = \theta(2,\lambda_n) = \frac{\pi}{2} + n\pi \quad \text{for} \quad h(S_1(1)) \le x \le h(S_2(0)). \tag{3.8}$$

If $S_1(1) = S_2(0)$, (3.8) holds trivially. If $S_1(1) < S_2(0)$, (3.8) follows from (3.3), taking into account that $\mathbf{1}_B(x) = 0$ on $[h(S_1(1)),h(S_2(0))]$. As

$$\mathbf{1}_B(x + h(S_2(0))) = \mathbf{1}_B(x) \quad \text{for} \quad h(S_1(0)) \le x \le h(S_1(1)),$$

we obtain

$$\theta(x + h(S_2(0)), R\lambda_n) = n\pi + \theta(x, R\lambda_n) \quad \text{for} \quad h(S_1(0)) \le x \le h(S_1(1)),$$

which yields

$$\theta(x + h(S_2(1)), R\lambda_n) = n\pi + \theta(h(S_1(1)), R\lambda_n) = \frac{\pi}{2} + 2n\pi = \theta(2, \lambda_{2n}).$$

Iterating this procedure leads to

$$\theta(2, R\lambda_n) = \theta(h(S_M(1)), R\lambda_n) = \frac{\pi}{2} + Mn\pi = \theta(2, \lambda_{Mn}),$$

which proves the assertion of the theorem. □

Corollary 3.1. *Denote as above* $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$ *the eigenvalues of the operator* $-\Delta_N^{\mu,x} = -\frac{d}{d\mu}\frac{d}{dx}$. *Then for any* $k = 1, 2, 3, \dots$, *the subsequence*

$$\left(\frac{(\lambda_{kM^n})^\gamma}{kM^n} \right)_{n \ge 0}$$

is a constant.

Proof. Fixing $k \in \mathbb{N}$ and taking into account that $\gamma = \frac{\log M}{\log R}$, it follows immediately from the latter theorem that

$$\frac{(\lambda_{kM^n})^\gamma}{kM^n} = \frac{(R^n \lambda_k)^\gamma}{kM^n} = \frac{\lambda_k^\gamma}{k} \quad \text{for any } n \ge 0,$$

which yields the assertion. □

Hence, it is sufficient, to calculate the first two eigenvalues λ_1 and λ_2 (λ_1 and λ_3, in case that $M = 2$) in order to decide whether $\lim_{n\to\infty} \frac{\lambda_n^\gamma}{n}$ exists or not. It readily verifies that $\lim_{n\to\infty} \frac{\lambda_n^\gamma}{n}$ exists if and only if $\lim_{x\to\infty} (x^{-\gamma} \cdot N_N^{\mu,\nu}(x))$ exists. The proof is an easy exercise, in particular because the eigenvalues are simple. Computing small eigenvalues can be done by approximating the fractal by finite unions of intervals. Volkmer (see [14]) found $\lambda_1 \approx 7.09$ and $\lambda_3 \approx 61.26$ in the case of the middle third Cantor set, hence he proved "real periodicity", i.e. oscillation of the term $(x^{-\gamma} \cdot N_N^{\mu,\nu}(x))$ as $x \to \infty$. We guess that the same holds for any fractal treated in Theorem 3.1 which is not the entire interval $[0, 1]$ equipped with the classical one–dimensional Laplacian.

Remark 5. The reader might wonder about a corresponding result for the Dirichlet eigenvalue counting function N_D. Note that the factor r_i^{-1} on the right hand side in (3.7) causes some difficulties if one would try to prove a analogue result by using the same techniques. However, as we have (see Proposition 5 in [5])

$$N_D^{\mu,\nu}(x) \le N_N^{\mu,\nu}(x) \le N_D^{\mu,\nu}(x) + 2, \quad x \ge 0,$$

the convergence result holds in the same way. An exact renormalization property for the Dirichlet eigenvalues can be obtained by the method of so–called "modified Prüfer angles" (see [14]).

§ 3.5. Examples, open problems, and a conjecture

Now we apply our result to some special self–similar sets and measures. Assume that the contractions $S_i : [0,1] \longrightarrow [0,1]$, $i = 1, \ldots, M$, all with the same ratio $r := \frac{1}{2M-1}$, map the interval in an "equidistant" way, i.e.

$$S_i(x) := rx + \frac{2(i-1)}{2M-1}, \qquad i = 1, \ldots, M.$$

Denote $L \subseteq [0,1]$ the unique nonempty compact self–similar set w.r.t. S and μ the unique probability measure which is self–similar w.r.t. S and the vector $\varrho = (\frac{1}{M}, \ldots, \frac{1}{M})$. Hence, the Hausdorff dimension of L equals $d_M := \frac{\log M}{\log(2M-1)}$ and μ is just the normalized d_M–dimensional Hausdorff measure restricted to L. Obviously, for $M = 2$ we obtain the middle third Cantor set.
From Subsection 2.2 it follows that the spectral exponent is given by

$$\gamma = \frac{\log M}{\log(M(2M-1))} = \frac{\log M}{\log M - \log r},$$

and the renormalization property of the Neumann eigenvalues reads

$$\lambda_{Mn} = \frac{M}{r}\lambda_n = M(2M-1)\lambda_n, \quad n = 0, 1, 2, \ldots .$$

One can also construct "more anisotropic" examples. It even holds that for any numbers $\alpha, \beta \in (0,1)$ with $\alpha+\beta < 1$ there are associated self–similar sets and measures satisfying the assumptions of Theorem 3.1 such that the Hausdorff dimension of L is given by the unique number $d \in (0,1)$ satisfying $\alpha^d + \beta^d = 1$. Just choose L and μ to be generated by the family $\{S_1(x) = \alpha x, S_2(x) = \beta x + (1-\beta)\}$ and the vector $\varrho = (\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})$. Obviously, similar constructions easily can be developed for more than two similitudes.

Finally, we want to mention, that our result holds for a much wider class than treated by Theorem 3.1. The proof of Theorem 3.1 is somehow done by "gluing together" localized eigenfunctions on the sets $S_1(L), \ldots, S_M(L)$ (i.e. on copies of L of "depth" one) to the same eigenvalue – which corresponds to an accumulation of the corresponding Prüfer angles. One could think about matching eigenfunctions on copies of the set L of *different depth.* In doing so, it would be sufficient that the products $\varrho_i r_i$ are rationally linked (instead of taking the same value as required in (3.6)). But those cases would lead in general to the non–lattice case, because the numbers $\log(\varrho_i r_i)$ would not be rationally linked at the same time. We are convinced that at this point, a further study of the subject requires a deep knowledge in number theory and zeta–functions.

Let us conclude the paper posing two open problems:

1. Find a self–similar set L and a self–similar measure μ supported on L such that in Theorem 2.1 the lattice case occurs, the periodic function G is a constant, L is not the interval, and μ is not the Lebesgue measure. We conjecture that such a set and such a measure do not exist.

2. Determine the Hausdorff dimension of the set B (cf. Remark 4). As it is no longer self–similar, the concept of similarity dimension (see [9]) does not apply. However, B still carries a highly recursive structure, which it should make possible to give at least some thresholds for its Hausdorff dimension.

References

[1] Atkinson, F. V.,*Discrete and continuous boundary problems,* Academic Press, 1964.

[2] Binding, P. and Volkmer, H., Prüfer angle asymptotics for Atkinson's semi–definite Sturm–Liouville eigenvalue problem, *Math. Nach.,***278** (2005), 1458–1475.

[3] Freiberg, U., Analytic properties of measure geometric Krein–Feller–operators on the real line, *Math. Nach.,***263** (2002), 34–47.

[4] Freiberg, U., Spectral asymptotics of generalized measure geometric Laplacians on Cantor like sets, *Forum Math.,***17** (2005), 87–104.

[5] Freiberg, U., Dirichlet forms on fractal subsets of the real line, *Real Analysis Exchange,***30**(2) (2004/2005), 589–604.

[6] Freiberg, U., Refinement of the spectral asymptotics of generalized Krein–Feller–operators, *submitted*

[7] Freiberg, U. and Zähle, M., Harmonic calculus on fractals – A measure geometric approach I, *Potential Anal.* **16** (2002), 265–277.

[8] Fujita, T., A fractional dimension, self similarity and a generalized diffusion operator, *Taniguchi Symp. PMMP Katata* (1985), 83–90.

[9] Hutchinson, J. E., Fractals and self–similarity, *Indiana Univ. Math. J.* **30** (1981), 713–747.

[10] Itô, K. and McKean, H. P. jr., *Diffusion processes and their sample paths,* Springer, 1965.

[11] Kac, I.S. and Krein, M.G., On the spectral functions of the string, *AMS Translations* **103** (1974), 19–102.

[12] Kigami, J.,*Analysis on fractals,* Cambridge University Press, 2001.

[13] Sabot, C., Integrated density of states of self–similar Sturm–Liouville operators and holomorphic dynamics in higher dimension, *Ann. I. H. Poincaré* **37**, 3 (2001), 275–311.

[14] Volkmer, H., Eigenvalues associated with Borel sets,*Real Analysis Exchange* **31**(1) (2005/2006), 111–124.

[15] Weyl, H., Das asymptotische Verteilungsgesetz der Eigenschwingungen eines beliebig gestalteten elastischen Körpers. *Rend. Cir. Mat. Palermo* **39** (1915), 1–50.