RIMS Kôkyûroku Bessatsu **B6**
(2008), 15–29

Reflection Positive Random Fields and Dirichlet Spaces

By

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Abstract

Focusing upon the notions of reflecting positive random fields, Dirichlet spaces and distribution valued symmetric Markov processes, a concise guide to understand the Nelson's Euclidean strategy of constructing quantum field theory is given.

§ 1. Introduction

The purpose of this note is to give a guide to understand how the reflection positive random fields and the Dirichlete spaces play the key role in the context of Nelson's Euclidean strategy of constructive quantum field theory (cf. e.g., [Si]).

In [AY] we have already introduced a general discussion on the reflection positive random fields indexed by $\mathcal{S}(\mathbb{R}^d)$, $d = 1 + (d-1) \in \mathbb{N}$, where 1 corresponds to the dimension of time and $d-1$ corresponds to the dimension of the space, and $d \in \mathbb{N}$ denotes the space time dimension. In the same paper such random fields have been constructed by convoluting the Wiener functional and some pseudo differential operators (cf. also the Remark in the present note).

The present note is a continuation of [AY], but here we focus not upon the space time random fields, discussed in [AY], but upon the $\mathcal{S}'(\mathbb{R}^{d-1})$-valued symmetric Markov processes by which the d-dimensional space time random fields are generated. Namely, we discuss how the reflection positive random fields indexed by $\mathcal{S}(\mathbb{R}^d)$ can be identified with some particular $\mathcal{S}'(\mathbb{R}^{d-1})$-valued symmetric Markov processes (cf., e.g., [Si]).

2000 Mathematics Subject Classification(s):
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In other words, the objective of the present note is to give a guide of clear understanding of Nelson's framework of Euclidean field theory through the arguments of the theory of symmetric Markov processes.

For this purpose, we use a model constructed by generalizing Nelson's Euclidean free field or the sharp time free field. By this model the essence of Nelson's Euclidean strategy can be clearly exposed.

The key words on the present discussion are the following:

(Key word 1) $\mathcal{S}'(\mathbb{R}^{d-1} \to \mathbb{R})$-valued symmetric Markov processes;

(Key word 2) Hypercontractive semi-groups (by which the above $\mathcal{S}'(\mathbb{R}^{d-1} \to \mathbb{R})$-valued processes are defined);

(Key word 3) Dirichlet forms satisfying a logarithmic Sobolev inequality (the generator of hypercontractive semi-groups correspond to such forms (cf. [G])).

Nevertheless, the models introduced here are the examples by which the key ideas of Nelson's Euclidean strategy are explained, except the well known free field and $P(\Phi)_2$ field models, they temselves are completly newly developed and the corresponding discussions include new results.

§ 2. Identification of Euclidean quantum fields on $\mathbb{R}^d$ with $\mathcal{S}'(\mathbb{R}^{d-1} \to \mathbb{R})$ valued Markov processes

Throughout this note, we denote by $d \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, the space-time dimension, and we understand that $d-1$ is the space dimension and 1 is the dimension of time. Correspondingly, we use the notations

$$\mathbf{x} \equiv (t, \vec{x}) \in \mathbb{R} \times \mathbb{R}^{d-1}.$$

Let $\mathcal{S}(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^{d-1})$) be the Schwartz space of rapidly decreasing test functions on the d dimensional Euclidean space $\mathbb{R}^d$ (resp. $d-1$ dimensional Euclidean space $\mathbb{R}^{d-1}$), equipped with the usual topology by which it is a Fréchet nuclear space. Let $\mathcal{S}'(\mathbb{R}^d)$ (resp. $\mathcal{S}'(\mathbb{R}^{d-1})$) be the topological dual space of $\mathcal{S}(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^{d-1})$).

The probability measures on $\mathcal{S}'(\mathbb{R}^d \to \mathbb{R})$ which are invariant with respect to the Euclidean transformations are called as **Euclidean random fields** (in this note we also consider the probability measures on $\mathcal{S}'(\mathbb{R}^d \to \mathbb{R})$ that are not necessarily invariant with respect to the Euclidean transformations on $\mathbb{R}^d$ but invariant with respect to the Euclidean transformations of the space variables, i.e. the transformations on $\mathbb{R}^{d-1}$, and in the sequel we call such fields as **partial Euclidean random fields**). The Euclidean random fields which admit an analytic continuation to relativistic quantum fields (**Wightman fields**) are called as **Euclidean quantum (random) fields**. Here the analytic continuation, very roughly speaking, means analytic continuation of the time variable $t \in \mathbb{R}$ of Euclidean fields to $\sqrt{-1}t$, and **Wightaman fields** are the fields that are invariant with respect to the transformations keeping the Lorentz scalar product unchanged (i.e. the restricted Poincaré invariance).

In this section we review how the Euclidean quantum random fields on $\mathbb{R}^d$, the probability measures on $\mathcal{S}'(\mathbb{R}^d \to \mathbb{R})$, and partial Euclidean random fields, are identified with the probability measures on the space $C\left(\mathbb{R} \to \mathcal{S}'(\mathbb{R}^{d-1} \to \mathbb{R})\right)$ which are generated by some $\mathcal{S}'(\mathbb{R}^{d-1} \to \mathbb{R})$ valued Markov processes (cf. **Key word 1** in this section).

In order to simplify the notations, in the sequel, by the symbol D we denote both d and $d-1$. In each discussion we exactly explain the dimension (space-time or space) of the field on which we are working.

Now, suppose that on a complete probability space $(\Omega, \mathcal{F}, P)$ we are given an isonormal Gaussian process $W^D = \{W^D(h), h \in L^2(\mathbb{R}^D; \lambda^D)\}$, where λ^D denotes the Lebesgue measure on $\mathbb{R}^D$ (cf., e.g., [AY]). Precisely, W^D is a centered Gaussian family of random variables such that

$$E[W^D(h)\, W^D(g)] = \int_{\mathbb{R}^D} h(\mathbf{x})\, g(\mathbf{x}) \lambda^D(d\mathbf{x}), \qquad h,\ g \in L^2(\mathbb{R}^D; \lambda^D). \tag{2.1}$$

We write

$$W^D_\omega(h) = \int_{\mathbb{R}^D} h(y) W^D_\omega(d\mathbf{y}), \qquad \omega \in \Omega$$

with $W^D_\omega(\cdot)$ a Gaussian generalized random variable (in the general notation of **Hida calculus** for the Gaussian white noise $W^D_\omega(d\mathbf{y})$ would be written as $\dot{W}^D(\mathbf{y})d\mathbf{y}$).

Since, we are considering a massive scalar field, we suppose that we are given a real mass parameter $m > 0$. Let Δ_d and resp. Δ_{d-1} be the d, resp. $d-1$, dimensional Laplace operator, and define the pseudo differential operators $L_{-\frac{1}{2}}$ and $H_{-\frac{1}{4}}$ as follows:

$$L_{-\frac{1}{2}} = (-\Delta_d + m^2)^{-\frac{1}{2}}. \tag{2.2}$$

$$H_{-\frac{1}{4}} = (-\Delta_{d-1} + m^2)^{-\frac{1}{4}}, \tag{2.3}$$

By the same symbols as $L_{-\frac{1}{2}}$ and $H_{-\frac{1}{4}}$, we also denote the integral kernels of the corresponding pseudo differential operators, i.e., the Fourier inverse transforms of the corresponding symbols of the pseudo differential operators.

By making use of stochastic integral expressions, we define two important random fields ϕ_N, the **Nelson's Euclidean free field**, and ϕ_0, the **sharp time free field**, as

follows:
For $d \geq 2$,

$$\phi_N(\cdot) \equiv \int_{\mathbb{R}^d} L_{-\frac{1}{2}}(\mathbf{x} - \cdot) W^d(d\mathbf{x}), \tag{2.4}$$

$$\phi_0(\cdot) \equiv \int_{\mathbb{R}^{d-1}} H_{-\frac{1}{4}}(\vec{x} - \cdot) W^{d-1}(d\vec{x}). \tag{2.5}$$

These definitions of ϕ_N and resp. ϕ_0 seems formal, but they are rigorously defined as $\mathcal{S}'(\mathbb{R}^d)$ and resp. $\mathcal{S}'(\mathbb{R}^{d-1})$ valued random variables through a limiting procedure (cf. [AY]), more precisely it has been shown that

$$P(\phi_N(\cdot) \in B_d^{a,b}) = 1, \quad \text{for } a, b \text{ such that } \min(1, \frac{2a}{d}) + \frac{2}{d} > 1, \ b > d \tag{2.6}$$

$$P(\phi_0 \in B_{d-1}^{a',b'}) = 1, \quad \text{for } a', b' \text{ such that } \min(1, \frac{2a'}{d-1}) + \frac{1}{d-1} > 1, \ b' > d-1. \tag{2.7}$$

Here for each a, b, $D > 0$, the Hilbert space $B_d^{a,b}$, which is a linear subspace of $\mathcal{S}'(\mathbb{R}^D)$, is defined by

$$B_D^{a,b} = \{(|\mathbf{x}|^2 + 1)^{\frac{b}{4}}(-\Delta_D + 1)^{+\frac{a}{2}} f : \ f \in L^2(\mathbb{R}^D; \lambda^D)\}, \tag{2.8}$$

where $\mathbf{x} \in \mathbb{R}^D$ and λ denotes the Lebesgue measure on $\mathbb{R}$, the scalar product of $B_d^{a,b}$ is given by

$$\begin{aligned} < u \,|\, v > = \int_{\mathbb{R}^D} & \left\{(-\Delta_D + 1)^{\frac{a}{2}}((1 + |\mathbf{x}|^2)^{-\frac{b}{4}} u(\mathbf{x}))\right\} \\ & \times \left\{(-\Delta_D + 1)^{\frac{a}{2}}((1 + |\mathbf{x}|^2)^{-\frac{b}{4}} v(\mathbf{x}))\right\} d\mathbf{x}, \quad u, v \in B_D^{a,b}. \end{aligned} \tag{2.9}$$

The following definition of $< \phi_N, f >$ and $< \phi_0, \varphi >$ gives a good explanation of (2.4) and (2.5). We denote

$$< \phi_N, f > \equiv \int_{\mathbb{R}^d} \left(L_{-\frac{1}{2}} f\right)(\mathbf{x})\, W^d(d\mathbf{x}), \qquad f \in \mathcal{S}(\mathbb{R}^d \to \mathbb{R}), \tag{2.10}$$

$$< \phi_0, \varphi > \equiv \int_{\mathbb{R}^{d-1}} \left(H_{-\frac{1}{4}} \varphi\right)(\vec{x})\, W^{d-1}(d\vec{x}), \qquad \varphi \in \mathcal{S}(\mathbb{R}^{d-1} \to \mathbb{R}). \tag{2.11}$$

Any probabilistic treatment of Euclidean quantum field theory starts from **Nelson's Euclidean free field** ϕ_N.

ϕ_N satisfies all the requirements under which it admits an analytic continuation to a quantum field that satisfies the Wightman axioms (cf.,e.g., [Si], [AY] and references therein). In particular, ϕ_N satisfies the following two important properties:

N-1) ϕ_N is Markovian with respect to time in the sense that

$$E[< \phi_N, f_1 > \cdots < \phi_N, f_k > | \mathcal{F}_{(-\infty,0]}] = E[< \phi_N, f_1 > \cdots < \phi_N, f_k > | \mathcal{F}_0],$$

$$\text{for any} \quad k \in \mathbb{N}, \quad f_j \in \mathcal{S}(\mathbb{R}^d \to \mathbb{R}), \quad j = 1, \cdots, k, \quad \text{such that}$$

$$\text{supp}[f_j] \subset \{(t, \vec{x}) | \, t \geq 0, \, \vec{x} \in \mathbb{R}^{d-1}\}, \quad j = 1, \cdots, k,$$

$\mathcal{F}_{(-\infty,0]} \equiv$ the σ field generated by the random variables $< \phi_N, g >$ such that

$$\text{supp}[g] \subset \{(t, \vec{x}) | \, t \leq 0, \, \vec{x} \in \mathbb{R}^{d-1}\},$$

$\mathcal{F}_0 \equiv$ the σ field generated by the random variables $< \phi_N, \varphi \times \delta_{\{0\}}(\cdot) >$, where φ are functions having only the space variable $\vec{x}$, i.e., $\varphi(\vec{x})$ such that $\varphi \in \mathcal{S}(\mathbb{R}^{d-1} \to \mathbb{R})$ and $\delta_{\{0\}}(t)$ is the Dirac point measure at time $t = 0$, namely

$$\text{supp}[\varphi \times \delta_{\{0\}}(\cdot)] \subset \{(t, \vec{x}) | \, t = 0, \, \vec{x} \in \mathbb{R}^{d-1}\}.$$

N-2) ϕ_N is an Euclidean invariant field, precisely, for each Euclidean group (rotation, translation, reflection) G, the probability distributions of two random variables

$$< \phi_N, f_1 > \cdots < \phi_N, f_k > \quad \text{and} \quad < \phi_N, G f_1 > \cdots < \phi_N, G f_k >$$

are identical with each other for any $f_j \in \mathcal{F}(\mathbb{R}^d \to \mathbb{R})$, $j = 1, \cdots, k$, $k \in \mathbb{N}$.

Remark 1. For ϕ_N, the random variable $< \phi_N, \varphi \times \delta_{\{0\}}(\cdot) >$ is well defined (cf. [AY]), precisely for any $t_0 \in \mathbb{R}$ and the Dirac point measure $\delta_{\{t_0\}}(\cdot)$ at time $t = t_0$

$$< \phi_N, \varphi \times \delta_{\{t_0\}}(\cdot) > \in \cap_{q \geq 1} L^q(\Omega; P).$$

□

Let θ be the time reflection operator:

$$(\theta f)(t, \vec{x}) = f(-t, \vec{x}),$$

then by N-1), for any $k \in \mathbb{N}$, $f_j \in \mathcal{S}(\mathbb{R}^d \to \mathbb{R})$, $j = 1, \cdots, k$, such that $\text{supp}[f_j] \subset \{(t, \vec{x}) | \, t \geq 0, \, \vec{x} \in \mathbb{R}^{d-1}\}$, $j = 1, \cdots, k$, we see that

$$\begin{aligned}
(2.12) \quad & E[(< \phi_N, f_1 > \cdots < \phi_N, f_k >) \, (< \phi_N, \theta f_1 > \cdots < \phi_N, \theta f_k >)] \\
&= E\Big[E[< \phi_N, f_1 > \cdots < \phi_N, f_k > | \, \mathcal{F}_{(-\infty,0]}] \, < \phi_N, \theta f_1 > \cdots < \phi_N, \theta f_k >\Big] \\
&= E\Big[E[< \phi_N, f_1 > \cdots < \phi_N, f_k > | \, \mathcal{F}_0] \, < \phi_N, \theta f_1 > \cdots < \phi_N, \theta f_k >\Big] \\
&= E\Big[E\Big[E[< \phi_N, f_1 > \cdots < \phi_N, f_k > | \, \mathcal{F}_0] \, < \phi_N, \theta f_1 > \cdots < \phi_N, \theta f_k > \Big| \, \mathcal{F}_0\Big]\Big] \\
&= E\Big[E[< \phi_N, f_1 > \cdots < \phi_N, f_k > | \, \mathcal{F}_0] \, E[< \phi_N, \theta f_1 > \cdots < \phi_N, \theta f_k > | \, \mathcal{F}_0]\Big].
\end{aligned}$$

But, by N-2) the Euclidean invariance of ϕ_N, in particular by the symmetric property, we also see that

$$E\Big[\{<\phi_N, f_1>\cdots<\phi_N, f_k> - <\phi_N, \theta f_1>\cdots<\phi_N, \theta f_k>\}$$
$$\times <\phi_N, \varphi\times\delta_{\{0\}}(\cdot)>\Big] = 0, \qquad \text{for any} \quad \varphi\in\mathcal{S}(\mathbb{R}^{d-1}\to\mathbb{R}).$$

This means that

$$E[<\phi_N, f_1>\cdots<\phi_N, f_k> \mid \mathcal{F}_0] = E[<\phi_N, \theta f_1>\cdots<\phi_N, \theta f_k> \mid \mathcal{F}_0] \qquad P-a.s., \tag{2.13}$$

hence, the right hand side of (2.12) can be rewritten as

$$E\Big[\Big(E[<\phi_N, f_1>\cdots<\phi_N, f_k> \mid \mathcal{F}_0]\Big)^2\Big] \geq 0.$$

Consequently, we see that ϕ_N satifies the following :

$$E[(<\phi_N, f_1>\cdots<\phi_N, f_k>)\,(<\phi_N, \theta f_1>\cdots<\phi_N, \theta f_k>)] \geq 0 \tag{2.14}$$

The property (2.14) is refered as the **reflection positivity**, and Nelson's Euclidean free field ϕ_N is a **reflection positive** random field. But from the above discussion (cf. (2.12) and (2.13)) we see that the property of reflection positivity is a property of symmetric Markov processes.

Remark 2. By N-1) and (2.13), $\{\phi_N(t,\cdot)\}_{t\in\mathbb{R}}$ can be understood as a symmetric "Markov process", moreover by the Euclidean invariance N-2) $\phi_N(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^d$ it is a **Markov field** (cf., e.g., [Si], [AY] and references therein).

□

In the above Remak 2, for $\{\phi_N(t,\cdot)\}_{t\in\mathbb{R}}$ we use the heuristic terminology "Markov process". In order to certify that $\{\phi_N(t,\cdot)\}_{t\in\mathbb{R}}$ is really a symmetric Markov process with the time parameter $t\in\mathbb{R}$, we have to give an answer for the following questions:

(Question 1) Are there a probability measure μ on $\mathcal{S}'(\mathbb{R}^{d-1}\to\mathbb{R})$ and a Markov semigroup T_t, $t\geq 0$ such that

$$T_t\,:\,L^q(\mu)\to L^q(\mu), \quad q\in[1,\infty],\ t\geq 0; \qquad T_t,\ t\geq 0 \text{ is positivity preserving;}$$

$$\|T_t\|_{L^q\to L^q}\leq 1, \qquad \forall q\in[1,\infty], \quad t\geq 0, \quad \text{i.e., } T_t \text{ is a contraction semigroup;}$$

for any φ_1, $\varphi_2\in\mathcal{S}(\mathbb{R}^{d-1}\to\mathbb{R})$, and any t_1, $t_2\geq 0$

$$\int_{\mathcal{S}'(\mathbb{R}^{d-1}\to\mathbb{R})} T_{t_1}\Big(\Big(T_{t_2}<\cdot,\varphi_2>_{\mathcal{S}',\mathcal{S}}\Big)(\cdot)<\cdot,\varphi_1>_{\mathcal{S}',\mathcal{S}}\Big)(\phi)\ \mu(d\phi)$$
$$= E[<\phi_N, \varphi_1\times\delta_{\{t_1\}}(\cdot)><\phi_N, \varphi_2\times\delta_{\{t_1+t_2\}}(\cdot)>] \quad ? \tag{2.15}$$

Here

$$< \phi, \varphi >_{\mathcal{S}',\mathcal{S}} \equiv < \phi, \varphi >_{\mathcal{S}'(\mathbb{R}^{d-1}\to\mathbb{R}),\mathcal{S}(\mathbb{R}^{d-1}\to\mathbb{R})},$$

is the dualization of $\mathcal{S}'(\mathbb{R}^{d-1} \to \mathbb{R})$ and $\mathcal{S}(\mathbb{R}^{d-1} \to \mathbb{R})$.

In other words, can we characterize the (symmetric) Markovian semi-group $T_{|t|}$ and the initial distribution μ by which $\{\phi_N(t, \cdot)\}_{t\in\mathbb{R}}$ is generated?

□

Question 1 corresponds to the consideration of **Key word 1**, $\mathcal{S}'(\mathbb{R}^{d-1} \to \mathbb{R})$-valued symmetric Markov processes. The answer for **Question 1** is affirmative. To state the exact answer, we prepare some notions.

Let μ_0 be the probability measure on $\mathcal{S}'(\mathbb{R}^{d-1} \to \mathbb{R})$ which is the probability law of the sharp time free field ϕ_0 on $(\Omega, \mathcal{F}, P)$ (cf. (2.7)), and μ_N be the probability measure on $\mathcal{S}'(\mathbb{R}^d \to \mathbb{R})$ which is the probability law of the Nelson's Euclidean free field on $\mathbb{R}^d$ (cf. (2.6)).

We denote

$$\phi_0(\varphi) \equiv < \phi_0, \varphi > \equiv \int_{\mathbb{R}^{d-1}} \left(H_{-\frac{1}{4}}\varphi\right)(\vec{x})\, W^{d-1}(d\vec{x}),$$

and

$$\begin{aligned} &: \phi_0(\varphi_1)\cdots\phi_0(\varphi_n) : \\ &= \int_{\mathbb{R}^{k(d-1)}} H_{-\frac{1}{4}}\varphi_1(\vec{x}_1)\cdots H_{-\frac{1}{4}}\varphi_1(\vec{x}_k) W^{d-1}(d\vec{x}_1)\cdots W^{d-1}(d\vec{x}_k) \in \cap_{q\geq 1} L^q(\mu_0) \end{aligned}$$

$$\text{for} \quad \varphi, \varphi_j \in \mathcal{S}(\mathbb{R}^{d-1} \to \mathbb{R}), \quad j = 1, \cdots, k, \quad k \in \mathbb{N}, \tag{2.16}$$

where (2.16) is the k-th multiple stochastic integral with respect to the isonormal Gaussian process W^{d-1} on $\mathbb{R}^{d-1}$.

Since, $: \phi_0(\varphi_1)\cdots\phi_0(\varphi_n) :$ is nothing more than an element of the n-th Wiener chaos of $L^2(\mu_0)$, it also admits an expression by means of the **Hermite polynomial** of $\phi_0(\varphi_j)$, $j = 1, \cdots, k$ (cf., e.g., [AY] and references therein).

Remark 3. From view point of the notational rigour, ϕ_0 and ϕ_N are the distribution valued random variables on the probability space $(\Omega, \mathcal{F}, P)$, hence the notation such as

$$: \phi_0(\varphi_1)\cdots\phi_0(\varphi_n) : \in \bigcap_{q\geq 1} L^q(\mu_0)$$

is incorrect. However in the above and in the sequel, since there is no ambiguity, for the simplicity of the notations we use the notations ϕ_0 and ϕ_N (with an obvious interpretation) to indicate the measurable functions X and resp. Y on the measure spaces $(\mathcal{S}'(\mathbb{R}^{d-1}), \mu_0, \mathcal{B}(\mathcal{S}'(\mathbb{R}^{d-1})))$ and resp. $(\mathcal{S}'(\mathbb{R}^d), \mu_N, \mathcal{B}(\mathcal{S}'(\mathbb{R}^d)))$ such that

$$P\Big(\big\{\omega \,:\, \phi_0(\omega) \in A\big\}\Big) = \mu_0\Big(\big\{\phi \,:\, X(\phi) \in A\big\}\Big), \quad A \in \mathcal{B}(\mathcal{S}'(\mathbb{R}^{d-1})),$$

$$P\Big(\big\{\omega \,:\, \phi_N(\omega) \in A'\big\}\Big) = \mu_N\Big(\big\{\phi \,:\, Y(\phi) \in A'\big\}\Big), \quad A' \in \mathcal{B}(\mathcal{S}'(\mathbb{R}^d)),$$

respectively, where $\mathcal{B}(S)$ denotes the Borel σ-field of the topological space S.

□

Let

$$H_{\frac{1}{2}} \equiv (-\Delta_{d-1} + m^2)^{\frac{1}{2}}, \tag{2.17}$$

and define the operator $d\Gamma(H_{\frac{1}{2}})$ on $L^2(\mu_0)$ such that (for the notations cf. Remark 3.)

$$\begin{aligned} &d\Gamma(H_{\frac{1}{2}})(: \phi_0(\varphi_1) \cdots \phi_0(\varphi_n) :) =: \phi_0(H_{\frac{1}{2}}\varphi_1)\phi_0(\varphi_2) \cdots \phi_0(\varphi_n) : + \cdots \\ &\cdots + : \phi_0(\varphi_1) \cdots \phi_0(\varphi_{n-1})\phi_0(H_{\frac{1}{2}}\varphi_k) : \end{aligned} \tag{2.18}$$

The following Proposition 2.1 is the answer for **Question 1** (cf., e.g., [Si]), and gives an explanation how the **Key word 1**, **2** and **3**, $\mathcal{S}'(\mathbb{R}^{d-1} \to \mathbb{R})$-valued symmetric Markov processes, **hypercontractive semigroup** and Dirichlet form with **logarithmic Sobolev inequality**, appear in the discussion of the Euclidean quantum random field:

Proposition 2.1. *i) The operator $d\Gamma(H_{\frac{1}{2}})$ on $L^2(\mu_0)$ with the natural domain is an essentially self adjoint non negative operator, and it is a generator of a Markovian semigroup (i.e. satisfying the properties of positivity preserving and $L^q(\mu_0)$ contraction ($q \in [0, \infty]$) given in Question 1), moreover this Markovian semigroup is* **hypercontractive** *(cf. [G]), presisely, for any $1 < q < p$ there exists $C_{p,q}$, $t_{p,q} > 0$ and any $t \geq t_{p,q}$ the following holds:*

$$\|T_t X\|_{L^p(\mu_0)} \leq \|X\|_{L^q(\mu_0)}, \quad \textit{where} \quad T_t \equiv e^{-td\Gamma(H_{\frac{1}{2}})}. \tag{2.19}$$

ii) By the probability measure μ_0 on $\mathcal{S}'(\mathbb{R}^{d-1} \to \mathbb{R})$ and the Markovian semigroup $\{T_t\}_{t>0}$, (2.15) in Question 1 is satisfied, namely the corresponging **symmetric Markov process** *generates the Nelson's Euclidean free field ϕ_N.*
iii) The operator $d\Gamma(H_{\frac{1}{2}})$ on $L^2(\mu_0)$ corresponds to a **pre-Dirichlet form** *as follows (hence, the discussion of the* **logarithmic Sobolev inequality** *is possible): For bounded smooth $F, G \in C_b^1(\mathbb{R}^n \to \mathbb{R})$, $n \in \mathbb{N}$, and $\{l_j\}_{j\in\mathbb{N}}$, a sequence of an O.N.B. of $L^2(\mathbb{R}^{d-1}; \lambda^{d-1})$ denote*

$$F(\phi_0) = F(<\phi_0, l_1>, \cdots, <\phi_0, l_n>), \quad G(\phi_0) = G(<\phi_0, l_1>, \cdots, <\phi_0, l_n>),$$

then

$$\begin{aligned}
&E\Big[\big\{d\Gamma(H_{\frac{1}{2}})F(\phi_0)\big\}\,G(\phi_0)\Big] \\
&= \sum_{j=1}^{\infty} E\Big[\big\{\lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(F(< \phi_0+\epsilon l_j, l_1 >, \cdots, < \phi_0+\epsilon l_j, l_n >) - F(\phi_0)\big)\big\} \\
&\times\big\{\lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(G(< \phi_0+\epsilon l_j, l_1 >, \cdots, < \phi_0+\epsilon l_j, l_n >) - G(\phi_0)\big)\big\}\Big] \\
&= E\Big[\sum_{j=1}^{n}(\frac{\partial}{\partial x_j}F)\,(\frac{\partial}{\partial x_j}G)\Big],
\end{aligned} \tag{2.20}$$

(for the notations cf. Remark 3.). □

Remark 4. Regardless of l_j being not an element of the **Cameron-Martin space** of the abstract Wiener space $(\mu_0, \mathcal{S}'(\mathbb{R}^{d-1}\to\mathbb{R}))$, the formula $F(< \phi_0+\epsilon l_j, l_1 >, \cdots, < \phi_0+\epsilon l_j, l_n >)$ in (2.20) is a measurable function (i.e., a random variable), since

$$\begin{aligned}
&F(< \phi_0+\epsilon l_j, l_1 >, \cdots, < \phi_0+\epsilon l_j, l_n >) \\
&= F(< \phi_0, l_1 > +\epsilon < l_j, l_1 >, \cdots, < \phi_0, l_n > +\epsilon < l_j, l_n >).
\end{aligned}$$

□

Without any precise definition of the terminologies concerning the **Wightman** fields, however, we give the following inportant result (cf., e.g., [Si], [AY] and references therein).

Proposition 2.2. *The field opertator $\Phi_t(\varphi)$, $t\in\mathbb{R}$, $\varphi\in\mathcal{S}(\mathbb{R}^{d-1}\to\mathbb{C})$ on the complexified $L^2(\mu_0)$ defined by*

$$\Phi_t(\varphi) = e^{itd\Gamma(H_{\frac{1}{2}})}\phi_0(\varphi)e^{-itd\Gamma(H_{\frac{1}{2}})}, \tag{2.21}$$

is the field operator of (Wightman) quantum free field on $\mathbb{R}^d$, and $e^{-itd\Gamma(H_{\frac{1}{2}})}$ is the time translation operator on it (for the notations cf. Remark 3.).

□

Since, we have seen that an Euclidean quantum random field, the Nelson's Euclidean field, can be constructed both by the direct method by means of the stochastic integral of the integral kernel of the operator $(-\Delta_d+m^2)^{-\frac{1}{2}}$ with respect to W^d and, as well, by the $\mathcal{S}'(\mathbb{R}^{d-1}\to\mathbb{R})$ valued symmetric Markov process, we may ask an another natural question, Question 2, below:

(Question 2) Nelson's Euclidean free field on $\mathbb{R}^d$ constructed from $\mathcal{S}'(\mathbb{R}^{d-1})$ valued symmetric Markov process in the discussion of Question 1 is the field that has **no**

interaction term, then is it possible to define a (partial) Euclidean field with an interaction term as a probability law of the trajectories of some $\mathcal{S}'(\mathbb{R}^{d-1})$ valued symmetric Markov process?

Precisely, can we construct a (partial) Euclidean quantum random field having an **interaction** term from some sharp time random field on $\mathbb{R}^{d-1}$ and some Markovian semigroup?

$\square$

In case when $\mathbf{d} = \mathbf{2}$ the answer for Question 2 is also affirmative, which we give in the following Proposition 2.3. The results are the well known $P(\phi)_2$ models and related models (cf., e.g., [Si]). The strategy of constructing (partial) Euclidean quantum random field from the probability laws of the trajectories of some Markov processes (giving the initial distributions) are refered as **stochastic quantization** (cf. references in [AY]), for which the **the Dirichlet space** arguments are crucial (cf. (2.20)).

Only for the next three propositions, we suppose that $\mathbf{d} = \mathbf{2}$. For each $p \in \mathbb{N}$, $T \geq 0$ and $r \in \mathbb{N}$ we define the random variables $v^{2p}(r)$ and $V^{2p}(r,T)$, which are potential terms on the sharp time free field and Nelson's Euclidean free field respectively, as follows:

$$
\begin{aligned}
v^{2p}(r) &= <: \phi_0^{2p} :, \Lambda_r > \\
&\equiv \int_{\mathbb{R}^{2p}} \left\{ \int_{-\infty}^{\infty} \Lambda_r(x) \prod_{k=1}^{2p} H_{-\frac{1}{4}}(x - x_k)\, dx \right\} W^1(dx_k) \\
&\equiv \int_{-\infty}^{\infty} \Lambda_r(x)\ : \phi_0^{2p} : (x)\, dx \in \bigcap_{q \geq 1} L^q(\mu_0),
\end{aligned} \tag{2.22}
$$

$$
\begin{aligned}
V^{2p}(r,T) &= \int_{-T}^{T} <: \phi_N^{2p} : (t,\cdot), \Lambda_r > \ dt \\
&\equiv \int_{-T}^{T} \int_{(\mathbb{R}^2)^{2p}} \left\{ \int_{-\infty}^{\infty} \Lambda_r(x) \prod_{k=1}^{2p} L_{-\frac{1}{2}}((t,x) - (t_k,x_k))\, dx \right\} W^2(d(t_k,x_k))\, dt \\
&\equiv \int_{-T}^{T} \int_{-\infty}^{\infty} \Lambda_r(x) : \phi_N^{2p} : (t,x)\, dx\, dt \in \bigcap_{q \geq 1} L^q(\mu_N),
\end{aligned} \tag{2.23}
$$

where for $r \in \mathbb{N}$, $\Lambda_r \in C_0^\infty(\mathbb{R} \to \mathbb{R}_+)$ is a given function such that $0 \leq \Lambda_r(x) \leq 1$ $(x \in \mathbb{R})$, $\Lambda_r \equiv 1$ $(|x| \leq r)$, $\Lambda_r \equiv 0$ $(|x| \geq r+1)$ (for the notations cf. Remark 3.).

We have the following important estimates (cf. eg., [Si]).

Proposition 2.3. *Let $d\Gamma(H_{\frac{1}{2}})$ be the positive self adjoint operator on $L^2(\mu_0)$ defined in Proposition 2.1. For each $p \in \mathbb{N}$ there exists some $\mathcal{S}(\mathbb{R})$ norm $|||\cdot|||$ and the* *multiplicative operator* $|v^{2p}(r)|$ *is dominated by* $d\Gamma(H_{\frac{1}{2}})+1$ *as follows*

$$|v^{2p}(r)| \leq (d\Gamma(H_{\frac{1}{2}})+1)|||\Lambda_r|||, \qquad \forall r\in\mathbb{N}. \tag{2.24}$$

For each $p\in\mathbb{N}$, $\lambda\geq 0$ *and* $r\in\mathbb{N}$ *the operator* $d\Gamma(H_{\frac{1}{2}})+\lambda v^{2p}(r)$ *on* $L^2(\mu_0)$ *is essentially self adjoint on the natural domain and bounded below:*
There exists a smallest eigenvalue $\alpha=\alpha_{2p,r,\lambda}>-\infty$ *and the corresponding eigenfunction* $\rho=\rho_{2p,r,\lambda}$ *of* $d\Gamma(H_{\frac{1}{2}})+v^{2p}(r)$ *such that*

$$\big(d\Gamma(H_{\frac{1}{2}})+v^{2p}(r)\big)\rho=\alpha\cdot\rho, \tag{2.25}$$

$$\rho(\phi)>0, \quad \mu_0 \text{ a.e. } \phi\in\mathcal{S}'(\mathbb{R}); \qquad d\Gamma(H_{\frac{1}{2}})+v^{2p}(r)\geq\alpha. \tag{2.26}$$

For each $p\in\mathbb{N}$, $\lambda\geq 0$, $r\in\mathbb{N}$ *and* $T\geq 0$

$$e^{-\lambda V^{2p}(r,T)}\in\bigcap_{q\geq 1}L^q(\mu_N). \tag{2.27}$$

(All notations follow the rule given by Remark 3.)

Because $v^{2p}(r)$ is defined through $H_{-\frac{1}{4}}$ (cf. (2.22)), (2.24) holds for $d\Gamma(H_{\frac{1}{2}})$ with $H_{\frac{1}{2}}$. (2.26) can be shown by a crucialy use of the **hypercontractivity** of $e^{-td\Gamma(H_{\frac{1}{2}})}$ (cf. (2.19)) and (2.24). (2.27) is also a consequence of the Nelson's hypercontractive bound on $L^q(\mu_N)$, $q\geq 1$.

Proposition 2.4. *Let* $\alpha_{2p,r,\lambda}$ *and* $\rho_{2p,r,\lambda}>0$ *be the eigenvalue and eigenfunction in Prop.2.3 respectively, and suppose that* $\rho_{2p,r,\lambda}$ *is normalized in order that*

$$E^{\mu_0}\left[(\rho_{2p,r,\lambda}(\cdot))^2\right]=1.$$

Let $\nu_{2p,r,\lambda}$ *be the probability measure on* $\mathcal{S}'(\mathbb{R})$ *such that*

$$\nu_{2p,r,\lambda}\equiv(\rho_{2p,r,\lambda})^2\mu_0,$$

and define a mapping $U : L^2(\mu_0)\to L^2(\nu_{2p,r,\lambda})$ *as follows:*

$$UX\equiv\frac{X}{\rho_{2p,r,\lambda}}, \qquad X\in L^2(\mu_0).$$

Then the operator $\tilde{T}_t$, $t\geq 0$, *on* $L^q(\nu_{2p,r,\lambda})$, $q\geq 0$, *defined by*

$$\tilde{T}_t\equiv U\exp\Big\{-t\big(d\Gamma(H_{\frac{1}{2}})+\lambda v^{2p}(r)-\alpha_{2p,r,\lambda}\big)\Big\}U^{-1}, \quad t\geq 0, \tag{2.28}$$

is Markovian contraction semigroup. By taking $\nu_{2p,r,\lambda}$ *the initial distribution,* $\tilde{T}_{|t|}$, $t \in \mathbb{R}$, *generates a partial Euclidean random field on* $\mathcal{S}'(\mathbb{R}^2)$ *the probability law of which is identical to*

$$d\mu_{V^{2p}(r,\infty)} = \lim_{T\to\infty} \frac{e^{-\lambda V^{2p}(r,T)} d\mu_N}{E^{\mu_N}[e^{-\lambda V^{2p}(N,T)}]}, \tag{2.29}$$

more precisely (cf. (2.15) in Question 1), for any $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R} \to \mathbb{R})$, *and any* $t_1, t_2 \geq 0$

$$\begin{aligned} &\int_{\mathcal{S}'(\mathbb{R}\to\mathbb{R})} \tilde{T}_{t_1}\Big(\Big(\tilde{T}_{t_2} < \cdot, \varphi_2 >_{\mathcal{S}',\mathcal{S}}\Big)(\cdot) < \cdot, \varphi_1 >_{\mathcal{S}',\mathcal{S}}\Big)(\phi)\ \nu_{2p,r,\lambda}(d\phi) \\ &\quad = E^{\mu_{V^{2p}(r,\infty)}}[< \phi, \varphi_1 \times \delta_{\{t_1\}}(\cdot) >< \phi, \varphi_2 \times \delta_{\{t_1+t_2\}}(\cdot) >], \end{aligned} \tag{2.30}$$

where $E^{\mu_{V^{2p}(r,\infty)}}[\,\cdot\,]$ *denots the expectation taken with respect to the measure* $\mu_{V^{2p}(r,\infty)}$. *(All notations follow the rule given by Remark 3.)*

Remark. 5 Nelson's Euclidean free field ϕ_N possesses an important property known from the Ising model, namely the *"ferromagnetic property"* which leads to *"ferromagnetic inequalities"*, e.g., for the evaluation of $\mu_{V(r,\infty)}$ given by (2.29), which is a perturbation of μ_N, the probability law of ϕ_N, one has the **GKS inequality** (Griffith, Kelly and Sherman) and the **FKG inequality** (Fortuin, Kastelyn and Ginibre), cf., eg. [Si]. Using these inequalities one obtain, e.g.,

$$\mu_{V(\infty,\infty)} = \lim_{r\to\infty} \mu_{V(r,\infty)}.$$

The random field on $\mathcal{S}'(\mathbb{R}^2)$ characterized by the probability measure $\mu_{V(\infty,\infty)}$ is known as the Euclidean $P(\phi)$ quantum field with the space time dimension $d = 2$, and is denoted by $P(\phi)_2$, cf. [GRS].

□

§ 3. Generalization of Proposition 2.4 to higher dimensions

In this section we propose a mathematical extension of Proposition 2.4, and give a generalized model of $P(\phi)_2$ to the higher space time dimensions. Here the basic tool of constructing the random field is "$\mathcal{S}'(\mathbb{R}^{d-1} \to \mathbb{R})$ valued symmetric Markov processes" (cf. (2.28)).

Let $d \in \mathbb{N}$ $(d \geq 2)$ be a given space time dimension, and ϕ_0 and ϕ_N be the corresponding sharp time free field and Nelson's Euclidean free field defined by (2.5) and (2.4) respectively, and μ_0 and μ_N be the probability laws of ϕ_0 and ϕ_N respectively. For real γ satisfying

$$\gamma \geq \frac{d-1}{4}, \tag{3.1}$$

let

$$H_{-\gamma} \equiv (-\Delta_{d-1} + m^2)^{-\gamma}. \tag{3.2}$$

For $r \in \mathbb{N}$, $\Lambda_{r,d-1} \in C_0^\infty(\mathbb{R}^{d-1} \to \mathbb{R}_+)$ is a given function such that

$$0 \le \Lambda_{r,d-1}(\vec{x}) \le 1 \ (\vec{x} \in \mathbb{R}^{d-1}), \quad \Lambda_{r,d-1} \equiv 1 \ (|\vec{x}| \le r), \quad \Lambda_{r,d-1} \equiv 0 \ (|\vec{x}| \ge r+1),$$

for $p \in \mathbb{N}$ define

$$\begin{aligned} v_{d-1}^{2p}(r) &= <: (H_{-\gamma+\frac{1}{4}}\phi_0)^{2p} :, \Lambda_{r,d-1} > \\ &\equiv \int_{(\mathbb{R}^{d-1})^{2p}} \left\{ \int_{\mathbb{R}^{d-1}} \Lambda_{r,d-1}(\vec{x}) \prod_{k=1}^{2p} H_{-\gamma}(\vec{x}-\vec{x}_k)\, d\vec{x} \right\} W^{d-1}(d\vec{x}_k) \\ &\equiv \int_{\mathbb{R}^{d-1}} \Lambda_{r,d-1}(\vec{x}) \ : (H_{-\gamma+\frac{1}{4}}\phi_0)^{2p} : (\vec{x})\, d\vec{x} \ \in \bigcap_{q \ge 1} L^q(\mu_0), \end{aligned} \tag{3.3}$$

and for $T \ge 0$,

$$\begin{aligned} V_d^{2p}(r,T) &= \int_{-T}^{T} <: (H_{-\gamma+\frac{1}{4}}\phi_N)^{2p} : (t,\cdot), \Lambda_{r,d-1} > \, dt \\ &\equiv \int_{-T}^{T} \int_{(\mathbb{R}^d)^{2p}} \left\{ \int_{\mathbb{R}^{d-1}} \Lambda_{r,d-1}(\vec{x}) \prod_{k=1}^{2p} H_{-\gamma+\frac{1}{4}}(\vec{x}-\vec{x}_k) L_{-\frac{1}{2}}((t,\vec{x})-(t_k,\vec{x}_k))\, d\vec{x} \right\} \\ &\times W^d(d(t_k,\vec{x}_k))\, dt \\ &\equiv \int_{-T}^{T} \int_{\mathbb{R}^{d-1}} \Lambda_{r,d-1}(\vec{x}) : (H_{-\gamma+\frac{1}{4}}\phi_N)^{2p} : (t,\vec{x})\, d\vec{x}\, dt \ \in \bigcap_{q \ge 1} L^q(\mu_N). \end{aligned} \tag{3.4}$$

(All notations follow the rule given by Remark 3.)

Under the assumption (3.1), we have (3.3) which corresponds to (2.22) in $P(\phi)_2$ theory. Since, for $d \ge 2$ by (3.1) and (3.2), the pseudo differential operator $H_{-\gamma}$, by which $v_{d-1}^{2p}(r)$ is defined, satisfies $\gamma \ge \frac{1}{4}$, similar to (2.24), we have the following estimate (3.5) for $v_{d-1}^{2p}(r)$ by $d\Gamma(H_{\frac{1}{2}})$ on $L^2(\mu_0)$ with the space dimension $d-1$. Also, by making use of the **hypercontractivity** of $e^{-td\Gamma(H_{\frac{1}{2}})}$ and (3.5), similar to (2.26), we have the following (3.7). (3.8) also can be shown through a similar discussion for (2.27), but all the results in this section (Theorems 3.1 and 3.2) are new developments.

Theorem 3.1. *Let $d\Gamma(H_{\frac{1}{2}})$ be the positive self adjoint operator on $L^2(\mu_0)$ defined in Proposition 2.1. Suppose that (3.1) is satisfied. For each $p \in \mathbb{N}$ there exists some $\mathcal{S}(\mathbb{R}^{d-1})$ norm $|||\cdot|||$ depending on γ and the multiplicative operator $|v_{d-1}^{2p}(r)|$ is dominated by $d\Gamma(H_{\frac{1}{2}})+1$ as follows*

$$|v_{d-1}^{2p}(r)| \le (d\Gamma(H_{\frac{1}{2}})+1)|||\Lambda_{r,d-1}|||, \qquad \forall r \in \mathbb{N}. \tag{3.5}$$

For each $p \in \mathbb{N}$, $\lambda \geq 0$ and $r \in \mathbb{N}$ the operator $d\Gamma(H_{\frac{1}{2}}) + \lambda v_{d-1}^{2p}(r)$ on $L^2(\mu_0)$ is essentially self adjoint on the natural domain and bounded below:
There exists a smallest eigenvalue $\alpha = \alpha_{d-1,2p,r,\lambda} > -\infty$ and the corresponding eigenfunction $\rho = \rho_{d-1,2p,r,\lambda}$ such that

$$\big(d\Gamma(H_{\frac{1}{2}}) + v_{d-1}^{2p}(r)\big)\rho = \alpha \cdot \rho, \tag{3.6}$$

$$\rho(\phi) > 0, \quad \mu_0 \text{ a.e. } \phi \in \mathcal{S}'(\mathbb{R}); \qquad d\Gamma(H_{\frac{1}{2}}) + v_{d-1}^{2p}(r) \geq \alpha. \tag{3.7}$$

For each $p \in \mathbb{N}$, $\lambda \geq 0$, $r \in \mathbb{N}$ and $T \geq 0$

$$e^{-\lambda V_{d-1}^{2p}(r,T)} \in \bigcap_{q \geq 1} L^q(\mu_N). \tag{3.8}$$

(All notations follow the rule given by Remark 3.)

Theorem 3.2. *Let $\alpha_{d-1,2p,r,\lambda}$ and $\rho_{d-1,2p,r,\lambda} > 0$ be the eigenvalue and eigenfunction in Theorem 3.1 respectively, and suppose that $\rho_{d-1,2p,r,\lambda}$ is normalized in order that*

$$E^{\mu_0}\big[\rho_{d-1,2p,r,\lambda}(\cdot))^2\big] = 1.$$

Let $\nu_{d-1,2p,r,\lambda}$ be the probability measure on $\mathcal{S}'(\mathbb{R})$ such that

$$\nu_{d-1,2p,r,\lambda} \equiv (\rho_{d-1,2p,r,\lambda})^2 \mu_0,$$

and define a mapping $U_{d-1} : L^2(\mu_0) \to L^2(\nu_{d-1,2p,r,\lambda})$ as follows:

$$U_{d-1}X \equiv \frac{X}{\rho_{d-1,2p,r,\lambda}}, \qquad X \in L^2(\mu_0).$$

Then the operator $\tilde{T}_t^{d-1}$, $t \geq 0$, on $L^q(\nu_{2p,r,\lambda})$, $q \geq 0$, defined by

$$\tilde{T}_t^{d-1} \equiv U_{d-1} \exp\Big\{-t\big(d\Gamma(H_{\frac{1}{2}}) + \lambda v_{d-1}^{2p}(r) - \alpha_{d-1,2p,r,\lambda}\big)\Big\} U_{d-1}^{-1}, \quad t \geq 0, \tag{3.9}$$

is a Markovian contraction semigroup. By taking $\nu_{d-1,2p,r,\lambda}$ the initial distribution, $\tilde{T}_{|t|}^{d-1}$, $t \in \mathbb{R}$, generates a partial Euclidean random field on $\mathcal{S}'(\mathbb{R}^d)$ the probability law of which is identical to

$$d\mu_{V_d^{2p}(r,\infty)} = \lim_{T \to \infty} \frac{e^{-\lambda V_d^{2p}(r,T)} d\mu_N}{E^{\mu_N}[e^{-\lambda V_d^{2p}(N,T)}]}. \tag{3.10}$$

More precisely (cf. (2.15) in Question 1), for any $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^{d-1} \to \mathbb{R})$, and any $t_1, t_2 \geq 0$

$$\int_{\mathcal{S}'(\mathbb{R}^{d-1}\to\mathbb{R})} \tilde{T}_{t_1}^{d-1}\Big(\Big(\tilde{T}_{t_2}^{d-1} < \cdot, \varphi_2 >_{\mathcal{S}',\mathcal{S}}\Big)(\cdot) < \cdot, \varphi_1 >_{\mathcal{S}',\mathcal{S}}\Big)(\phi)\ \nu_{d-1,2p,r,\lambda}(d\phi)$$
$$= E^{\mu_{V_d^{2p}(r,\infty)}}[< \phi, \varphi_1 \times \delta_{\{t_1\}}(\cdot) >< \phi, \varphi_2 \times \delta_{\{t_1+t_2\}}(\cdot) >], \tag{3.11}$$

where $E^{\mu_{V_d^{2p}(r,\infty)}}[\,\cdot\,]$ *denotes the expectation taken with respect to the measure* $\mu_{V_d^{2p}(r,\infty)}$. *(All notations follow the rule given by Remark 3.)*

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